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# A New Approach in the Calibration of Stochastic Volatility Models

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**Introduction**

The value of a share represents the current value of the future cash flows that it ensures. It is anything but certain: who issues a share doesn't make a commitment against those who subscribe it, if not to get them to participate in the profits of a business activity.
Therefore it is easy to understand that the evaluation of the shares is a matter of great complexity. Investors and financial analysts usually rely on models that can explain the trend value of shares.
In these analysis advanced models are often used: they are based on stochastic differential equations in which the uncertainty of financial market is represented by a random process.
Through these models, in absence of arbitrage, the operators try to outline the prices of derivatives linked to an underlying entity traded on the market (e.g. Options) and develop techniques in order to reduce the risk in such investments.

In these models, the volatility represents a fundamental factor for the price evaluation of financial primary assets as well as of financial derivatives. The well-known formula of Black & Scholes (1973) is frequently used for the evaluation of European options but the disadvantage of this formula relies on the assumption of a constant volatility like a constant parameter with respect the time. This formula is also used to evaluate the underlying volatility (called implied volatility) observing option prices. Unfortunately, the implied volatility observed on the market changes significantly in time. Thus, models describing time-varying volatility are suitable for realistic applications.

This problem has been investigated in several research contributions: Scott (1977) created an econometric model to estimate the volatility parameter in its stochastic process; Johnson and Shanno (1987) used Monte Carlo methods to simulate the underlying asset price ($S_t$) and the volatility processes; Heynen and others (1994) analyzed many econometric models (ARCH(1), GARCH(1,1) and EGARCH(1,1)) for the volatility dynamics; Xu and Taylor (1994) used un AR(1) model and a Kalman filter technique in order to understand the volatility behavior through the implied volatility.

In many research papers the use of *mean-reverting* random processes has been highlighted. In fact, there is an empirical evidence that this kind of dynamics represents a good driver for asset volatility dynamics.
Moreover, empirical analyzes reveal that large part of financial time series have probability distributions very different from a normal distribution. This probability distributions assigns a higher probability to events far from the mean (*fat tail*). Leptokurtosis is compatible with variance variability of asset.

In this thesis we address the problem of parameter estimation in one of the most popular stochastic volatility model: the Heston's model (1993). This model assumes that the price of the asset is described by a geometric Brownian motion with stochastic variance. The variance is driven by a *square-root* stochastic differential equation. This last derives by Cox Ingersoll Ross model (1985), a well known model for the evolution of interest rates.

We propose a new approach for the estimation of the parameters in general stochastic volatility models. Then, in order to establish the validity of our technique, we apply our method to the Heston case, and we present numerical comparisons with existing techniques.

**Heston Stochastic Volatility Model**

We suppose that the stock price $S$ and its variance $v$ satisfy the following SDEs:

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW^1(t) \quad (1)$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW^2(t) \quad (2)$$

$$(3)$$

where $W^1(t)$ and $W^2(t)$ are standard BM with instantaneous correlation $\rho \in [-1, 1]$, i.e. $Cov(W^1(t), W^2(t)) = \rho t$, and $\mu \in \mathbb{R}$ is the instantaneous drift of stock price returns. The stochastic process that describes volatility is an example of *square-root process* introduced by Cox, Ingersoll and Ross in 1985 to describe the interest rate. Its deterministic component $\kappa(\theta - v(t))$ produces an oscillation around the parameter $\theta > 0$, called *long-term volatility.* For this reason we say that the process is mean-reverting, and the parameter

$\kappa > 0$ is called *reversion speed.* In the stochastic term $\sigma\sqrt{v(t)}dW^2(t)$, the constant $\sigma > 0$ is called *the volatility of volatility* and it gives the intensity of the noise generated by $W^2(t)$.
For the volatility process to remain strictly positive, the parameters $\kappa > 0$, $\theta > 0$ and $\sigma > 0$ must also verify a fundamental constraint, the Feller condition [54]:

$$\frac{2\kappa\theta}{\sigma^2} > 1. \tag{4}$$

One of the reasons for the popularity of the Heston model is that it provides a closed-form solution for pricing of vanilla options. This is of great benefit in particular when calibrating against market prices. In this section we will derive the closed-form for the price of a Call option according to the Heston model following Heston's original 1993 approach. In the Black-Scholes case, there is only one source of randomness – the stock price $S = S(t)$- which can be hedged with stock. In the present case, random changes in volatility also need to be hedged in order to form a riskless portfolio.

**Financial Market Modeling with Random Parameters**

In the Chapter 3 of this thesis, we will propose a new mathematical framework to price financial instruments derivatives, where the underlying stochastic model depends on some random parameters governed by a pre-defined probability law. For a fixed parameter value, we consider a stochastic Ito process driven by a Brownian motion on the same probability space for the assets dynamics in the market.

The motivation for the introduction of this new point of view in financial modeling is mainly based on the fact that the price of a call option obtained in the framework of a stochastic volatility model depends on the value $v_0$, the initial volatility, that unfortunately acts like an hidden stochastic variable. The most simple approach adopted to resolve the estimation of this hidden variable, is considering $v_0$ as an additional parameter in the calibration procedure.

In fact, in valuing financial derivatives, the no-arbitrage price of a European-type derivative can be found by a representation formula, where the price is given as a conditional expectation under a *risk-neutral* probability measure. In our approach, we extend that framework, by proving a new version of the fundamental theorem of asset pricing (Harrison Kreps (1979)) for processes depending on random parameters.

This allows to state a no arbitrage pricing formula similar to the classical one, without conflict with classical theory.

Although the application of this new arbitrage context is related to a specific stochastic volatility model, the theoretical results presented in this work, exhibit a much more general significance in financial modeling.

**Random Parameters Modeling**

Let $q : (\Omega_0, \mathcal{F}_0) \to (\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p))$ a random variable defined on a complete probability space $(\Omega_0, \mathcal{F}_0, f_0)$ with a probability law $\mu_0$ defined as follows:

$$\mu_0(B) = f_0(\{\omega_0 \in \Omega_0 : q(\omega_0) \in \mathcal{B}\}), \ \ \forall B \in \mathcal{B}(\mathbb{R}^p). \tag{5}$$

Let $(\tilde{\Omega}_0, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}}_0)$ be the product probability space:

1. $\tilde{\Omega}_0 := \Omega \times \mathbb{R}^p$ and $\tilde{\Omega}_0 \ni \tilde{\omega} = (\omega, q)$
2. $\tilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^p)$
3. $\tilde{\mathbb{P}}_0 := \mathbb{P} \otimes \mu_0$

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a complete probability space where is defined a $d$-dimensional BM $W(t)$ which usually represents stock prices in a market.

We consider a $d$-dimensional market with a set of risky non-dividend-paying assets, with price at time $t$ given by $S_i^q(t)$, $i = 1, \dots k$, and a riskless asset, a bond, with price at time $t$ given by $S_0(t)$. They verify the following $q$-depending stochastic differential equations, for every fixed $q \in \mathbb{R}^p$:

$$\begin{cases} dS_i^q(t) = \mu_i(t,q)S_i^q(t)dt + \sum_{j=1}^d \sigma_{i,j}(t,q)S_i^q(t)dW_j(t), \\ \qquad\qquad S_i^q(0) = s_i^q > 0, \qquad\qquad\qquad \text{for } t \in [0,T] \end{cases} \tag{6}$$

and

$$\begin{cases} dS_0(t) &= \quad r_t S_0(t)dt, \\ S_0(0) &= \quad s_0 > 0 \quad \text{for } t \in [0,T]. \end{cases} \tag{7}$$

In particular we focus on interest rate $r_t$ that is a real-valued progressively measurable process and, without loss of generality, we assume that $s_0 = 1$ so that $S_0(t) = \exp(\int_0^t r_u du)$. We assume that the initial prices are known, given $q \in \mathbb{R}^p$.

**Portfolio and Arbitrage**

An essential feature of market modeling is based on the absence of arbitrage opportunities. This assumption can be interpreted as a market equilibrium condition.

Now we introduce the notion of Portfolio and Arbitrage in the market (6)-(7).

**Definition 0.0.1.** *A portfolio in this market is a progressively measurable process $\theta(t)$ with values in $\mathbb{R}^{k+1}$. The value of this portfolio at time t will be given by $\langle \theta(t), S^q(t) \rangle$.*

**Definition 0.0.2.** *The porfolio $\theta(t)$ is a self-financing portfolio if $\theta \in \Lambda(S)$ and its value verifies*

$$\langle \theta(t), S^q(t) \rangle = \langle \theta(0), S(0) \rangle + \int_0^t \theta(s) dS^q(s) \tag{8}$$

*for all $0 \leq t \leq T$, $\tilde{\mathbb{P}}_0$ almost surely.*

**Definition 0.0.3.** *A portfolio $\theta \in \Theta(S^q)$ is an arbitrage if*

$$\langle \theta(0), S^q(0) \rangle \leq 0 \leq \langle \theta(T), S^q(T) \rangle, \quad \tilde{\mathbb{P}}_0\text{-}a.e.$$

*and*

$$\mathbb{P}(\langle \theta(T), S^q(T) \rangle > 0) > 0, \quad \mu_0\text{-}a.e$$

In absence of arbitrage there is a proportional relationship between average rate of security prices variation and risk related to the same security values volatility. This concept is expressed by the following result. We define $V^q(t) := \frac{S^q(t)}{S_0(t)}$ and $\tilde{V}^q(t) := (V_1^q, \ldots, V_k^q)$.

$\sigma_{\tilde{V}}$ and $\mu_{\tilde{V}}$ are respectively the diffusion process and the drift of $\tilde{V}^q$.

**Theorem 0.0.4.** *Let the financial market represented by $V^q$ be arbitrage free. Then, there exists a $\mathbb{R}^d$-valued progressively measurable process $\lambda$ with respect to $(\tilde{\mathcal{F}}_t)_t$, such that for almost every $(\tilde{\omega}, t) \in \tilde{\Omega}_0 \times [0, T]$, it holds*

$$\sigma_{\tilde{V}}(t,q)\lambda(t,q) = \mu_{\tilde{V}}(t,q) \tag{9}$$

**$q$-Depending Risk Neutral Probability Measures**

By Girsanov theorem, we define a probability measure $\tilde{\mathbb{Q}}_0 \sim \tilde{\mathbb{P}}_0$ on the product space $\tilde{\Omega}_0$ by its Radon–Nikodym derivative:

$$\frac{d\tilde{\mathbb{Q}}_0}{d\tilde{\mathbb{P}}_0}|_{\tilde{\mathcal{F}}_t} = \delta_0^\lambda(t,q) = e^{-\int_0^t \lambda(s,q)dW_s - \frac{1}{2}\int_0^t |\lambda(s,q)|^2 ds}, \tag{10}$$

where $\lambda$ is given by Theorem 0.0.4. Thus, given a product set $A \times B$, where $A \in \mathcal{F}_t$ and $B \in \mathcal{B}(\mathbb{R}^p)$, $\tilde{\mathbb{Q}}_0$ acts as follows

$$\tilde{\mathbb{Q}}_0(A \times B) = \int_B \int_A \delta_0^\lambda(\omega, q) d\mathbb{P}(\omega) d\mu_0(q) = \int_B \mathbb{Q}^q(A) d\mu_0(q) \tag{11}$$

$\mathbb{Q}^q$ is obviously a probability measure on $\Omega$ which is equivalent to $\mathbb{P}$, $\mu_0$-a.e. Therefore we have the following mapping:

$$\mathbb{R}^p \to \{\text{Probability measures equivalent and absolutely continuous with respect to } \mathbb{P}\}$$
$$q \mapsto \mathbb{Q}^q, \quad \mathbb{Q}^q(A) = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A \delta_0^\lambda(\omega, q)]$$

Given the discounted price process on $\tilde{\Omega}_0$, $\tilde{V}^q$, the following relation holds true:

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}_0}[\tilde{V}^q(\omega,T)] &= \mathbb{E}^{\tilde{\mathbb{P}}_0}[\frac{d\tilde{\mathbb{Q}}_0}{d\tilde{\mathbb{P}}_0}\tilde{V}^q(\omega,T)] = \\
&= \int_{\mathbb{R}^p} d\mu_0(q) \int_\Omega \delta_0^\lambda(\omega,q)\tilde{V}^q(\omega,T)d\mathbb{P}(\omega) = \\
&= \int_{\mathbb{R}^p} \mathbb{E}^{\mathbb{Q}^q}[\tilde{V}^q(\omega,T)]d\mu_0(q)
\end{aligned}$$

$\{\tilde{V}^q(\omega,t)\}_{t\in[0,T]}$ is a stochastic process $\mu_0$-a.s. and a martingale under the measure $\mathbb{Q}^q$, i.e.

$$\mathbb{E}^{\mathbb{Q}^q}[\tilde{V}^q(\omega,T)|\mathcal{F}_t] = \tilde{V}^q(\omega,t) \quad \forall t < T, \quad \forall q, \quad \mathbb{P} - a.s. \tag{12}$$

In this thesis we prove that $\{\tilde{V}^q(\omega,t)\}_{t\in[0,T]}$ is a martingale under $\tilde{\mathbb{Q}}_0$, that is:

$$\mathbb{E}^{\tilde{\mathbb{Q}}_0}[\tilde{V}^q(\omega,T)|\tilde{\mathcal{F}}_t] = \tilde{V}^q(\omega,t) \quad \forall t < T, \tilde{\mathbb{P}}_0 - a.s. \tag{13}$$

In this framework, the extended no-arbitrage theorem and the contingent claim price represent by 13 is consistent with classical theory and, in this context, the range of implied prices is increased with respect to the case where $q$ is a constant parameter.
In order to prove 13, we use the following:

**Definition 0.0.5.** *Let $\Omega$ be a nonempty set, and let $D$ be a collection of subsets of $\Omega$. Then $D$ is a Dynkin system if*

1. $\Omega \in D$
2. *if* $A \in D$, *then* $A^c \in D$
3. *if* $A_1, A_2, A_3, \ldots$ *is a sequence of subsets in* $D$ *such that* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$, *then* $\bigcup_{n=1}^{\infty} A_n \in D$

**Theorem 0.0.6** (Dynkin's $\pi$-$\lambda$ Theorem [74])**.** *If $P$ is a $\pi$-system and $D$ is a Dynkin system with $P \subseteq D$, then $\sigma(P) \subseteq D$.*

**Averaged call price for the Heston's stochastic volatility model**

We apply the theoretical framework described in the Chapter 3 of the thesis to the case of the Heston stochastic volatility model. In addition, we state rigorous results in order to derive a closed-form formula for vanilla options (called *Averaged call price formula*).
Furthermore, we show how our extended version of the Heston pricing model is in fact more effective in the calibration of option prices in comparison with a traditional procedure often used in practice. In particular the estimation method as been applied to a real dataset of option prices written on the S& P500 index. Stochastic dynamics of stock prices is commonly described by a geometric (multiplicative) Brownian motion, which gives a log-normal

probability distribution function (PDF) for stock price changes (returns). However, numerous observations show that the tails of the PDF decay slower than the log–normal distribution predicts, see Bouchaud and Potters (2001).

There is empirical evidence and a set of stylized facts indicating that volatility, instead of being a constant parameter, is driven by a stochastic process. Due to the apperant contradiction of constant volatility assumption of the Black-Sholes model as illustrated by the volatility skew observed in practice, the stochastic volatility models were proposed and applied to the option pricing problems. We consider the simple stochastic volatility model proposed by Heston (1993).

The choice of Heston's model is motivated by the fact that it has a closed-form expression for the characteristic function of its transitional probability density function from which options can be efficiently priced, a feature of Heston's model that has received considerable attention in the literature. Heston's model is the most popular one because of its three main features: it does not allow negative volatility, it allows the correlation between asset returns and volatility and it has a closed-form pricing formula.

The calibration of the Heston model faces at least three difficulties. First, because volatility is random, an exact likelihood function cannot be computed, which means that the standard econometric method cannot be applied to estimate the underlying asset return diffusion process. Second, the data are observed at discrete times, but the model is built under a continuous-time framework. There must be a map between the continuous-time diffusion process suggested by the theory and the discrete time estimation used in practice. Third, in pricing options under stochastic volatility, the market is not complete; one needs to know the volatility risk premium before pricing the option.

In a time-series analysis, the fitting of stochastic volatility models to index returns is a well established field of research. For example, Ait-Sahalia and Kimmel (2007), (2010) develop closed-form approximations to the log-likelihood function; Eraker (2001), (2004), Eraker, Johannes and Polson (2003) Jacquier, Polson and Rossi (2004) propose the use of MCMC methods. Instead, filtering methods are used by Bates (1996), Johannes, Polson and Stroud (2009), Christoffersen, Jacobs and Mimouni (2010) and Hurn, Lindsay and McClelland (2012). The ability to combine a long time series data on the underlying's price with the available options with the aim to estimate the dynamics of a stochastic volatility model is, however, not new. See for instance Ait-Sahalia, Wang and Yared (2001), Jones (2003) and Eraker (2004). Expecially particle filter methods are implied for the unobserved state (volatility) and integration over the unobserved states is achieved by Monte Carlo integration Johannes, Polson and Stroud (2009). Unfortunately, the computational complexity of this approach is driven by the requirement that each particle in the filter must be used to price all the required options.

Since volatility is not a tradable asset, the option pricing formula for any stochastic volatility model will involve a volatility risk term. Therefore, in this chapter we describe and implement an estimation technique where the price of an option is based on the extended version of the market illustrated in Chapter 3. The advantage of this method is that the initial volatility, which is usually incorporated in the parameter set, is now considered as a random parameter driven by a suitable probability distribution function. All the model risk-neutral parameters, including those involved in the distribution of the initial volatility, can be identified by calibration procedure from cross sectional option prices. Actually, the question remains open whether the implied parameters truly reflect the original information contained in the underlying asset return distribution. As Bates (1996) points out, the major problem of the implied estimation method is the lack of an associated statistical theory. The implied methodology solely based on option prices is thus purely objective driven.

It should be stressed that the Heston model is used only as a specific example to allow the econometric methodology to be fully developed. Our technique itself is, not limited to any particular model and the extension to other models, eventually involving jumps, is a matter of detail alone and requires no further significant conceptual development. The method is illustrated using the S&P 500 Index from – to – and options written on the index over that period. All the parameters of the Heston's model of stochastic volatility are estimated with good precision.

**Heston model Calibration**

The Heston model has essentially six parameters that need estimation: $\kappa$, $\theta$, $\sigma$, $\rho$, $\mu$ and $v_0$. Research contributions have shown that the implied parameters that produce the correct vanilla option prices and their time-series estimate counterparts are different [5]. So one cannot just use empirical estimates for the parameters.

This leads to a complication that plagues stochastic volatility models in general. A common solution is to find those parameters which produce the correct market prices of vanilla options. This is called an inverse problem, as we solve for the parameters indirectly through some implied structure.

The most popular approach to solve this inverse problem is to minimize the error or discrepancy between model prices and market prices. This usually turns out to be a non-linear least-squares optimization problem. More specifically, the squared differences between vanilla option market prices and that of the model are minimized over the parameter space, i.e., we evaluate

$$\min_{\Omega} F(\Omega) = \min_{\Omega} \sum_{i=1}^{N} \omega_i \Big( C_i^{Model}(T_i, K_i; \Omega) - C_i^{Market}(K_i, T_i) \Big)^2 \qquad (14)$$

where $\Omega$ is a vector of parameter values, $C_i^{Model}(T_i, K_i; \Omega)$ and $C_i^{Market}(K_i, T_i)$ are the $i^{th}$ option prices from the model and market, respectively, with strike $K_i$ and maturity $T_i$, $N$ is the number of options used for calibration, and the $\omega_i$'s are weights (the choice of these weights will be discussed later).

The minimization above is not as trivial as it would seem. In general, $F(\Omega)$ is neither convex nor does it have any particular structure. This poses some complications:

- Finding the minimum of $F(\Omega)$ is not as simple as finding those parameter values that make the gradient of $F(\Omega)$ zero. This means that a gradient based optimization method will prove to be futile.
- Hence, finding a global minimum is difficult (and very dependent on the optimization method used).

- Unique solutions to (14) need not necessarily exist, in which case only local minima can be found. This has some implications regarding the stationarity of parameter values which are important in these types of models.

This is therefore an inverse, ill-posed problem termed the calibration problem. There are several calibration methods that have been experimented. The *regularization method* involves adding a penalty function, $p(\Omega)$, to (14) such that

$$\min_{\Omega} \sum_{i=1}^{N} \omega_i \Big( C_i^{Model}(T_i, K_i; \Omega) - C_i^{Market}(K_i, T_i) \Big)^2 + \alpha p(\Omega) \qquad (15)$$

is convex. The parameter $\alpha$ here is called the regularization parameter. Since we cannot hope to determine the exact solution to our problem because of its very nature, we attempt to find an approximation which is as close to the true solution as possible. To achieve this we are moved to replace our problem with one which is close to the original, but does not possess the ill conditioning which the makes the original intractable. For a detailed discussion refer to Chiarella et al. [47]. When applied to a given set of market prices, these methods yield a single set of model parameters calibrated to the market but also require the extra step of determining the regularization parameter [48].

**Hidden State Variable Estimation**

The price of a call option obtained in the framework of a stochastic volatility model depends on the value $v_0$, the initial volatility, that unfortunately acts like an hidden stochastic variable. The most simple approach adopted to resolve the estimation of this hidden variable, is considering $v_0$ as an additional parameter in the calibration procedure. An alternative approach can be perfomed with at-the-money implied variance, based on the results of Gatheral [26].

We propose a new method where $v_0$ is considered as a random variable with a given probability density function $\Pi$, satisfying suitable assumptions.

Consider the actual price of a call option with maturity $T_i$ and strike $K_i$ obtained in the framework of the Heston model that we denote here as

$$C_i^H(\Omega, v_0) = C^H(S, T_i, K_i; \Omega, v_0), \tag{16}$$

where $\Omega = (\kappa, \theta, \sigma, \rho)$ are the Heston's model parameters. Averaging over volatility, we can express the actual call price as

$$\mathbb{E}_\Pi[C_i^H(\Omega, v_0)] = \int_0^{+\infty} dv_0 C_i^H(\Omega, v_0)\Pi(v_0). \tag{17}$$

By the above expression, we remark that if we consider $v_0$ to be a positive constant and we replace $\Pi(v_0)$ with the Dirac delta function $\delta(v - v_0)$ centered at $v_0$, then (17) simplifies:

$$\int_0^{+\infty} dv C_i^H(\Omega, v)\delta(v - v_0) = C_i^H(\Omega, v_0). \tag{18}$$

Thus, the case of a constant initial volatility can be loosely seen as a special case of considering $v_0$ a random variable when the latter has the delta distribution. We wonder which approach yields the best approximation result of the model against option prices observed in the market.

To this end, we consider the set of probability density functions (pdf). Precisely, let $\mathcal{P}$ be the set of all non-negative Lebesgue-integrable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 0$ a.e. for $x \leq 0$, and

$$\int_0^{\infty} f(x)dx = 1. \tag{19}$$

Given a basket of $N$ call option prices related to different strikes and maturities $\{(K_i, T_i)\}_{i=1,\dots,N}$, let $\mathcal{P}'$ be a non-empty subset of $\mathcal{P}$ and define

$$J(\Omega, v_0) \quad := \quad \sum_{i=1}^{N} \omega_i |C_i^M - C_i^H(\Omega, v_0)|^2, \tag{20}$$

$$J'(\Omega, f) \quad := \quad \sum_{i=1}^{N} \omega_i |C_i^M - \mathbb{E}_f[C_i^H(\Omega, v_0)]|^2. \tag{21}$$

Let $\jmath$, $\jmath'$ be respectively the infimum of $J$ over $(\Omega, v_0)$, with $v_0 \in (0, +\infty)$ and the infimum of $J'$ over $(\Omega, f)$, where $f$ belongs to $\mathcal{P}'$.

We remind that the parameters in $\Omega$ are still supposed to verify the conditions: $\kappa, \theta, \sigma > 0$, $\rho \in (-1, 1)$ along with the Feller condition 4.

The following result states that if $\mathcal{P}'$ includes a sequence of densities weakly converging to the Dirac delta centered at an arbitrary $\bar{v} \geq 0$, then the calibration obtained through the $v_0$-averaged call price improves the value of the objective functional.

**Theorem 0.0.7.** *If $\mathcal{P}' \subseteq \mathcal{P}$ is such that for every $\bar{v} > 0$, there exists a sequence $\{f_n\}_n \subset \mathcal{P}'$ satisfying*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x) g(x) dx = g(\bar{v}) \tag{22}$$

*for all bounded, continuous functions $g : \mathbb{R} \to \mathbb{R}$. Then $\jmath' \leq \jmath$.*

We remark that (22) is equivalent to the weak converge of $\{f_n\}_n$ to $\delta(\cdot - \bar{v})$

In light of Theorem 0.0.7 and inspired by the work of Dragulescu and Yakovenko [51], we put our attention on the subset $\mathcal{G} \subset \mathcal{P}$ of the pdfs associated with the Gamma distribution:

$$\mathcal{G} = \left\{ g_{\alpha,\beta} \in \mathcal{P} \,:\, g_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} 1_{x>0}, \quad \alpha, \beta > 0 \right\}. \tag{23}$$

It is easy to see that $\mathcal{G}$ satisfies the condition (22) of Theorem 0.0.7. Indeed, let $\bar{v} \geq 0$ and $g_n \in \mathcal{G}$ be given by

$$g_n(x) = g_{\alpha_n,\beta_n}, \qquad \alpha_n = n, \qquad \beta_n = \frac{n}{\bar{v}}. \tag{24}$$

The characteristic function of $g_n$ is

$$\phi_n(t) = \left(1 - \frac{\imath t}{\beta_n}\right)^{-\alpha_n} = \left(1 - \frac{\imath t \bar{v}}{n}\right)^{-n}, \tag{25}$$

for all $t \in \mathbb{R}$. $g_n$ converges weakly to $\delta(\cdot - \bar{v})$, since $\phi_n(t) \to e^{\imath t \bar{v}}$, for any $t \in \mathbb{R}$, where

$$\phi(t) = \int_{\mathbb{R}} e^{\imath t x} \delta(x - \bar{v}) dx = e^{\imath t \bar{v}}, \tag{26}$$

which is the characteristic function associated to the delta function.

Thus, Theorem 0.0.7 states that $\jmath \geq \jmath_{\mathcal{G}}$, where

$$\jmath_{\mathcal{G}} = \inf_{(\Omega,\alpha,\beta)} \sum_{i=1}^{N} \omega_i |C_i^M - \mathbb{E}_g[C_i^H(\Omega, v_0)]|^2. \tag{27}$$

The calibration is simply achieved by adding to the set of parameters that characterize Heston model two real parameters $\alpha,\ \beta > 0$ that describe the distribution of the initial volatility $v_0$.

**The Averaged Call Price Formula**

When $\Pi$ is the pdf of the initial volatility $v_0$, the averaged call price at time $t = 0$ of a call option is given by

$$\tilde{C}(\Omega, K, T, r) = \mathbb{E}_\Pi[C^H(\Omega, K, T, r, v_0)] = \int_0^{+\infty} dv_0 C^H(\Omega, K, T, r, v_0)\Pi(v_0). \tag{28}$$

The theoretical results presented in Chapter 3 show that formula (28) represents a no arbitrage price in the extended market defined by the product probability space $\Omega \times \mathbb{R}$, endowed with the physical measure which is the product of the original probability $\mathbb{P}$ and the probability law related to $\Pi$. In Chapter 4, we prove two representation theorems that give simplified forms for the average call price above, both reducing the expression of $\tilde{C}(\Omega, K, T, r)$ to a single integration. This simplification will be of great convenience for numerical computation. The result makes use of Heston's original call price formula.

**Theorem 0.0.8.** *(***The Averaged Call Price Formula***) If $\Pi \in \mathcal{P}$ satisfies $\mathbb{E}_\Pi[v_0] < \infty$, then the call price in (28) the following*

$$\tilde{C}(\Omega, K, T, r) = S_0 Q_1(T, S_0) - e^{-r\tau} K Q_2(T, S_0) \tag{29}$$

*where*

$$Q_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\Big[\frac{e^{C_j(T,\phi)+\imath\phi\log(\frac{S_0}{K})} M_\Pi\big(D_j(T,\phi)\big)}{\imath\phi}\Big] d\phi \tag{30}$$

*for $j = 1,\ 2$, with $M_\Pi$ given by:*

$$M_\Pi(z) = \int_0^\infty e^{zv}\, \Pi(v)\, dv, \tag{31}$$

**Estimation Results**

In order to see whether the new model can survive from the volatile market, the data set of option prices is used from SPX index at the close of market form September 01, 2010 to September 30, 2010. It is considered only the call options that verify the no arbitrage condition. Moreover we only test the model with call option prices, which has $0.9 < M < 1.1$ where $M$ is the moneyness defined by $\frac{K}{S_0}$. Overall we considered 8315 call prices divided into 21 starting dates and 9 expiry dates. In Figure 1 is shown the prices

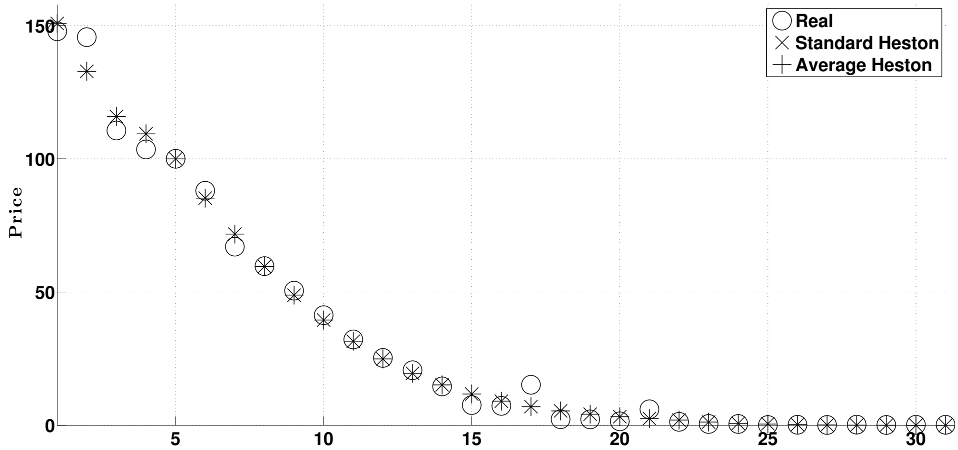


Figure 1: Call price value for a data set with starting date September 07, 2010 and expiry date June 18, 2011

obtained from the standard Heston pricing formula and from the averaged call price formula. The parameters estimated for both models is reported in Table 1.

To evaluates the performance of the two pricing methods, we use four error estimators: average prediction error (APE), average absolute error (AAE), root mean-square error (RMSE), and average relative pricing error (ARPE), reported in Table 2, where:

$$AAE = \sum_{i=1}^{N} \frac{|V^{Model}(S, T_i, K_i; \Omega) - V_i^{Market}|}{N}, \tag{32}$$

$$APE = \sum_{i=1}^{N} \frac{|V^{Model}(S, T_i, K_i; \Omega) - V_i^{Market}|}{V^{Model}(S, T_i, K_i; \Omega)}, \tag{33}$$

$$ARPE = \frac{1}{N} \sum_{i=1}^{N} \frac{|V^{Model}(S, T_i, K_i; \Omega) - V_i^{Market}|}{V^{Model}(S, T_i, K_i; \Omega)}, \tag{34}$$

$$RMSE = \sqrt{\sum_{i=1}^{N} \frac{(V^{Model}(S, T_i, K_i; \Omega) - V_i^{Market})^2}{N}}. \tag{35}$$

| | $k$ | $\theta$ | $\sigma$ | $\rho$ | $v_0$ | |
|---|---|---|---|---|---|---|
| **Heston** | 0.0252 | 4.4944 | 0.4599 | -0.6062 | 0.0024 | |
| | $k$ | $\theta$ | $\sigma$ | $\rho$ | $\alpha$ | $\beta$ |
| **Averaged Heston Formula** | 0.1178 | 0.9404 | 0.4567 | -0.6057 | 0.0016 | 2.7953 |

Table 1: Model parameters estimated for a data set with starting date September 07, 2010 and expiry date June 18, 2011

| | **APE** | **AAE** | **RMSE** | **ARPE** |
|---|---|---|---|---|
| **Heston** | 2.1905 | 16.0666 | 0.5182 | 3.5480 |
| **Average Heston** | 2.1878 | 15.9517 | 0.5145 | 3.5478 |

Table 2: Error estimator for a data set with starting date September 07, 2010 and expiry date June 18, 2011

**Conclusions and future directions**

The model of Heston (1993) is a mathematical tool still widely used as a basis for the valuation of financial derivatives. In this thesis, we have described a new method for the calibration of the Heston model in order to improve the effectiveness of such model. Thus, we have introduced a pricing method based on a market model where some parameter are defined by random variables. In Chapter 3, we have established theoretical results that allow to derive a new no arbitrage pricing relation in that extended market context, the application of these results to the Heston model being given in Chapter 4. Our method overcomes the problem of the non-observability of the initial volatility and it is inspired by a previous work of Dragulescu and Yakovencko [51] for the estimation of the historical probability density function.
In particular, these authors propose a probabilistic model for the initial volatility which is based on the stationary distribution associated with the volatility process. We formalize a generalization of this insight in a rigorous way in order to reduce the estimation error of the parameters on both the historical estimate and on the calibration of option prices. Thus, our results improve the descriptive ability of the Heston model.

Among the directions of future research, we aim investigate two feautures related to stochastic volatility models. The first concerns the construction of a new filtering technique for the volatility process. This issue has been widely discussed in the literature but has not yet provided exhaustive answers. In order to address this problem, we will propose a technique based on an integration between a polynomial filter and the methodology presented in Chapter 4. The second problem concerns with the application of the method illustrated in Chapters 4 to the extensions of the Heston model based on the use of jump stochastic processes as, for example, in the Bates model [6].

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