Candidate Reduced Order Models for Structural Parameter Estimation

M. I. Friswell
Department of Mechanical and Production Engineering, Aston University, United Kingdom

This paper considers the reduction of high order models generated by a finite element analysis. The aim is to provide a low order model which retains the effect of parameter changes and so may be used to update the unknown physical structural parameters of the modelled system in the time or frequency domains. After reviewing the methods of structural parameter identification, the available order reduction algorithms and their properties are introduced. Modal truncation is determined to be the most suitable method and the standard algorithm is extended to accommodate unknown parameters. Finally the effect of the reduction process on the modelled receptances is demonstrated.

1 Introduction

Structural parameter estimation is the identification of the physical structural parameters such as mass, damping, and stiffness, or possibly geometric parameters. In practical applications an estimate for these parameters will exist and the estimation process updates these estimates. In linear systems this becomes the estimation of the mass, damping, and stiffness matrices. No papers have been published that directly tackle the major problem of the high model order of theoretical finite element models. This paper argues that in many practical situations reducing the order of a linear model, while maintaining its dependence on unknown parameters, can be beneficial and suggests some order reduction methods.

Irrespective of the method used the parameters to be estimated should be carefully selected. Generally the dimension of the measurement vector is considerably smaller than the order of the finite element model. The input to output relationship, even in the absence of noise, could be reproduced by an infinite number of mass, damping, and stiffness matrices. Thus to obtain accurate estimates of physical parameters the choice of which parameters to estimate is important. If mass, damping, and stiffness matrices are available, for example, from a theoretical finite element analysis, then updated matrices "closest" to those of the initial model could be chosen. But even this does not consider the special structure of a typical finite element model. If a homogeneous continuum is discretized then the only unknowns would be the mass, damping, and stiffness properties of the material and the geometry of the modelled item. The elements of the matrices defining the system model will not be independent of each other. In principle it is unreasonable and unnecessary to identify whole mass, damping, and stiffness matrices. Wei et al. (1988) consider the selection of physical parameters in more detail. This reasoning may be adapted slightly to consider a discrete system comprising of masses, springs, and dampers although the parameters of such relatively low order systems may be obtained by standard methods of parameter estimation. Fritzen (1986) considers the application of some of these estimation methods to mechanical systems.

The methods of structural parameter estimation fall into three categories: methods applied directly to the time domain data, methods using the measured frequency response functions, and methods using the estimated natural frequencies, damping, and mode shapes. Time domain methods have been used extensively in control and other low order applications. Their use in identifying physical structural parameters is limited by the high order of structural models. Reduced order models may produce practical algorithms particularly for the monitoring of time varying parameters.

Mottershead, Lees, and Stanway (1987) use a linear, frequency domain filter to estimate the mass and stiffness matrices from receptance data. This algorithm updates a condensed theoretical model and is not directly applicable to finite element models with many more degrees of freedom than the number of measurement locations. The method is easily extended to update unknown physical parameters but requires the unmeasured state variables of the system to be estimated. The method loses much of its computational speed advantage when a state estimator is included. Such an estimator is likely to be numerically ill-conditioned if a large number of states are estimated from a small number of measurements, even if all the states were theoretically observable. It would be possible to generalize the nonlinear filter described by Mottershead and Stanway (1986) although for models with a large number of degrees of freedom computation times would become impractical.

Methods using the measured natural frequency, damping, and mode shapes, or modal model, are numerous and can produce good results. Obviously the amount of data is reduced as one goes from the time domain through the frequency domain to the modal properties. Deriving the modal properties effec-

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tively chooses a reduced order model which has the same number of degrees of freedom as the number of measured modes. In many instances the modal model is difficult to obtain accurately, especially when there are heavily damped, closely coupled, or complex modes present or the data has frequency shifts due to the mass loading of a roving accelerometer. Also any dependence of the unknown parameters on the structural response cannot be included in the procedure to obtain the modal properties. Caesar (1986) summarizes methods based on minimizing a norm of the difference between the estimated mass and stiffness matrices and the initial matrices obtained analytically, with the constraint that the system has the measured modal properties. Friswell (1988) and Collins et al (1974) give details of the statistical identification of structural parameters based on a minimum variance estimator. Janter et al (1988) outline an algorithm which updates physical parameters by improving the correlation between the theoretical and experimental modal models subject to practical constraints. Reduced order models have no application to these methods as the computation involved in the reduction process will be as great as the computational effort of the method.

2 An Overview of Order Reduction

Before describing the reduction process the validity of the approach will be discussed. Finite element models of realistic structures are generally high order and produce a correspondingly high number of natural frequencies, damping coefficients, and mode shapes. The natural frequency of most of these modes will be outside of the frequency range of interest in practical applications. For example, when measurements of the structure are taken using a computerized data acquisition system the resulting frequency response functions (inertance, mobility, or receptance) have an upper limit on the usable frequency range determined by the sampling rate through the Nyquist Frequency. Thus it should be possible to reduce the number of degrees of freedom in the theoretical model for little loss of accuracy over the measured frequency range. This assumes sufficient degrees of freedom are included to provide at least the same number of modes, within the frequency range, in the reduced model as were in the original model. The accuracy of the response function of the reduced order model within the frequency range of interest will be improved by including a reasonable number of modes outside the measured frequency range. In many practical applications this would produce enormous savings.

Methods of order reduction have been used extensively in control and filter applications to reduce the cost of designing or implementing a high order controller or filter. The oldest and least computationally demanding algorithms are based on Padé approximations or continued fractions, for example, Shamash (1975). These methods are not suitable to reduce the order of structural models because they substantially alter the eigenvalues, or natural frequencies, of the system, which can usually be measured quite accurately. Obviously, when the full model is predicting the system natural frequencies adequately, the reduced order model should also predict the lower natural frequencies adequately.

Static condensation, for example, Guyan (1965) and Irons (1965), has been used to reduce the order of structural problems. Equations that do not include an external force term are used to eliminate spatial variables. Generally these methods must be handled with extreme care as important natural frequencies may be changed considerably, or omited altogether (Thomas, 1982). Paz (1984) suggested a method of dynamic condensation that is really limited to solving the theoretical eigenproblem.

Modal truncation, or reducing the model order by retaining only the modes with the the lowest natural frequencies, is slightly more complex and computationally more demanding. It has the advantage that the lower natural frequencies remain unchanged and providing that enough modes are included the reduced model can approximate the full model sufficiently accurately. This method shows the most promise and is developed further in this paper.

There has been considerable interest recently on methods based on balanced realizations and the Hankel singular values of a system. Moore (1981) proposed the balanced realization approach based on the transformation given by Laub (1980). Glover (1984) develops optimal Hankel-norm approximations for multivariable systems. These methods of reduction are inappropriate for the identification of structural parameters for three reasons. First, the large dimension of a finite element model makes the computation times involved prohibitive. Second, the methods do not allow for unknown parameters. The linearization of the equations and the solution of a series of balanced realizations or Hankel-norm approximations could extend the methods at the expense of additional computation. Finally the lower eigenvalues of the system are not guaranteed to remain unchanged. For practical structural systems, including the example in this paper, the resulting reduced order model is indistinguishable from that derived from modal truncation.

Nonlinear programming may be used to directly minimize a cost function related to the difference between the full and reduced order models. This would be so computationally demanding it is not considered further.

3 Modal Truncation

An order reduction method based on a transformation using a subset of the current estimated eigenvectors of the full model will now be formally derived. The standard method does not allow for unknown parameters. If the transformation is obtained using the current parameter estimate, then the low frequency eigenvalues of the reduced order model are correct to first order in parameter variations. The next section demonstrates the use of a transformation dependent on the parameter variations. This section also introduces the structural model used and most of the notation.

The n degree of freedom system model with p unknown parameters is assumed to be given by

$$\begin{aligned} M(\theta)\mathbf{x}'' + C(\theta)\mathbf{x}' + K(\theta)\mathbf{x} &= B_n\mathbf{u} \\ \mathbf{y} &= D_n\mathbf{x} \end{aligned} \tag{1}$$

where $'$ denotes differentiation with respect to time

- $\theta = (\theta_1, \theta_2, \ldots, \theta_p)^T$ is the p dimensional vector of unknown parameters
- $\mathbf{x}$ is the n dimensional vector of generalized coordinates
- $\mathbf{y}$ is the m dimensional measured displacement vector
- $\mathbf{u}$ is the q dimensional vector of input forces
- $M(\theta)$ is the mass matrix for the model which is dependent on the unknown parameters
- $C(\theta)$ is the viscous damping matrix for the model which is dependent on the unknown parameters
- $K(\theta)$ is the stiffness matrix for the model which is dependent on the unknown parameters
- B_n is the matrix allocating the input force to the correct degrees of freedom
- D_n is the matrix determining the position of measurement transducers

The stiffness matrix in equation (1) may be complex to allow for hysteretic damping. The mass, damping, and stiffness matrices are all functions of the unknown parameters. Comments on the choice of physical parameters to update were given in the introduction. The number of parameters will obviously depend on the particular system and how the physical system is modelled. In principle all the physical parameters input to a finite element package could be updated. In some cases the model can be formulated so that the mass, damping, and stiffness matrices will be linear functions of unknown parameters which could, in turn, be functions of the unknown physical parameters. If the matrices involve more complicated functions, then a Taylor Series may be used to expand the matrices as linear functions in the parameters which are valid for small parameter variations from the current parameter estimate. The mass, damping, and stiffness matrices may be written in either case as

$$\begin{aligned} M(\theta) &= M_{n0}+\theta_1 M_{n1}+\ .\ .\ +\theta_p M_{np} \\ C(\theta) &= C_{n0}+\theta_1 C_{n1}+\ .\ .\ +\theta_p C_{np} \\ K(\theta) &= K_{n0}+\theta_1 K_{n1}+\ .\ .\ \theta_p K_{np} \end{aligned} \qquad (2)$$

Equations (1) and (2) may be combined and rewritten as the first order differential equation

$$[M_0+\theta_1 M_1+\ .\ .\ +\theta_p M_p]z'+[K_0+\theta_1 K_1+\ .\ .\ +\theta_p K_p]z=B\mathbf{u} \qquad (3)$$

$$\mathbf{y}=Dz$$

where

$$M_i = \begin{bmatrix} 0 & M_{ni} \\ M_{ni} & C_{ni} \end{bmatrix} \qquad i=0,\ .\ .\ .\ ,p$$

$$K_i = \begin{bmatrix} -M_{ni} & 0 \\ 0 & K_{ni} \end{bmatrix} \qquad i=0,\ .\ .\ .\ ,p$$

$$B = \begin{bmatrix} 0 \\ B_n \end{bmatrix}$$

$$D = [0\ D_n]$$

$$z = \begin{bmatrix} \mathbf{x}' \\ \mathbf{x} \end{bmatrix}$$

Let the current estimate of the unknown parameters be

$\theta_e=(\theta_{e1},\theta_{e2},\ .\ .\ .\ ,\theta_{ep})^T$ and

$$\begin{aligned} \delta\theta &= (\delta\theta_1,\delta\theta_2,\ .\ .\ .\ ,\delta\theta_p)^T=\theta-\theta_e \\ &= (\theta_1-\theta_{e1},\theta_2-\theta_{e2},\ .\ .\ .\ ,\theta_p-\theta_{ep})^T \end{aligned}$$

$$M_e=M_0+\theta_{e1}M_1+\ .\ .\ .\ +\theta_{ep}M_p$$

$$K_e=K_0+\theta_{e1}K_1+\ .\ .\ .\ +\theta_{ep}K_p$$

so that equation (3) may be written as

$$\begin{aligned} &[M_e+\delta\theta_1 M_1+\ .\ .+\delta\theta_p M_p]z' \\ &\qquad +[K_e+\delta\theta_1 K_1+\ .\ .+\delta\theta_p K_p]z=B\mathbf{u} \end{aligned} \qquad (4)$$

The system eigenvalues and eigenvectors evaluated at the latest parameter estimates are given by the solutions of

$$[M_e\lambda_i+K_e]\phi_i=0 \quad \text{for } i=1,\ .\ .\ .\ ,2n \qquad (5)$$

where

- λ_i is the ith eigenvalue arranged in ascending order of natural frequency
- ϕ_i is the corresponding eigenvector normalized so that $\phi_i{}^T M_e\phi_k = \delta_{ik}$, the Kronecker delta

Let the reduced order model have r degrees of freedom, which is generally much less than the n degrees of freedom of the full order model. Then for the reduction of the equations only the first $2r$ eigenvalues and eigenvectors are required and these can be assembled into matrices defined as

$$\Lambda_0=-\text{diag}(\lambda_1,\lambda_2,\ .\ .\ .\ ,\lambda_{2r})$$

$$\Phi_0=[\phi_1,\phi_2,\ .\ .\ .\ ,\phi_{2r}] \qquad (6)$$

The matrix Φ_0 will be the transformation matrix used to reduce the original model to a model with r degrees of freedom. The 0 subscript on Λ_0 and Φ_0 is to show that these eigenvalue and eigenvector matrices are correct to zeroth order in $\delta\theta$. If Λ and Φ are the eigenvalue and eigenvector transformation matrices correct to first order in $\delta\theta$ then they are given by

$$\Lambda=\Lambda_0+\delta\theta_1\Lambda_1+\delta\theta_2\Lambda_2+\ .\ .\ +\delta\theta_p\Lambda_p$$

$$\Phi=\Phi_0+\delta\theta_1\Phi_1+\delta\theta_2\Phi_2+\ .\ .+\delta\theta_p\Phi_p \qquad (7)$$

where the matrices Λ_i and Φ_i may be determined from the properties of the eigenvalues and eigenvectors neglecting terms of second order in $\delta\theta$. The next section considers the evaluation of these matrices and their use in a modal truncation. Applying the transformation

$$z=\Phi_0\mathbf{w} \qquad (8)$$

where $\mathbf{w}$ is the reduced order state vector of dimension $2r$, to equation (3) and premultiplying by $\Phi_0{}^T$ produces the reduced order equation

$$\begin{aligned} &[M_{R0}+\theta_1 M_{R1}+\ .\ .+\theta_p M_{Rp}]\mathbf{w}' \\ &\qquad +[K_{R0}+\theta_1 K_{R1}+\ .\ .+\theta_p K_{Rp}]\mathbf{w}=B_R\mathbf{u} \\ &\qquad \mathbf{y}=D_R\mathbf{w} \end{aligned} \qquad (9)$$

where

$$\begin{aligned} M_{Ri} &= \Phi_0{}^T M_i\Phi_0 \\ K_{Ri} &= \Phi_0{}^T K_i\Phi_0 \\ B_R &= \Phi_0{}^T B \\ D_R &= D\Phi_0 \end{aligned}$$

This is the equation of the r degree of freedom model that can now be used in parameter identification routines. Note that from the definitions M_{R0} and K_{R0} are not diagonal but

$$\begin{aligned} M_{R0}+\theta_{e1}M_{R1}+\ .\ .+\theta_{ep}M_{Rp} &= I_{2r} \\ K_{R0}+\theta_{e1}K_{R1}+\ .\ .+\theta_{ep}K_{Rp} &= \Lambda_0 \end{aligned} \qquad (10)$$

where I_{2r} is the $2r$ dimensional identity matrix.

4 First Order Modal Truncation

This section considers the differences between the basic transformation used in the previous section and the transformation, given in equation (7), based on eigenvectors that are correct to first order in the parameter variations. In fact the first $2r$ eigenvalues given by the reduced model, equation (9), are correct to first order. The Λ_i matrices in equation (7) are given by, (Nelson 1976)

$$\Lambda_i=\text{diag}\left(\frac{\partial\lambda_1}{\partial\Theta_i},\ \frac{\partial\lambda_2}{\partial\Theta_i},\quad ,\frac{\partial\lambda_{2r}}{\partial\Theta_i}\right) \qquad (11)$$

where

$$\frac{\partial\lambda_k}{\partial\Theta_i}=-\phi_k{}^T(\lambda_k M_i+K_i)\phi_k$$

The normalized eigenvectors $\mathbf{u}_k$ of the reduced model, equation (9), at the current parameter estimate are unit vectors in the direction of the kth coordinate. Thus the variation in the

kth eigenvalue of the reduced model due to the ith parameter is given by

$$\frac{\partial \lambda_k}{\partial \Theta_i} = -\mathbf{u}_k{}^T(\lambda_k M_{Ri} + K_{Ri})\mathbf{u}_k$$
$$= -\mathbf{u}_k{}^T \Phi_0{}^T(\lambda_k M_i + K_i)\Phi_0 \mathbf{u}_k$$
$$= -\phi_k{}^T(\lambda_k M_i + K_i)\phi_k \qquad (12)$$

Let the first order transformation and reduced equations be given by

$$z = \phi v$$

$$v' + \Lambda v = \Phi^T B\mathbf{u}$$
$$= [\Phi_0{}^T + \delta\Theta_1 \Phi_1{}^T + \delta\Theta_2\Phi_2{}^T + \ldots + \delta\Theta_p\Phi_p{}^T]B\mathbf{u} \qquad (13)$$

$$\mathbf{y} = D\Phi v$$
$$= D[\Phi_0 + \delta\Theta_1\Phi_1 + \delta\Theta_2\Phi_2 + \ldots + \delta\Theta_p\Phi_p]v$$

where v is the reduced order state. The evaluation of the reduced order model requires the sensitivities of the first $2r$ eigenvectors to the unknown parameters, Φ_i. Nelson (1976) and Chen and Garba (1980) discuss various methods to compute these sensitivities.

The advantages of using this first order modal truncation as opposed to the zeroth order truncation given in the previous section is that now the eigenvectors are correct to the first order in the parameter variations. Of course it is still a further approximation to the full order system since not all the modes are included. For many applications the zeroth order approximation will produce a reduced order model which is almost as good as the model based on the first order modal truncation. What are the errors involved if zeroth order modal truncation is used? The derivative of the hth eigenvector may be written as the linear combination of the $2n$ eigenvectors of the full model evaluated at the current parameter estimate given by equation (14) [see Chen and Garba (1980)].

$$\frac{\partial \phi_h}{\partial \Theta_i} = \sum_{k=1}^{2n} {}_i a_{hk}\phi_k \qquad (14)$$

where

$${}_i a_{hk} = \frac{1}{\lambda_k - \lambda_h}\phi_k^T[\lambda_h M_{Ri} + K_{Ri}]\phi_h \quad \text{if } k \neq h$$

$${}_i a_{hh} = \frac{1}{2}[\phi_h^T M_{Ri}\phi_h]$$

It is easily shown, using a method similar to that in equation (12), that the eigenvector derivatives of the zeroth order approximation consist of only the first $2r$ terms of this series. Providing enough modes are included in the reduced order model the modulus of the term $\lambda_k - \lambda_h$ for $k > 2r$ will be large for all modes whose natural frequencies lie in the frequency range of interest. In general this will mean that the derivatives of the eigenvectors with low associated natural frequencies will be determined more accurately than those with higher natural frequencies. Thus, in the frequency response functions, at low frequencies the difference between the zeroth and first order approximation will be small and become larger at higher frequencies. The order of the reduced model should be chosen so that this error is acceptable over the measured frequency range.

The first order approximation does produce diagonal "mass" and "stiffness" matrices that may give a computational advantage in some parameter estimation routines. This advantage is minimized by the relatively short computer time that would be used to diagonalize the reduced model obtained from the zeroth order transformation.

5 Numerical Example with Proportional Viscous Damping

So far the structural could be modelled using general viscous or hysteretic damping. The computation is eased when the structure is modelled using proportional viscous damping where the damping matrix in equation (1) is given by

$$C(\Theta) = \alpha M(\Theta) + \beta K(\Theta) \qquad (15)$$

for some, possibly unknown, constants α and β. The equations may be reduced using a subset of the normalized eigenvectors of equation (1), ψ_i, given by

$$[M_{ne}\mu_i{}^2 + K_{ne}]\psi_i = 0 \quad \text{for } i = 1, \ldots, n \qquad (16)$$

where $\psi_i{}^T M_{ne}\psi_k = \delta_{ik}$

$$M_{ne} = M_{n0} + \Theta_{e1}M_{n1} + \ldots + \Theta_{ep}M_{np}$$
$$K_{ne} = K_{n0} + \Theta_{e1}K_{n1} + \ldots + \Theta_{ep}K_{np}$$

μ_i are the eigenvalues of equation (16) (purely imaginary)

Because proportional damping is a particular case of the systems considered in the previous two sections, the methods are exactly the same but are implemented more easily using eigenvectors defined by equation (16) and their derivatives with respect to the unknown parameters. Since the dimensions of the matrices are halved and real matrix algebra may be used, the computational savings are substantial.

Consider a ten degree of freedom system whose damping matrix is proportional to the mass matrix although the numerical value of the constant has only been estimated. Force is applied at one position and the response is measured at only one location. The mass matrix is assumed fixed and the stiffness matrix is a function of a second parameter. The system has a single input and produces a single output. This example will only show the effect of the reduced order modelling on the magnitude of the receptance of the system and does not implement any parameter estimation algorithms.

The reduced models are obtained on the parameter estimate $\Theta = (0.01, 3.0)$. Figure 1 shows the receptance of the system over a frequency range that includes three modes. Also shown is receptance of the system reduced by including only the first four degrees of freedom. The major discrepancies are where the magnitude of the response is small and where an experimental receptance would be susceptible to noise. Although the approximation is reasonably accurate at the current estimated parameter values, the reduced model must retain

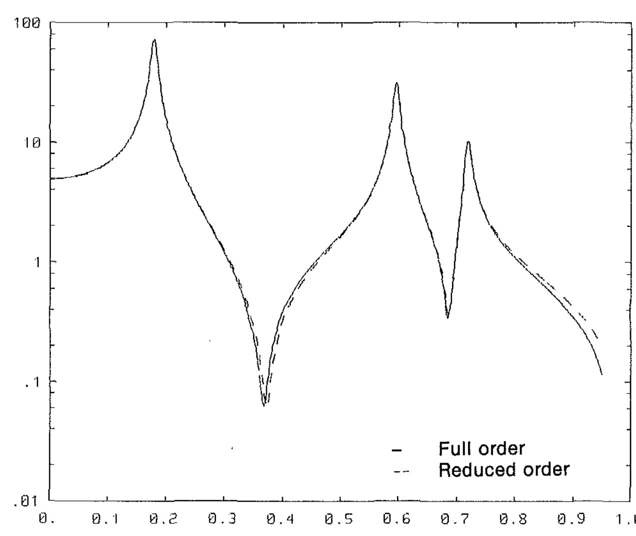

Fig. 1 The effect of modal truncation

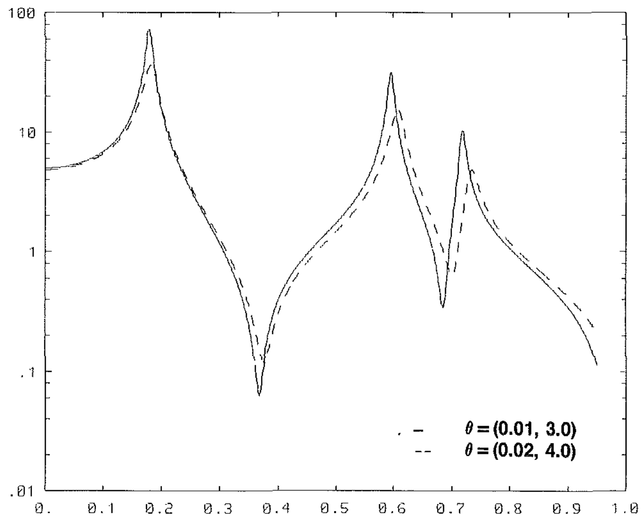

Fig. 2 The receptance at different parameter values

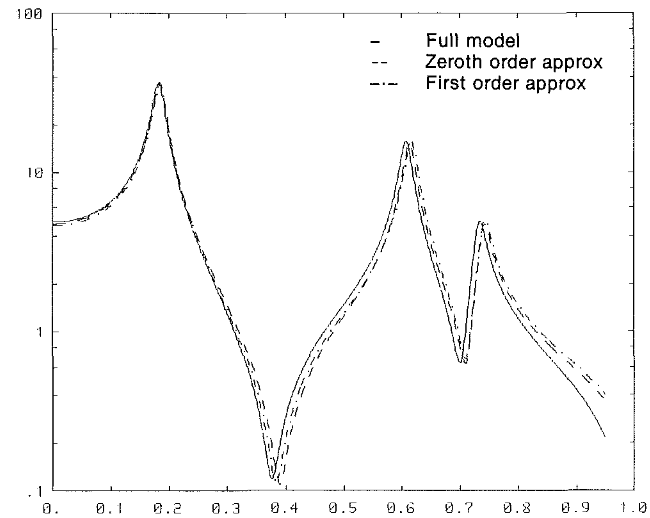

Fig. 3 The effect of parameter variations on modal truncation

this accuracy over a range of parameter values. Figure 2 shows the receptance of the full model for parameter values of (0.01, 3.0) and (0.02, 4.0). The damping and natural frequencies have obviously changed. Figure 3 shows the receptance of the full model for a parameter vector of (0.02, 4.0) and the zeroth and first order reduced models evaluated at the same parameter values. Even though the parameter change is large, and could not be described as first order, the first natural frequency is still accurate. The second and third modes suffer more inaccuracy because of their magnitude relative to that of the fourth and last mode modelled in the reduced equation. The extra complexity in calculating the eigenvector sensitivities to the parameter variations does not yield a significantly better approximation.

6 Discussion

This paper has considered the problems of using high order finite element models to identify physical structural parameters with time and frequency domain methods. In practical situations a reduced order model must be used to produce an algorithm that is computationally feasible. Modal approximations based either on current estimated eigenvectors or a first order eigenvector expansion in the parameter variations have been recommended as the most suitable order reduction method. The main reason for this recommendation is that modal truncation is the only method that guarantees no change in the natural frequencies of interest.

Throughout the paper there has been little mention of the parameter estimation algorithms that would use the reduced order models. Whichever method is used the integration of reduction and estimation will have to be considered carefully. How often is the reduced order model to be updated to allow for the change in parameters caused by the estimation procedure? The answer will depend on the stability and convergence rate of the combined reduction-estimation algorithm. Indeed the stability and convergence of a combined algorithm must be checked. Does the reduction process produce a bias on the parameter estimates? The amount of bias will depend on the parameter estimation algorithm but hopefully it would be insignificant.

There are many questions still to be answered before the estimation of structural parameters using reduced order models may be routinely applied. This paper has suggested the type of reduced order models which could form the basis of the total estimation algorithms.

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