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From convergent dynamics to incremental stability

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***Abstract*— This paper advocates that the convergent systems property and incremental stability are two intimately related though different properties. Sufficient conditions for the convergent systems property usually rely upon first showing that a system is incrementally stable, as e.g. in the celebrated Demidovich condition. However, in the current paper it is shown that incremental stability itself does not imply the convergence property, or vice versa. Moreover, characterizations of both properties in terms of Lyapunov functions are given. Based on these characterizations, it is established that the convergence property implies incremental stability for systems evolving on compact sets, and also when a suitable uniformity condition is satisfied.**

I. Introduction

The notion of convergent dynamics originates from the Russian literature. It requires the existence of a unique uniformly asymptotically stable solution to a differential equation, which is bounded in forward and backward time, see e.g. [3], [16].

This convergence property has many useful applications. One particular advantage is that there are sufficiency criteria which guarantee this property without explicit knowledge of the unique uniformly asymptotically stable solution. The celebrated Demidovich condition [3] is of this type, see [8] for a brief introduction, or the monograph [10] for more advanced results on the convergence property. The Demidovich condition does in fact establish another quite related property, namely incremental stability. Roughly speaking, a system is incrementally stable, if all solutions are uniformly asymptotically stable, see [17], [1]. Both, convergence and incremental stability, are instrumental in a range of control problems such as, e.g., output regulation and synchronization [11], [2], [15], [7].

There is a lot of renewed interest in both properties in recent years, in parts due to their potential usefulness in understanding problems of synchronization of many systems. It may seem that incremental stability would be more a restrictive condition on a system than the convergence property. This, however, is not entirely true. We will present examples to show that none of the two concepts on its own implies the other.

Historically, a number of results are known that allow to deduct the convergence property from incremental stability, under additional assumptions, just like the Demidovich condition in [3], [8]. In this paper, we go in the other direction and present criteria that guarantee convergent systems to be also incrementally stable.

To this end, we develop a Lyapunov characterization for incremental stability, which is similar but not identical to the characterization by Angeli [1], owed to circumstance that we consider time-varying systems. This will lead us to a natural criterion for incremental stability. We will also see that for systems whose dynamics evolve on compact sets convergence always implies incremental stability. There are, however, other notions of incremental stability that we do not focus on, e.g., a notion that is invariant under changes of coordinates, see the recent work [18].

This paper is organized as follows: First we review the notions of convergent systems and of incremental stability and develop Lyapunov characterizations for both properties. Then we present the examples showing that none of the two per se implies the other. In a final section, we present the main results, allowing to pass from convergence to incremental stability.

a) Notation: By $\mathbb{R}_+$ we denote the real half line $[0,\infty)$. Throughout the paper, we will denote by $\mathcal{K}$ the class of continuous and strictly increasing functions $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ for which $\kappa(0) = 0$. A function ρ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for any fixed $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$ and $\beta(s,\cdot)$ is non-increasing with $\lim_{t\to\infty} \beta(s,t) = 0$.

II. Convergent systems

We start by introducing the definition of convergent systems. Consider hereto a system

$$\dot{x}(t) = f(t, x) \tag{1}$$

with $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ measurable in t and locally Lipschitz in $x \in \mathbb{R}^n$, uniformly for t in compact sets (this assumption guarantees uniqueness and local existence of solutions, cf. [13]). We say that a set $A \subset \mathbb{R}^n$ is *positively invariant* under (1) if $x^0 \in A$ implies $x(t, t^0, x^0) \in A$ for all $t \geq t^0$. Let $\mathcal{X} \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$.

Definition 1 (convergent dynamics; cf. [10], [8]): System (1) is *uniformly convergent in a positively invariant set* $\mathcal{X}$ if

1) all solutions $x(t, t^0, x^0)$ exist for all $t \geq t^0$ for all initial conditions $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$;
2) there exists a unique solution $\overline{x}(t)$ in $\mathcal{X}$ defined and bounded for all $t \in \mathbb{R}$;
3) the solution $\overline{x}(t)$ is uniformly[1] asymptotically stable in $\mathcal{X}$, i.e., there exists a function $\beta \in \mathcal{KL}$ such that for all $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$ and $t \geq t^0$,
$$\|x(t, t^0, x^0) - \overline{x}(t)\| \leq \beta\big(\|x^0 - \overline{x}(t^0)\|, t - t^0\big). \tag{2}$$

System (1) is *globally uniformly convergent* if it is uniformly convergent in $\mathbb{R}^n$.

For a uniformly convergent system, the unique, bounded uniformly asymptotically stable solution $\overline{x}(t)$ is called a *steady-state solution*. As the convergence property states essentially the asymptotic stability of a single (but perhaps not known)

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[1]In Definition 1 the uniqueness of the solution $\overline{x}(t)$ is in fact a consequence of its *uniform* asymptotic stability, cf. [10, p.15, Property 2.15].

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trajectory, the following characterization is evident from a standard converse Lyapunov result in [6, Theorem 23].

Theorem 2: Assume that system (1) is globally uniformly convergent. Assume that the function f is continuous in (t,x) and $\mathcal{C}^1$ with respect to the x variable. Assume also that the Jacobian $\frac{\partial}{\partial x} f(t,x)$ is bounded, uniformly in t. Then there exists a $\mathcal{C}^1$ function $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$, functions α_1, α_2, and $\alpha_3 \in \mathcal{K}_\infty$, and a constant $c \geq 0$ such that

$$\alpha_1(\|x-\overline{x}(t)\|) \leq V(t,x) \leq \alpha_2(\|x-\overline{x}(t)\|) \tag{3}$$

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x-\overline{x}(t)\|) \tag{4}$$

and

$$V(t,0) \leq c, \quad t \in \mathbb{R}. \tag{5}$$

Conversely, if a differentiable function $V\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_+$ and functions $\alpha_i \in \mathcal{K}_\infty$, $i=1,2,3$, and $c \geq 0$ are given such that for some trajectory $\overline{x}\colon \mathbb{R}\to\mathbb{R}^n$ estimates (3)–(5) hold, then system (1) must be globally uniformly convergent and the solution $\overline{x}$ is the unique bounded solution as in Definition 1.

Condition (5) is to ensure the existence of a positively invariant, compact set. Note that the result does require the explicit knowledge of the asymptotically stable solution $\overline{x}$.

III. Incremental stability

Next, we introduce the second stability concept considered in this paper.

Definition 3 (incremental stability; cf. [1]): System (1) is *incrementally asymptotically stable* (IS for short) *in a positively invariant set* $\mathcal{X} \subset \mathbb{R}^n$ if there exists a function $\beta \in \mathcal{KL}$ such that for any $\xi^1, \xi^2 \in \mathcal{X}$ and $t \geq t^0$,

$$\|x(t,t^0,\xi^1) - x(t,t^0,\xi^2)\| \leq \beta(\|\xi^1-\xi^2\|, t-t^0)\,. \tag{6}$$

In the case $\mathcal{X} = \mathbb{R}^n$ we say that system (1) is *globally* incrementally stable (GIS), or just incrementally stable.

This definition of incremental stability implicitly requires that solutions to (1) exist for all forward times. Also note that in contrast to the definition given here, most existing notions of incremental stability are defined only for systems with right-hand sides not depending explicitly on time.

In [1], a characterization of GIS in terms of a merely continuous Lyapunov function has been derived for systems of the form

$$\dot{x} = f(x,d), \tag{7}$$

where d is an arbitrary, measurable disturbance function taking values in a closed subset $\mathcal{D}$ of $\mathbb{R}^m$. However, the formulation (7) does not encode an explicit dependence of the right-hand side f on time as is the case in (1), and subsequently the Lyapunov function shown to exist in [1] does not depend on time either. To characterise GIS for systems of the form (1), we present the following result for time-varying systems, extending that of [1].

Theorem 4: System (1) is GIS if and only if there exist a continuous function $W\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, functions α_1, α_2, α_3 of class $\mathcal{K}_\infty$ such that

1) the inequalities

$$\alpha_1(\|x^1-x^2\|) \leq W(t,x^1,x^2) \leq \alpha_2(\|x^1-x^2\|) \tag{8}$$

hold for all x^1, $x^2 \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

2) along trajectories of (1) for any ξ^1, $\xi^2 \in \mathbb{R}^n$, and any $t \geq t^0$ it holds that

$$\begin{aligned} &W\big(t, x(t,t^0,\xi^1), x(t,t^0,\xi^2)\big) - W(t^0,\xi^1,\xi^2) \\ &\quad\leq -\int_{t^0}^{t} \alpha_3\big(\|x(\tau,t^0,\xi^1) - x(\tau,t^0,\xi^2)\|\big)d\tau. \end{aligned} \tag{9}$$

The proof is rather technical and long, while following in spirit the same steps of the corresponding result in [1] with the added difficulty of the time-varying nature of the problem. Yet, the proof is not the same, and there are some non-trivial technicalities involved. The proof of Theorem 4 is given in the appendix.

In this result, we may trade the unboundedness of α_3 for a Lipschitz-like property of the Lyapunov function W as formalized in the next corollary.

Corollary 5: If system (1) is GIS then there exist a continuous function $W\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, and a positive definite function α_3 such that the inequalities (8) and (9) hold. Moreover, there exists a function $\gamma \in \mathcal{K}_\infty$ so that for all $z^1, z^2 \in \mathbb{R}^n\times\mathbb{R}^n$ and all t^0,

$$|W(t^0,z^1) - W(t^0,z^2)| \leq \gamma(\|z^1-z^2\|). \tag{10}$$

IV. Examples

In this section, we present example systems which are 1) uniformly convergent but not GIS or 2) GIS but not uniformly convergent.

Our first example is a uniformly convergent system that is not GIS, or, more precisely, a system with globally asymptotically stable equilibrium at the origin that is not incrementally stable. The trajectories of this system spiral counter-clockwise into the origin, but the further away from the origin a solution starts, the faster the angular velocity is. So, the solution $\overline{x}(t) \equiv 0$ is globally asymptotically stable, which is shown using a quadratic Lyapunov function, while two solutions starting at $t=0$ with an appropriately chosen distance $\epsilon > 0$ away from each other get separated arbitrarily much in finite time, if they both start far away from the origin.

Example 6: Consider the system

$$\dot{x} = A(x)x\,, \qquad x \in \mathbb{R}^2\,, \tag{11}$$

where $A(x) \in \mathbb{R}^{2\times 2}$ is defined by

$$A(x) = (x^\top x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \mathrm{sat}_1(x^\top x)\, I\,,$$

where $I \in \mathbb{R}^{2\times 2}$ denotes the identity matrix and $\mathrm{sat}_r : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{sat}_r(s) = \begin{cases} -r & \text{if } s \leq -r \\ s & \text{if } |s| < r \\ r & \text{if } s \geq r \end{cases}.$$

Consider the standard quadratic Lyapunov function $V(x) = \frac{1}{2}x^\top x$ with $\nabla V(x) = x^\top$. Then

$$\begin{aligned} \langle \nabla V(x), A(x)x\rangle &= x^\top x(-x_1x_2 + x_1x_2) - \mathrm{sat}_1(x^\top x)x^\top x \\ &= -\mathrm{sat}_1(x^\top x)x^\top x < 0 \end{aligned}$$

for all $x \neq 0$, proving global asymptotic stability of the origin with respect to (11). Hence the system is globally uniformly convergent. Rewriting system (11) in polar coordinates yields, in the region where $r > 1$,

$$\begin{aligned} \dot{r} &= -r \\ \dot{\phi} &= r^2\,, \end{aligned}$$

which has solutions for initial values (in polar coordinates) $(r^0, \phi^0)^\top$, $r^0 > 1$, explicitly given by

$$r(t) = r^0 e^{-t}$$
$$\phi(t) = \phi^0 + \frac{(r^0)^2}{2}\left(1 - e^{-2t}\right), \tag{12}$$

for $t \geq 0$ such that $r(t) > 1$.

Claim: With $M = \frac{2\pi e}{e-1}$ there exist points ξ^1, ξ^2 with $\|\xi^1 - \xi^2\| \leq M$ such that for any $R > 1$ sufficiently large,

$$\|x(1/2, 0, \xi^1) - x(1/2, 0, \xi^2)\| = \frac{\sqrt{R+M} + \sqrt{R}}{\sqrt{e}}. \tag{13}$$

This implies that there cannot exist a $\mathcal{KL}$ function β such that (6) holds and hence the system is not GIS.

Proof of the claim: We argue constructively. Let $R > 1$ be large enough such that solutions starting in $\xi^1 = \left(\sqrt{R+M}, 0\right)^\top$ and $\xi^2 = \left(\sqrt{R}, 0\right)^\top$ satisfy $\|x(t, 0, \xi^i)\| > 1$ for all $t \in [0, 1/2]$, $i = 1, 2$. Observe that $\|\xi^1 - \xi^2\| = \left(M + \sqrt{R}\left(2\sqrt{R} - 2\sqrt{R+M}\right)\right)^{1/2} \leq \sqrt{M}$. Using (12), at time $t = 1/2$ the difference of the respective angle functions $\phi_i(t) = \phi(t, 0, \xi^i)$, $i = 1, 2$, satisfies

$$\phi_1(1/2) - \phi_2(1/2) = (R+M)/2(1-e^{-2t}) - R/2(1-e^{-2t})$$
$$= \frac{M}{2}(1 - 1/e) = \pi. \tag{14}$$

Denote correspondingly $r_i(t) = r(t, 0, \xi^i)$, $i = 1, 2$. Using (14),

$$\|x(1/2, 0, \xi^1) - x(1/2, 0, \xi^2)\| = r_1(1/2) + r_2(1/2)$$
$$= \sqrt{R+M}e^{-1/2} + \sqrt{R}e^{-1/2} = \frac{\sqrt{R+M} + \sqrt{M}}{\sqrt{e}},$$

where the first equality is owed to the fact that $x(1/2, 0, \xi^1)$ and $x(1/2, 0, \xi^2)$ are vectors pointing in opposite directions.

The second example is a system which is GIS but not uniformly convergent.

Example 7: Consider

$$\dot{x}(t) = t - x, \quad x \in \mathbb{R}, \tag{15}$$

which has the explicit solution

$$x(t, t^0, x^0) = x^0 e^{-t+t^0} + (t-1) - (t^0 - 1)e^{t^0 - t}.$$

Obviously, the solution passing through $x^0 = 0$ at $t^0 = 0$ is unbounded. Hence the system cannot be globally convergent (since otherwise the same solution would have to be attracted to a bounded solution as $t \to \infty$). Taking any $\xi_1, \xi_2 \in \mathbb{R}$ then

$$\frac{d}{dt}\left[x(t, t^0, \xi_1) - x(t, t^0, \xi_2)\right] = -\left(x(t, t^0, \xi_1) - x(t, t^0, \xi_2)\right),$$

which implies

$$\|x(t, t^0, \xi_1) - x(t, t^0, \xi_2)\| \leq \|\xi_1 - \xi_2\| e^{-t},$$

which, in turn, represents a $\mathcal{KL}$-estimate on the difference between any two solutions. So the system (15) is GIS.

This in turn implies that the solution passing through $x^0 = 0$ is globally attractive, and hence no bounded solution can exist, so the system cannot be convergent on a subset of $\mathbb{R}^n$.

On the one hand, the above examples clearly show that the stability notions of convergence and incremental stability are different. On the other hand, the classes of GIS and convergent systems also have nonempty intersection: for example, any linear system $\dot{x} = Ax$ with A Hurwitz or any nonlinear systems satisfying the Demidovich condition [3] or the conditions in [16] for Luré-type systems satisfies both properties.

V. Main results

The following theorem is a new sufficiency condition for incremental stability and provides a condition under which a uniformly convergent system is also incrementally stable.

Theorem 8: Suppose system (1) is uniformly convergent on a compact set $\mathcal{X}$. Then, it is also incrementally stable on that set.

Proof: For future reference we denote $d_\mathcal{X} := \max_{x,y \in \mathcal{X}} \|x - y\|$, the diameter of $\mathcal{X}$. Note that without loss of generality we may assume that the closure of the trajectory $\overline{x}$ (which is a compact set) is contained in $\mathcal{X}$, i.e., $\overline{\bigcup_{t \in \mathbb{R}}\{\overline{x}(t)\}} \subset \mathcal{X}$.

We are going to show that differences of solutions satisfy the uniform attraction and stability properties for restricted initial conditions.

Uniform attraction: For any $\epsilon > 0$ there exists a $T > 0$ such that for any $\xi \in \mathcal{X}$, $\|x(t, t^0, \xi) - \overline{x}(t)\| \leq \beta(d_\mathcal{X}, t - t^0) \leq \epsilon/2$ if $t - t^0 \geq T$. By the triangle inequality it follows that for any $\xi, \eta \in \mathcal{X}$, $\|x(t, t^0, \xi) - x(t, t^0, \eta)\| \leq \epsilon$ if $t - t^0 \geq T$. This shows that all solutions starting in $\mathcal{X}$ are mutually uniformly attractive.

Uniform stability: The following argument follows ideas in the proof of [13, Theorem 55]. Let $\xi^1, \xi^2 \in \mathcal{X}$ and $t^0 \in \mathbb{R}$ be arbitrary. In view of item 3 of Definition 1 we have that $\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leq 2\beta(d_\mathcal{X}, t - t^0)$ for all $t > t^0$, i.e., there exists a $\mathcal{KL}$ function $\widehat{\beta}$ such that

$$\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leq \widehat{\beta}(d_\mathcal{X}, t - t^0) \text{ for all } t > t^0.$$

Thus there exists a compact set $\mathcal{Y} \supset \mathcal{X}$ which contains all solutions with initial values in $\mathcal{X}$ (in fact, $\mathcal{X}$ is positively invariant, so $\mathcal{Y} = \mathcal{X}$). Write $x^1(t) := x(t, t^0, \xi^1)$ and $x^2(t) := x(t, t^0, \xi^2)$. Regarding

$$x^1(t) - x^2(t) = \xi^1 - \xi^2 + \int_{t^0}^{t} [f(s, x^1(s)) - f(s, x^2(s))] \mathrm{d}s$$

for all $t \geq t^0$, we have due to the local Lipschitz condition on f and the compactness of $\mathcal{X}$ that there exists a locally integrable function $\alpha\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$, cf. [13, Appendix C], such that for all $t \geq t^0$,

$$\|x^1(t) - x^2(t)\| \leq \|\xi^1 - \xi^2\| + \int_{t^0}^{t} \alpha(s)\|x^1(s) - x^2(s)\| \mathrm{d}s$$

Thus, with Gronwall's inequality we arrive at

$$\|x^1(t) - x^2(t)\| \leq \|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}$$

for all $t \geq t^0$. As $\|x^1(t) - x^2(t)\| \leq \widehat{\beta}(d_\mathcal{X}, t - t^0)$ for all $t \geq t^0$, we arrive at

$$\|x^1(t) - x^2(t)\| \leq \min\left\{\|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}, \widehat{\beta}(d_\mathcal{X}, t - t^0)\right\}.$$

From there we can obtain a $\mathcal{KL}$ function $\widetilde{\beta}$ such that

$$\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leq \widetilde{\beta}(\|\xi^1 - \xi^2\|, t - t^0)$$

for all $\xi^1, \xi^2 \in \mathcal{X}$, $t^0 \in \mathbb{R}$ and $t \geq t^0$. ■

Remark 9: Let us briefly revisit Example 6 in view of the statement of Theorem 8. Example 6 concerns a system that is globally uniformly convergent, but not GIS. Since the system is globally uniformly convergent, it is also uniformly convergent on compact sets and Theorem 8 shows that it is also incrementally stable on compact sets. On compact sets one cannot choose arbitrary R in (13) that allow for arbitrarily large distances between solutions at $t = 1/2$, which start with the distance M at $t = 0$. Hence, despite being not

GIS system (11) is indeed incrementally stable on compact sets.

If system (1) does not evolve in a compact set then additional conditions on the vector field f allow to infer one stability property from the other. Let us now formulate a condition under which a *globally* convergent system is also *globally* IS. In general, while also for convergent systems all trajectories approach each other, they may do so non-uniformly, as could be seen from Example 6.

The combined lesson of the example and Theorem 8 is that problems can only occur "far away" from the unique bounded solution $\overline{x}$. In order to infer GIS from global convergence, it is natural to require a uniformity of the attraction of any two solutions only outside an arbitrarily large set. A generic result based on that idea is as follows.

Theorem 10: Suppose system (1) is globally uniformly convergent. Assume further that there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$, i.e. $P = P^\top > 0$, a constant $C > 0$, and a continuous positive function $\alpha_4\colon [C,\infty) \to (0,\infty)$ such that for all times $t \in \mathbb{R}$ and all $x^1, x^2 \in \mathbb{R}^n$

$$(x^1 - x^2)^\top P\left(f(t,x^1) - f(t,x^2)\right) \leq \begin{cases} -\alpha_4(\|x^1 - x^2\|) & \text{if } \max\{\|x^1\|,\|x^2\|\} \geq C, \\ 0 & \text{otherwise.}\end{cases} \quad (16)$$

Then (1) is GIS.

Proof: By Theorem 2, which is on the characterization of the uniform convergence property, there exists a Lyapunov function V satisfying (3) and (4). The solution $\overline{x}$ is bounded on $\mathbb{R}$, i.e. there exists a $C_2 \geq 0$ such that $\|\overline{x}(t)\| \leq C_2$ for all $t \in \mathbb{R}$. Without loss of generality we may assume that $C - C_2 > 0$, if necessary by enlarging C for which (16) is satisfied. There also exist positive constants c_P, C_P such that for all $x^1, x^2 \in \mathbb{R}^n$, $c_P\|x^1 - x^2\|^2 \leq (x^1 - x^2)^\top P(x^1 - x^2) \leq C_P\|x^1 - x^2\|^2$.

Denote $K := \left\{(x^1,x^2) \in \mathbb{R}^n \times \mathbb{R}^n\colon \max\{\|x^1\|,\|x^2\|\} \leq C\right\}$. On the compact set K we have $V(t,x^1) + V(t,x^2) \leq \alpha_2(\|x^1 - \overline{x}(t)\|) + \alpha_2(\|x^2 - \overline{x}(t)\|) \leq 2\alpha_2(C + C_2)$.

Let us define $W(t,x^1,x^2) := \frac{1}{2}b\big(V(t,x^1) + V(t,x^2)\big)(x^1 - x^2)^\top P(x^1 - x^2)$ where $b(s) = s/(1+s)$ is a bounded class $\mathcal{K}$ function.

We have

$$W(t,x^1,x^2) \leq \frac{1}{2}C_P\|x^1 - x^2\|^2 =: \widetilde{\alpha}_1(\|x^1 - x^2\|)$$

since $b(s) \leq 1$ for all $s \geq 0$. We also have

$$\begin{aligned} W(t,x^1,x^2) &\geq \frac{1}{2}b\big(\alpha_1(\|x^1 - \overline{x}\|) + \alpha_1(\|x^2 - \overline{x})\big)c_P\|x^1 - x^2\|^2 \\ &\geq \frac{1}{2}b\left(\alpha_1\left(\frac{1}{2}\|x^1 - \overline{x}\| + \frac{1}{2}\|x^2 - \overline{x}\|\right)\right)c_P\|x^1 - x^2\|^2 \\ &\geq \frac{1}{2}b\left(\alpha_1\left(\frac{\|x^1 - x^2\|}{2}\right)\right)c_P\|x^1 - x^2\|^2 \\ &=: \widetilde{\alpha}_2(\|x^1 - x^2\|). \end{aligned}$$

So W is positive definite and radially unbounded in the distance $\|x^1 - x^2\|$.

Denoting $\dot{V}(x^i) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x^i) \leq -\alpha_3(\|x^i - \overline{x}^i(t)\|)$ as per (4) and $\frac{d}{ds}b(s)$ by $b'(s)$, we compute the time-derivative of W as

$$\begin{aligned} \dot{W} &:= \frac{d}{dt}W(t,x^1(t),x^2(t)) \\ &= b'\big(V(t,x^1) + V(t,x^2)\big)[\dot{V}(x^1) + \dot{V}(x^2)] \\ &\quad \cdot \frac{1}{2}(x^1 - x^2)^\top P(x^1 - x^2) \\ &\quad + b\big(V(t,x^1) + V(t,x^2)\big) \\ &\quad \cdot (x^1 - x^2)^\top P\left(f(t,x^1) - f(t,x^2)\right). \end{aligned} \quad (17)$$

On the set K the first in the right hand side of (17) term is bounded from above by

$$-\frac{1}{2}\alpha_3\left(\frac{\|x^1 - x^2\|}{2}\right)\frac{c_P\|x^1 - x^2\|^2}{1 + \big(2\alpha_2(C + C_2)\big)^2}$$

while the second term in the right hand side of (17) is nonpositive due to (16). Outside of K the first term could be arbitrarily small in magnitude as $b'(s) \to 0$ for $s \to \infty$, while still negative, so that outside of K (17) is bounded from above by

$$\begin{aligned} &b\big(2\alpha_2(C - C_2)\big)(x^1 - x^2)^\top P\left(f(t,x^1) - f(t,x^2)\right) \\ &\leq -\alpha_4(\|x^1 - x^2\|)b\big(2\alpha_2(C - C_2)\big), \end{aligned}$$

again due to (16). If follows that $\dot{W}$ is bounded from above by a function which is negative definite with respect to the set where $x^1 = x^2$. A standard scaling argument (see [14]) with $U = \rho(W)$ for a suitable function $\rho \in \mathcal{K}_\infty$ turns this into a smooth Lyapunov function satisfying $\dot{U} \leq -\alpha_5(U)$ with $\alpha_5 \in \mathcal{K}_\infty$. This function U in particular satisfies (8) and (9). Hence, by virtue of Theorem 4 we conclude that system (1) is indeed GIS. ■

Examples of systems to which Theorem 10 is applicable include all so-called quadratically convergent systems, see [9], i.e., globally convergent systems where the convergence property is characterized by a quadratic Lyapunov-type function. This also includes systems satisfying the convergence conditions in [16], [3].

On the other hand, it is also clear from Theorem 10 that system (11) considered in Example 6 cannot satisfy assumption (16). This can be seen as follows: We consider two cases. First, if $P = aI_2$ with $a > 0$, let $x^1 = [0, 2\mu]^\top$ and $x^2 = [\mu, 0]^\top$ with $\mu > 1$, then (in view of $\operatorname{sat}_1({x^1}^\top x^1) = \operatorname{sat}_1({x^2}^\top x^2) = 1$)

$$\begin{aligned} &(x^1 - x^2)^\top P(f(t,x^1) - f(t,x^2)) \\ &= a\left[({x_1}^\top x^1 - {x^2}^\top x^2){x^1}^\top \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} x^2 + ({x^1}^\top x^1 + {x^2}^\top x^2) + {x^2}^\top x^1\right] \\ &= a[(4\mu^2 - \mu^2)2\mu^2 - 5\mu^2] = a\mu^2[6\mu^2 - 5] > 0. \end{aligned}$$

Second, for any given symmetric positive definite matrix $P \in \mathbb{R}^{2\times 2}$, $P \neq aI_2$, let $x^1 = -x^2 = \xi = [\xi_1, \xi_2]^\top \in \mathbb{R}^2$ be sufficiently far away from the origin, then

$$\begin{aligned} &(x^1 - x^2)^\top P(f(t,x^1) - f(t,x^2)) \\ &= 4(\xi^\top\xi)\xi^\top P\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\xi - 4\operatorname{sat}_1(\xi^\top\xi)\xi^\top P\xi \\ &= 4\xi^\top\xi[P_{1,2}(\xi_1^2 - \xi_2^2) - (P_{1,1} - P_{2,2})\xi_1\xi_2] - 4\xi^\top P\xi \\ &= 4\big[P_{1,2}(\xi_1^4 - \xi_2^4) + (P_{2,2} - P_{1,1})\xi_1\xi_2(\xi_1^2 + \xi_2^2) \\ &\quad - (P_{1,1}\xi_1^2 + P_{1,2}\xi_1\xi_2 + P_{2,2}\xi_2^2)\big], \end{aligned}$$

which always can be made positive for some $\xi \in \mathbb{R}^2$: if $P_{1,2} > 0$ then let $\xi_2 = 0$ and $|\xi_1|$ sufficiently large, if $P_{1,2} < 0$ then let $\xi_1 = 0$ and $|\xi_2|$ sufficiently large, if $P_{1,2} = 0$ then let $\mathrm{sgn}(P_{2,2} - P_{1,1})\mathrm{sgn}(\xi_1)\mathrm{sgn}(\xi_2) > 0$ and $|\xi_1| = |\xi_2|$ sufficiently large. The above holds in particular for $x_1, x_2 \in \mathbb{R}^2$ arbitrarily far away from the origin, thus there is no $C > 0$ for which (16) can be satisfied.

VI. Conclusions

The global uniform convergence property and global incremental asymptotic stability are very related and yet different properties. This paper in particular contributes examples of systems that are globally uniformly convergent but not globally incrementally stable (and vice versa). These examples further illuminate the essential differences between these stability notions. Moreover, we present results that state sufficient conditions under which a convergent system is also incrementally stable.

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Appendix

Proof of Theorem 4: The proof is similar to the proof given by Angeli [1], but there are some significant and non-obvious differences that we will elaborate on. The main difference and technical difficulty lies in the fact that while the systems (15) considered in [1] can depend on a time-varying perturbation, they may not depend on time explicitly. In contrast, our characterization of incremental stability is for systems depending explicitly on time. The main differences are thus related to the uniformity of the decay of the Lyapunov function. This boils down to a different definition for $U(t^0, z^0)$ in step 3 of the proof, as compared to Angeli's proof. Another difference is the use of Sontag's Lemma on $\mathcal{KL}$-functions in step 7, where another argument was used in the original proof. Finally, we use a scaling argument similar to the one used in [14] in order to obtain a decay rate of class $\mathcal{K}_\infty$ in step 8.

The 'if'-part of the proof follows standard arguments (see, e.g., [4, Theorem 3.2.7]) and is thus omitted. In the following we treat the 'only if'-part.

Let us adopt the following notation for this proof. We consider

$$\dot{x} = f(t, x) \tag{18}$$

and

$$\dot{z} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f(t, x_1) \\ f(t, x_2) \end{pmatrix} \tag{19}$$

as in [1]. We have that the diagonal $\Delta := \{(x^\top, x^\top)^\top : x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$ is GAS w.r.t. system (19) if and only if system (18) is GIS, as is shown in Lemma 2.3 in [1][2]. The distance of a point $z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to the diagonal Δ is given by $\|z\|_\Delta := \inf_{w \in \Delta} \|w - z\|$ and it is shown in [1] that this equals $\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1 - x_2\|$. Now to the details of the proof:

1) First we define
$$g(t^0, z^0) := \sup_{t \geq t^0} \|z(t, t^0, z^0)\|_\Delta \tag{20}$$
which satisfies for the $\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1 = \mathrm{id}$ and $\widetilde{\alpha}_2 = \beta(\cdot, 0)$, where β comes from the definition of GIS, the estimate
$$\widetilde{\alpha}_1(\|z\|_\Delta) \leq g(t, z) \leq \widetilde{\alpha}_2(\|z\|_\Delta) \tag{21}$$
for all $z \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. Observe that the supremum in (20) is in fact a maximum, since $\|z(\cdot, t^0, z^0)\|_\Delta$ is continuous and tends to zero as time tends to infinity. The function g also satisfies the continuity property
$$\begin{aligned} |g(t, z^1) - g(t, z^2)| &\leq \sqrt{2}\beta(2\|z^1 - z^2\|_\Delta, 0) \\ &=: \widetilde{\gamma}(\|z^1 - z^2\|_\Delta), \end{aligned} \tag{22}$$
for all $z^1, z^2 \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. This can be proved as per Fact 2.5 in [1].

2) Along solutions the function g is obviously non-increasing: For $s > 0$ we have
$$g(t^0, z^0) \geq g(t^0 + s, z(t^0 + s, t^0, z^0)).$$

3) Now define
$$U(t^0, z^0) := \sup_{s \geq 0} g(t^0 + s, z(t^0 + s, t^0, z^0))k(s),$$
where k is any continuously differentiable, positive, increasing function for which there exist $1 \leq c_1 < c_2$ such that $k(t) \in [c_1, c_2]$ for all $t \in \mathbb{R}_+$, and the derivative of k is bounded from below by some positive and decreasing function d, i.e. $\dot{k}(t) \geq d(t)$ for all $t \in (0, \infty)$. Necessarily $d(t) \to 0$ as $t \to \infty$, since otherwise (and because $d(t) \geq 0$) k would grow without bound.

[2]Note that [1, Lemma 2.3] holds also true for (explicitly) time-dependent nonlinear systems (19), although in [1] "disturbance-dependent" systems are considered.

4) In view of $c_2 \geq k(t) \geq c_1 \geq 1$ for all $t \in \mathbb{R}_+$ and (21) it follows that
$$U(t^0, z^0) \geq g(t^0, z^0) \geq \|z^0\|_\Delta \tag{23}$$
and
$$U(t^0, z^0) \leq c_2 \widetilde{\alpha}_2(\|z^0\|_\Delta). \tag{24}$$
Using the relation $\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1 - x_2\|$, the inequalities (23) and (24) establish
$$\overline{\alpha}_1(\|x_1 - x_2\|) := \frac{1}{\sqrt{2}}\|x_1 - x_2\| \leq U(t^0, x_1, x_2) \text{ and}$$
$$U(t^0, x_1, x_2) \leq c_2 \widetilde{\alpha}_2\Big(\frac{\|x_1 - x_2\|}{\sqrt{2}}\Big) =: \overline{\alpha}_2(\|x_1 - x_2\|). \tag{25}$$

5) From the definition of U it follows that for all $t^0 \in \mathbb{R}$ and any $z^1, z^2 \in \mathbb{R}^{2n}$ and for all $\epsilon > 0$ there exists an $s_\epsilon = s_{\epsilon,t^0,z^1} \geq 0$ such that $U(t^0, z^1) \leq \epsilon + g(t^0 + s_\epsilon, z(t^0 + s_\epsilon, t^0, z^1))k(s_\epsilon)$. This inequality yields, in view of $k(t) \leq c_2$ for all $t \in \mathbb{R}_+$ and (22), in a few steps (refer to Angeli's proof in [1]) that $U(t^0, z^1) - U(t^0, z^2) \leq \epsilon + \widetilde{\gamma}(\beta(\|z^1 - z^2\|, 0))c_2$. With ϵ arbitrary and using a symmetry argument we arrive at $|U(t^0, z^1) - U(t^0, z^2)| \leq \gamma(\|z^1 - z^2\|)$, where $\gamma(r) = \widetilde{\gamma}\big(\beta(r, 0)\big)c_2$.

6) By definition, U is non-increasing along solutions. We will now show that U strictly decreases along solutions of (19).
By the definition of U, for all $r > 0$ and $z^0 \in \mathbb{R}^{2n}$ with $\|z^0\|_\Delta = r$, for all $t^0 \in \mathbb{R}$, all $h > 0$, and all $\epsilon > 0$, there exists an $s = s_{\epsilon,h,t^0,z^0} \geq 0$ such that we can show that
$$\begin{aligned} &U(t^0 + h, z(t^0 + h, t^0, z^0)) \\ &\leq U(t^0, z^0)\left[1 - \frac{k(h+s) - k(s)}{c_2}\right] + \epsilon. \end{aligned} \tag{26}$$

7) Now we would like to let $h \searrow 0$ and $\epsilon \to 0$ in (26) to obtain an estimate on the decay of U along solutions of (19). For this we have to ensure that s in (26) does not grow without bound when ϵ and h tend to zero.
Claim: For all $r > 0$ there exists a $T = T(r) > 0$ such that s in (26) satisfies $s \leq T$, independent of the choice of $h > 0$ and $\epsilon > 0$.
Proof: We start by recalling a known fact. From Sontag's Lemma on $\mathcal{KL}$-functions [12] it is known that for any $\beta \in \mathcal{KL}$ there exist functions $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ such that for all $r, t \in \mathbb{R}_+$,
$$\beta(r, t) \leq \kappa_1\big(\kappa_2(r)e^{-t}\big). \tag{27}$$
A simple consequence of (27) is that for any $\delta > 0$ we have
$$\beta(r, t) < \delta \text{ whenever } t > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\delta)}. \tag{28}$$
Now we prove the claim. We know from estimates (23) and (24) that
$$0 < r = \|z^0\|_\Delta \leq U(t^0, z^0) \leq c_2 \widetilde{\alpha_2}(r).$$
Continuity and monotonicity properties of U along trajectories of (19) with $\|z^0\|_\Delta = r$ yield that for some $\nu > 0$, $\mu > 0$,
$$\begin{aligned} \nu + \epsilon &< U(t^0, z^0) - \mu \\ &< U(t^0 + h, z(t^0 + h, t^0, z^0)) \\ &\leq U(t^0, z^0) \end{aligned} \tag{29}$$
for all $0 < h < \overline{h} = \overline{h}(\epsilon)$ if $\epsilon > 0$ is sufficiently small, which we will henceforth assume.
Let $\delta = \nu / c_2$ and let us assume that no finite $T > 0$ as in the claim exists. Then for every integer $N > 0$ there must exist an $s > N$ such that (26) holds for this s, i.e., we can show that
$$U(t^0 + h, z(t^0 + h, t^0, z^0)) \leq \beta(\|z^0\|_\Delta, h + s)c_2 + \epsilon$$
$$< \nu + \epsilon \text{ whenever } s > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\nu / c_2)} \text{ due to (28).}$$
Considering (29) we arrive at the contradiction
$$\nu + \epsilon < U(t^0 + h, z(t^0 + h, t^0, z^0)) < \nu + \epsilon$$
thus proving the claim. □
Hence we have shown that we can pass to an appropriate limit in (26) as $h \searrow 0$ and $\epsilon \to 0$, since $s = s_{\epsilon,h,t^0,z^0}$ in (26) remains bounded.

8) Following essentially the same arguments as in [1] we obtain for some positive definite function $\widetilde{\alpha}_3$,
$$\begin{aligned} \dot{U}(t^0, z^0) &:= \limsup_{h \searrow 0} \\ &\frac{U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0)}{h} \\ &\leq -\widetilde{\alpha}_3(\|z^0\|_\Delta). \end{aligned}$$
At this stage it is left to show that we can modify U such that the function $\widetilde{\alpha}_3$ can be taken to be of class $\mathcal{K}_\infty$. The argument we are going to make follows the idea in [14].
To this end let $\mu, \rho \in \mathcal{K}_\infty$ such that $\rho' = \mu$ and that $s \mapsto (\mu \circ \alpha_1^{-1})(s)\widetilde{\alpha}_3(s)$ is bounded from below by some class $\mathcal{K}_\infty$ function α_3. This is always possible.
Define $W := \rho(U)$ and verify using (25) that it satisfies bounds (8) with $\alpha_i = \rho \circ \overline{\alpha}_i$, $i = 1, 2$. Compute
$$\begin{aligned} \dot{W}(t^0, z^0) &:= \limsup_{h \searrow 0} \\ &\frac{W(t^0 + h, z(t^0 + h, t^0, z^0)) - W(t^0, z^0)}{h} \\ &= \limsup_{h \searrow 0} \rho'(U(\tau_{t^0,h}, z(\tau_{t^0,h}, t^0, z^0))) \cdot \\ &\frac{U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0)}{h} \\ &\leq -\rho'\big(\overline{\alpha}_1^{-1}(\|z^0\|_\Delta)\big) \cdot \widetilde{\alpha}_3(\|z^0\|_\Delta) \\ &\leq -\alpha_3(\|z^0\|_\Delta), \end{aligned} \tag{30}$$
with $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$ and where in equation (30) we have used the mean value theorem to obtain a sequence $\tau_{t^0,h} \overset{h \to 0}{\longrightarrow} t^0$ of points in $(t^0, t^0 + h)$, followed by continuity of ρ' and U with respect to time.

9) Now, following again the same arguments as in [1] we obtain for $t \geq t^0$, $W(t, z(t, t^0, z^0)) - W(t^0, z^0) \leq -\int_{t^0}^{t} \alpha_3(\|z(s, t^0, z^0)\|_\Delta)\mathrm{d}s$, which proves the inequality (9) in the theorem. This completes the proof of the theorem. ■

Proof of Corollary 5: Just take instead of W the function U defined in the preceding proof at the end of step 5, it satisfies all the requirements by construction. Without loss of generality, the function γ can be taken to be class $\mathcal{K}_\infty$. ■