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The Symmedian Point and Concurrent Antiparallel Images

Shao-Cheng Liu

Abstract. In this note, we study the condition for concurrency of the GP lines of the three triangles determined by three vertices of a reference triangle and six vertices of the second Lemoine circle. Here G is the centroid and P is arbitrary triangle center different from G. We also study the condition for the images of a line in the three triangles bounded by the antiparallels through a given point to be concurrent.

1. Antiparallels through the symmedian point

Given a triangle ABC with symmedian point K, we consider the three triangles AB_aC_a, A_bBC_b, and A_cB_cC bounded by the three lines ℓ_a, ℓ_b, ℓ_c antiparallel through K to the sides BC, CA, AB respectively (see Figure 1). It is well known [4] that the 6 intercepts of these antiparallels with the sidelines are on a circle with center K. In other words, K is the common midpoint of the segments B_aC_a, C_bA_b and A_cB_c. The circle is called the second Lemoine circle.

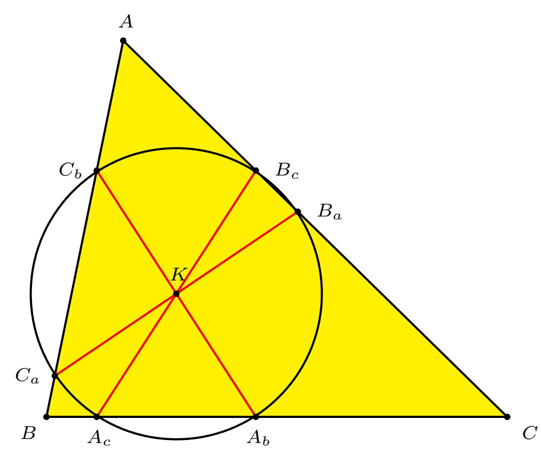

Figure 1.

Triangle AB_aC_a is similar to ABC, because it is the reflection in the bisector of angle A of a triangle which is a homothetic image of ABC. For an arbitrary triangle center P of ABC, denote by P_a the corresponding center in triangle AB_aC_a; similarly, P_b and P_c in triangles A_bBC_b and A_cB_cC.

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Now let P be distinct from the centroid G. Consider the line through A parallel to GP. Its reflection in the bisector of angle A intersects the circumcircle at a point Q', which is the isogonal conjugate of the infinite point of GP. So, the line G_aP_a is the image of AQ' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$, and it passes through a trisection point of the segment KQ' (see Figure 2).

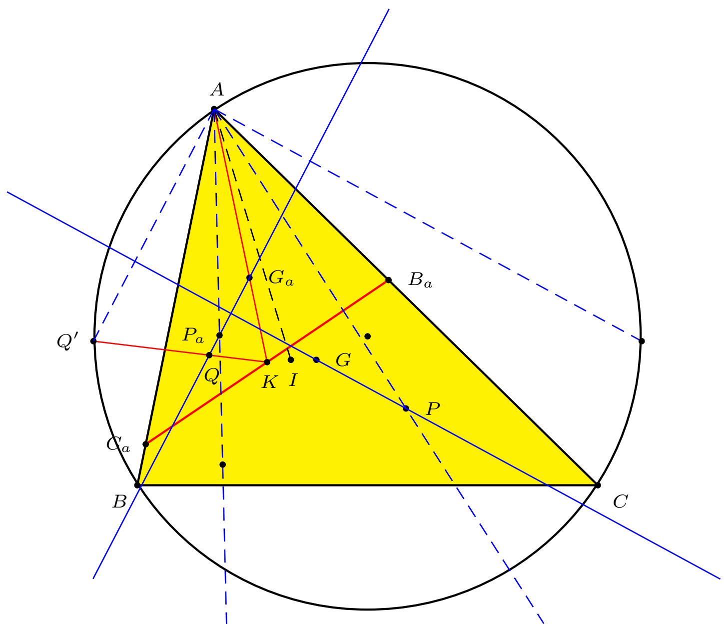

Figure 2.

In a similar manner, the reflections of the parallels to GP through B and C in the respective angle bisectors intersect the circumcircle at the same point Q'. Hence, the lines G_bP_b and G_cP_c also pass through the point Q, which is the image of Q' under the homothety $\mathsf{h}\left(K, \frac{1}{3}\right)$. It is clear that the point Q lies on the circumcircle of triangle $G_aG_bG_c$ (see Figure 3). We summarize this in the following theorem.

Theorem 1. *Let P be a triangle center of ABC, and P_a, P_b, P_c the corresponding centers in triangles AB_aC_a, BC_bA_b, CA_cB_c, which have centroids G_a, G_b, G_c respectively. The lines G_aP_a, G_bP_b, G_cP_c intersect at a point Q on the circumcircle of triangle $G_aG_bG_c$.*

Here we use homogeneous barycentric coordinates. Suppose $P = (u : v : w)$ with reference to triangle ABC.

(i) The isogonal conjugate of the infinite point of the line GP is the point

$$Q' = \left(\frac{a^2}{-2u+v+w} : \frac{b^2}{u-2v+w} : \frac{c^2}{u+v-2w}\right)$$

on the circumcircle.

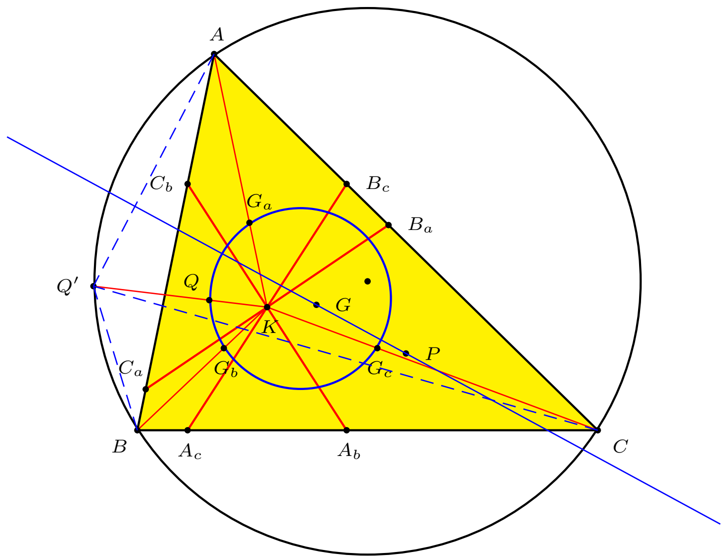

Figure 3.

(ii) The lines G_aP_a, G_bP_b, G_cP_c intersect at the point

$$Q = \left(\frac{a^2}{v+w-2u}\left(a^2 + \frac{b^2(w-u)}{w+u-2v} + \frac{c^2(v-u)}{u+v-2w}\right) : \dots : \dots\right).$$

which divides KQ' in the ratio $KQ : QQ' = 1 : 2$.

2. A generalization

More generally, given a point $T = (x : y : z)$, we consider the triangles intercepted by the antiparallels through T. These are the triangles AB_aC_a, A_bBC_b and A_cB_cC with coordinates (see [1, §3]):

$$\begin{aligned}
B_a &= (b^2x + (b^2 - c^2)y : 0 : c^2y + b^2z),\\
C_a &= (c^2x - (b^2 - c^2)z : c^2y + b^2z : 0),\\
C_b &= (a^2z + c^2x : c^2y + (c^2 - a^2)z : 0),\\
A_b &= (0 : a^2y - (c^2 - a^2)x : a^2z + c^2x),\\
A_c &= (0 : b^2x + a^2y : a^2z + (a^2 - b^2)x),\\
B_c &= (b^2x + a^2y : 0 : b^2z - (a^2 - b^2)y).
\end{aligned}$$

Now, for a point P with coordinates $(u : v : w)$ with reference to triangle ABC, the one with the same coordinates with reference to triangle AB_aC_a is

$$\begin{aligned}
P_a = &\big(b^2c^2(x+y+z)u + c^2(b^2x + (b^2 - c^2)y)v + b^2(c^2x - (b^2 - c^2)z)w :\\
&b^2(c^2y + b^2z)w : c^2(c^2y + b^2z)v\big).
\end{aligned}$$

By putting $u = v = w = 1$, we obtain the coordinates of the centroid

$$G_a = \left(3b^2c^2x + c^2(2b^2 - c^2)y - b^2(b^2 - 2c^2)z) : b^2(c^2y + b^2z) : c^2(c^2y + b^2z)\right)$$

of AB_aC_a. The equation of the line G_aP_a is

$$\begin{aligned}
&(c^2y + b^2z)(v - w)\mathbb{X} \\
&+ (c^2(x + y + z)u + (-2c^2x - c^2y + (b^2 - 2c^2)z)v + (c^2x - (b^2 - c^2)z)w)\mathbb{Y} \\
&- (b^2(x + y + z)u + (b^2x + (b^2 - c^2)y)v - (2b^2x + (2b^2 - c^2)y + b^2z)w)\mathbb{Z} \\
&= 0.
\end{aligned}$$

By cyclically replacing (a, b, c), (u, v, w), (x, y, z), and $(\mathbb{X}, \mathbb{Y}, \mathbb{Z})$ respectively by (b, c, a), (v, w, u), (y, z, x), and $(\mathbb{Y}, \mathbb{Z}, \mathbb{X})$, we obtain the equation of the line G_bP_b. One more applications gives the equation of G_cP_c.

Proposition 2. *The three lines G_aP_a, G_bP_b, G_cP_c are concurrent if and only if*

$$f(u, v, w)(x + y + z)^2 \left(b^2c^2(v - w)x + c^2a^2(w - u)y + a^2b^2(u - v)z\right) = 0,$$

where

$$f(u, v, w) = \sum_{\text{cyclic}} \left((2b^2 + 2c^2 - a^2)u^2 + (b^2 + c^2 - 5a^2)vw\right).$$

Computing the distance between G and P, we obtain

$$f(u, v, w) = 9(u + v + w)^2 \cdot GP^2.$$

This is nonzero for $P \neq G$. From this we obtain the following theorem.

Theorem 3. *For a fixed point $P = (u : v : w)$, the locus of a point T for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$b^2c^2(v - w)\mathbb{X} + c^2a^2(w - u)\mathbb{Y} + a^2b^2(u - v)\mathbb{Z} = 0.$$

Remarks. (1) The line clearly contains the symmedian point K and the point $(a^2u : b^2v : c^2w)$, which is the isogonal conjugate of the isotomic conjugate of P.

(2) The locus of the point of concurrency is the line

$$\sum_{\text{cyclic}} b^2c^2(v - w)((c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2)\mathbb{X} = 0.$$

This line contains the points

$$\left(\frac{a^2}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right)$$

and

$$\left(\frac{a^2u}{(c^2 + a^2 - b^2)(u - v)^2 + (a^2 + b^2 - c^2)(u - w)^2} : \cdots : \cdots\right).$$

Theorem 4. *For a fixed point $T = (x : y : z)$, the locus of a point P for which the GP-lines of triangles AB_aC_a, A_bBC_b, and A_cB_cC are concurrent is the line*

$$\left(\frac{y}{b^2} - \frac{z}{c^2}\right)\mathbb{X} + \left(\frac{z}{c^2} - \frac{x}{a^2}\right)\mathbb{Y} + \left(\frac{x}{a^2} - \frac{y}{b^2}\right)\mathbb{Z} = 0.$$

Remark. This is the line containing the centroid G and the point $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$.

More generally, given a point $T = (x : y : z)$, we study the condition for which the images of the line

$$\mathcal{L} : \qquad u\mathbb{X} + v\mathbb{Y} + w\mathbb{Z} = 0$$

in the three triangles AB_aC_a, A_bBC_b and A_cB_cC are concurrent. Now, the image of the line $\mathcal{L}$ in AB_aC_a is the line

$$\begin{aligned} &-u(c^2y + b^2z)\mathbb{X} + ((c^2x - (b^2 - c^2)z)u - c^2(x + y + z)w)\mathbb{Y} \\ &+ ((b^2x + (b^2 - c^2)y)u - b^2(x + y + z)v)\mathbb{Z} = 0. \end{aligned}$$

Similarly, we write down the equations of the images in A_bBC_b and A_cB_cC. The three lines are concurrent if and only if

$$\begin{aligned} &((b^2 + c^2 - a^2)(v - w)^2 + (c^2 + a^2 - b^2)(w - u)^2 + (a^2 + b^2 - c^2)(u - v)^2) \\ &\cdot (x + y + z)^2 \left(\sum_{\text{cyclic}} u \cdot a^2(c^2y + b^2z)\right) = 0. \end{aligned}$$

Since the first two factors are nonzero for nonzero (u, v, w) and (x, y, z), we obtain the following result.

Theorem 5. *Given $T = (x : y : z)$, the antiparallel images of a line are concurrent if and only if the line contains the point*

$$T' = \left(\frac{y}{b^2} + \frac{z}{c^2} : \frac{z}{c^2} + \frac{x}{a^2} : \frac{x}{a^2} + \frac{y}{b^2}\right).$$

Here are some examples of correspondence:

T	T'	T	T'	T	T'
X_1	X_{37}	X_{19}	X_{1214}	X_{69}	X_{1196}
X_2	X_{39}	X_{20}	X_{800}	X_{99}	X_{1084}
X_3	X_6	X_{40}	X_{1108}	X_{100}	X_{1015}
X_4	X_{216}	X_{55}	X_1	X_{110}	X_{115}
X_5	X_{570}	X_{56}	X_9	X_{111}	X_{2482}
X_6	X_2	X_{57}	X_{1212}	X_{887}	X_{888}

References

[1] N. Dergiades and P. Yiu, Antiparallels and Concurrent Euler Lines, *Forum Geom.*, 4(2004) 1–20.

[2] C. Kimberling, Encyclopedia of Triangle Centers, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[3] P. Yiu, *Introduction to the Geometry of the Triangles*, Florida Atlantic University Lecture Notes, 2001, available at `http://math.fau.edu/Yiu/Geometry.html`.

[4] P. Yiu, A Tour of Triangle Geometry (2004), available at
http://math.fau.edu/Yiu/Geometry.html.

Shao-Cheng Liu: 2F., No.8, Alley 9, Lane 22, Wende Rd., 11475 Taipei, Taiwan
E-mail address: liu471119@yahoo.com.tw