On the Controllability of a Class of Discrete Distributed Systems

Hasna Jourhmane[1], Mostafa Rachik[2], Ahmed Abdelhak*[3], Jamal Bouyaghroumni[4] and Hanane Ferjouchia[5]

[1,2] Departement of mathematics, University Hassan II-Mohammedia, faculty of sciences Ben M'sik, Casablanca, Morroco.

*[3] Departement of mathematics, University Ibn Tofail, faculty of sciences, B.P 133, Kénitra, Morroco.

[4,5] Departement of mathematics, University Hassan II-Mohammedia, faculty of sciences Ben M'sik, Casablanca, Morroco.

Abstract

We consider a class of linear discrete-time systems controlled by a continuous time input. Given a desired final state x_d, we investigate the optimal control which steers the system, with a minimal cost, from an initial state x_0 to x_d. We consider both discrete distributed systems and finite dimensional ones. We use a method similar to the Hilbert Uniqueness Method (HUM) to determine the control and the Galerkin method to approximate it, we also give an example to illustrate our approach.

Keywords: Discrete linear systems, Hilbert Uniqueness Method, Optimal Control, Galerkin Method.

1 Introduction

This paper is devoted to the study of the controllability problem corresponding to the discrete -time varying distributed systems described by

$$\begin{cases} x_{i+1} = \phi x_i + \int_{t_i}^{t_{i+1}} B_i(\theta)u(\theta)d\theta, \\ x_0 \text{ given in } X \end{cases} \qquad (S)$$

for $i = 0,..,N-1$, where $x_i \in X$, $u \in L^2(0,T,U)$, $\phi \in \mathcal{L}(X)$, $B_i(\theta) \in \mathcal{L}(U,X)$, $(X, \| \ \|)$ and $(U, \| \ \|)$ are Hilbert spaces and $(t_i)_i$ is a subdivision of the interval $[0,T]$ such that $t_0 = 0$ and $t_N = T$. Moreover, we suppose that the applications $\theta \to B_i(\theta)$, $i = 0,\dots,N-1$ are continuous.

In other words, given a desired final state x_d, we investigate the optimal control which steers the system (S) from x_0 to x_d with a minimal cost $J(u) = \|u\|$. As an example of systems described by (S), we consider the linear continuous system given by

$$x(t) = S(t)x_0 + \int_0^t S(t-r)Bu(r)dr, \ \ t \geq 0 \qquad (1)$$

where $S(t)$ is a strongly continuous semi group on the Hilbert space X and $B \in \mathcal{L}(U,X)$. In order to make the system accessible by a computer we proceed to a sampling of time (see for example [8, 12, 13]), this means, we put

$$[0,T] = \bigcup_{i=0}^{N-1} [t_i, t_{i+1}]$$

where

$$\begin{cases} t_0 &= 0 \\ t_{i+1} &= t_i + \delta, \end{cases}$$

with $\delta = \frac{T}{N}$ and $N \in \mathbb{N}^*$.

If we take $x_i = x(t_i)$ then

$$\begin{aligned} x_{i+1} &= x(t_{i+1}) \\ &= S(t_{i+1})x_0 + \int_0^{t_{i+1}} S(t_{i+1}-r)Bu(r)dr \\ &= S(t_i+\delta)x_0 + \int_0^{t_i} S(t_i+\delta-r)Bu(r)dr \\ &\quad + \int_{t_i}^{t_{i+1}} \underbrace{S(t_{i+1}-)B}_{B_i(r)} u(r)dr \\ &= S(\delta)[S(t_i)x_0 + \int_0^{t_i} S(t_i-r)Bu(r)dr] \\ &\quad + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr \end{aligned}$$

then

$$x_{i+1} = \underbrace{S(\delta)}_{\phi} x(t_i) + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr$$

and consequently

$$x_{i+1} = \phi x_i + \int_{t_i}^{t_{i+1}} B_i(r)u(r)dr$$

which is a system described by (S).
In many works (see [6, 8, 13]) and under the hypothesis

$$u(t) = u_i \quad \forall t \in [t_i, t_{i+1}[, \tag{2}$$

(the hypothesis (2) means that, $u(t)$ is assumed to be constant in the interval $[t_i, t_{i+1}[$), the sampling of system (S) leads to the difference equation

$$x_{i+1} = Lx_i + Mu_i$$

where $L = \phi$ and $M = \int_{t_i}^{t_{i+1}} B_i(r)dr$.
This last discrete version has been used by several authors ([5, 3, 7, 11, 15, 16]). In some situations, the control law could have fast variations during time. Consequently the hypothesis (2) becomes inappropriate, this shows the importance of our system (S).

In this chapter, we use a technique similar to the Hilbert Uniqueness Method, introduced by Lions J.L. (see [9, 10]), in order to treat the controllability problem. The section 4 contain a method for approximating the optimal control and an example that illustrate the developed results. In the section 5, we study this problem in finite dimensional case.

2 Preliminary results

The final state of system (S) can be written as follows

$$x_N = \phi^N x_0 + Hu$$

where

$$\begin{aligned} H \ : \ L^2(0,T,U) &\rightarrow X \\ u &\mapsto \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)u(\theta)d\theta. \end{aligned} \tag{3}$$

Definition 2.1 *We say that (S) is weakly controllable on $\{0,\dots,N\}$ if $\overline{Im\, H} = X$. ($Im\, H$ means the range of H).*

Remark 1 *(S) is weakly controllable if and only if $Ker\, H^* = \{0\}$.*

Lemma 1 *The operator H is bounded and its adjoint operator H^* is given by , for all $x \in X$*

$$H^*x(\theta) = B^*_{j-1}(\theta)(\phi^*)^{N-j}x, \tag{4}$$

forall $\theta \in]t_{j-1}, t_j[$ and all $j = 1,\dots,N$.

Proof
Let $u \in L^2(0,T,U)$, $x \in X$

$$\begin{aligned} &< Hu, x >=< \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \phi^{N-j} B_{j-1}(\theta)u(\theta)d\theta, x > \\ &= \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} < u(\theta), B^*_{j-1}(\theta)(\phi^*)^{N-j}x > d\theta \\ &= \sum_{j=1}^{N} \int_0^T < u(\theta), B^*_{j-1}(\theta)(\phi^*)^{N-j}x.\mathcal{X}_{]t_{j-1},t_j[}(\theta) > d\theta \\ &= \int_0^T < u(\theta), \sum_{j=1}^{N} B^*_{j-1}(\theta)(\phi^*)^{N-j}x.\mathcal{X}_{]t_{j-1},t_j[}(\theta) > d\theta \\ &= \int_0^T < u(\theta), H^*x(\theta) > d\theta \end{aligned}$$

hence

$$H^*x(\theta) = \sum_{j=1}^{N} B^*_{j-1}(\theta)(\phi^*)^{N-j}x.\mathcal{X}_{]t_{j-1},t_j[}(\theta) \tag{5}$$

which implies (4). ∎

Consider on $X \times X$ the bilinear form given by

$$< x, y >_F =< H^*x, H^*y >, \quad \forall x, y \in X \tag{6}$$

clearly, if (S) is weakly controllable, then $< .,. >_F$ describes an inner product on X. Let $\|.\|_F$ be the corresponding norm and F the completion of X with respect to the norm $\|.\|_F$.

Remark 2

$$\|x\|_F \leq \|H^*\|\|x\|, \quad \forall x \in X.$$

In the following, we suppose that (S) is weakly controllable.
Define the operator Λ by

$$\begin{array}{rccc} \Lambda & : X & \rightarrow & X \\ & x & \mapsto & HH^*x \end{array}$$

then

$$Ker\,\Lambda = Ker\,H^*$$

moreover

$$| < \Lambda\, x, y > | \leq \|x\|_F \|y\|_F, \quad \forall x, y \in F$$

then, it is classical that Λ can be extended, in a single way by an isomorphism, denoted also Λ, defined from F onto F' (see [10, 14]). Moreover, F is a Hilbert space with respect to the inner product

$$< x, y >_F = < \Lambda x, y >_{F',F} \quad \forall x, y \in F \tag{7}$$

where $< \Lambda x, y >_{F',F}$ means the range of y by the operator Λx. From (6) we deduce that

$$\|H^*x\| = \|x\|_F, \quad \forall x \in X$$

hence H^* is a bounded operator from $(X, \|.\|_F)$ onto $(L^2(0,T,U), \|.\|)$, so it has a bounded extension, denoted H_*, defined from F onto $L^2(0,T,U)$.

Lemma 2 *ImH can be identified to a subset of F'.*

Proof
Let $x \in Im\,H$, and consider the map

$$\begin{array}{rccc} \varphi_x & : X & \rightarrow & \mathbb{R} \\ & y & \mapsto & < x, y > \end{array}$$

there exists $u \in L^2(0,T,U)$ such that $x = Hu$, hence forall $y \in X$ we have

$$\begin{array}{rcl} |\varphi_x(y)| & = & | < x, y > | = | < Hu, y > | \\ & = & | < u, H^*y > | \leq \|u\|\|y\|_F. \end{array}$$

Consequently, φ_x has a bounded extension, denoted by $\overline{\varphi_x}$, which belongs to F'. Let j be the map defined by

$$\begin{array}{rccc} j & : Im\,H & \rightarrow & F' \\ & x & \mapsto & \overline{\varphi_x} \end{array}$$

clearly j is linear and injective ■

The operator HH_* is defined from F onto $Im\ H$, using lemma, (2) we can consider that HH_* is defined from F onto F'.

Proposition 2.1 *The operators Λ and HH_* are equal.*

Proof
Let $\overline{x} \in F$ be arbitrary, we have

$$\begin{array}{rcl} | < HH_*\overline{x}, y >_{F',F} | & = & | < HH_*\overline{x}, y > |, \quad \forall y \in X \\ & = & | < H_*\overline{x}, H^*y > | \\ & \leq & \|H_*\overline{x}\|\|H^*y\| \\ & \leq & \|H_*\overline{x}\|\|y\|_F \end{array}$$

by density of X on F, we deduce that

$$| < HH_*\overline{x}, \overline{y} >_{F',F} | \leq \|H_*\overline{x}\|\|\overline{y}\|_F, \quad \forall \overline{y} \in F$$

hence

$$\|HH_*\overline{x}\|_{F'} \leq \|H_*\overline{x}\| \leq \|H_*\|\|\overline{x}\|_F$$

which implies that HH_* is bounded. On the other hand

$$HH_*x = HH^*x = \Lambda x, \quad \forall x \in X$$

by density of X and continuity of both HH_* and Λ from F onto F', we deduce that

$$HH_*\overline{x} = \Lambda\overline{x}, \quad \forall \overline{x} \in F.$$

■

Lemma 3 *The inner product corresponding to $\|.\|_F$ is*

$$< x, y >_F = < H_*x, H_*y >, \quad \forall x, y \in F$$

Proof
From (7) and Proposition 2.1, we deduce

$$< x, y >_F = < HH_*x, y >_{F',F}, \quad \forall x, y \in F$$

but

$$\begin{array}{rcl} < HH_*x, y >_{F',F} & = & < HH_*x, y >, \quad \forall y \in X \\ & = & < H_*x, H^*y > \\ & = & < H_*x, H_*y > . \end{array}$$

if $y \in F$, $\exists (y_n) \subset X$ such that $\|y_n - y\| \rightarrow 0$. We have,

$$< HH_*x, y_n >_{F',F} = < H_*x, H_*y_n >, \quad \forall n \in \mathbb{N}$$

when $n \rightarrow +\infty$, we obtain

$$< HH_*x, y >_{F',F} = < H_*x, H_*y >, \quad \forall y \in F$$

■

Remark 3
From lemma 3, we deduce that if (S) is weakly controllable then $Ker\,H_* = \{0\}$.

3 The optimal control

We first characterize the set of all reachable states at time N from a given initial state x_0.

Proposition 3.1 *The reachable set at time N, from a given initial state x_0, is given by*

$$R(N) = \phi^N x_0 + F'.$$

Proof
If $z \in \phi^N x_0 + F'$, then $z - \phi^N x_0 \in F'$, hence there exists $f \in F$ such that $z - \phi^N x_0 = \Lambda f$, which implies that

$$z = \phi^N x_0 + HH_* f = \phi^N x_0 + Hu$$

where $u = H_* f$, thus z is reachable.
Conversely, if z is reachable, say that $z = \phi^N x_0 + Hu$, then

$$z - \phi^N x_0 = Hu$$

that is $z - \phi^N x_0 \in Im\, H \subset F'$ hence $z \in \phi^N x_0 + F'$.
■

Theorem 3.1 *If $x_d - \phi^N x_0 \in F'$, then the control $u^* = H_* f$, where f is the unique solution of the algebraic equation*

$$\Lambda f = x_d - \phi^N x_0 \tag{8}$$

steers the system from the initial state x_0 to the final state x_d at time N with a minimal cost $J(u) = \|u\|$, moreover $\|u^*\| = \|f\|_F$.

Proof
Let $u^* = H_* f$, where f verify (8), f exists since $x_d - \phi^N x_0 \in F'$. We have,

$$\phi^N x_0 + Hu^* = \phi^N x_0 + \Lambda f = x_d$$

hence u^* steers (S) from x_0 to x_d at time N. Suppose that v steers (S) from x_0 to x_d at time N, then

$$\phi^N x_0 + Hv = x_d = \phi^N x_0 + Hu^*$$

hence,

$$Hv = Hu^*$$

which implies that

$$< H(v - u^*), f_n >= 0; \quad \forall n$$

where $(f_n)_n$ is a sequence, of elements in X, which converges towards f with respect to the norm $\|.\|_F$.
Consequently,

$$< v - u^*, H^* f_n >= 0, \quad \forall n$$

or

$$< v - u^*, H_* f_n >= 0, \quad \forall n$$

when $n \to +\infty$, we deduce that

$$< v - u^*, H_* f >= 0$$

or

$$< v - u^*, u^* >= 0$$

thus

$$< v, u^* >= \|u^*\|^2$$

which implies that

$$\|u^*\|^2 \leq \|v\| \|u^*\|$$

$$\|u^*\| \leq \|v\|.$$

■

4 A numerical approach

In order to determine the optimal control u^*, we need to resolve the algebraic equation

$$\Lambda f = x_d - \phi^N x_0 \quad \text{on } F'. \tag{9}$$

In this section, we propose a numerical approach to approximate f. Suppose that $x_d - \phi^N x_0 \in F'$ and that X is a separable space. Let $(w_i)_{i \geq 1}$ be a basis of X.
Equation (9) is equivalent to

$$< \Lambda f, y >_{F',F} = < x_d - \phi^N x_0, y >_{F',F}, \quad \forall y \in X \tag{10}$$

Remark 4 *Since the bilinear form*

$$(u, v) \to < \Lambda u, v >_{F',F}$$

is coercive on $F \times F$ and the map

$$y \to < x_d - \phi^N x_0, y >_{F',F}$$

belongs to F', one can think to apply the Galerkin method to approximate f. But this involves some difficulties because the map $y \mapsto < x_d - \phi^N x_0, y >_{F',F}$ is known on X but almost unknown on F, also $(u, v) \mapsto < u, v >_F$ is known on $X \times X$ but almost unknown on $F \times F$.

Equation (10) is equivalent to

$$< f, y >_F = < x_d - \phi^N x_0, y >, \quad \forall y \in X \tag{11}$$

Remark that in equation (11), the solution f belongs to F and the variable y is in X. In the following, we will prove that by applying the Galerkin method to equation (11), we can construct a sequence (f_n) which converges strongly on F towards f.
Let X_m be the subspace of X spanned by the vector $w_1, w_2, \ldots, w_m$ and $f_m \in X$, the solution of

$$< f_m, y >_F = < x_d - \phi^N x_0, y >, \quad \forall y \in X_m \tag{12}$$

Since $\|.\|$ and $\|.\|_F$ are equivalent on X_m, the bilinear form $(u, v) \mapsto < u, v >_F$ is continuous and coercive on $X_m \times X_m$, moreover, $y \mapsto < x_d - \phi^N x_0, y >$ is bounded on X_m. From the Lax-Milgram theorem, see([1, 2]), we deduce that f_m exists and is unique. Using (12) we have

$$< f_m, f_m >_F = < x_d - \phi^N x_0, f_m > \tag{13}$$

Since $x_d - \phi^N x_0 \in F'$, there exists a constant c such that

$$| < x_d - \phi^N x_0, y >_{F',F} | \leq c\|y\|_F, \quad \forall y \in F$$

hence,

$$| < x_d - \phi^N x_0, y > | \leq c\|y\|_F, \quad \forall y \in X \tag{14}$$

from (13) and (14), we deduce that

$$\|f_m\|_F^2 \leq < f_m, f_m > \leq c\|f_m\|_F$$

i.e.

$$\|f_m\|_F \leq c, \quad \forall m.$$

Consequently, (f_m) admits a subsequence $(f_{m'})_{m'}$ which converges weakly to a certain $f_* \in F$, we will denote this weak convergence by

$$f_{m'} \rightharpoonup f_*. \tag{15}$$

Let $\mathcal{C}$ denote the set of all finite combinations of $w_i, \quad i \geq 1$. Suppose that $v \in \mathcal{C}$, then v belong to $X_{m'}$ for m' sufficiently large, hence

$$< f_{m'}, v >_F = < x_d - \phi^N x_0, v > .$$

From (15), we deduce that

$$\begin{aligned} \lim_{m' \to +\infty} < f_{m'}, v >_F &= < f_*, v >_F \\ &= < x_d - \phi^N x_0, v >, \ \ v \in \mathcal{C} \end{aligned}$$

let $x \in X$, since $\mathcal{C}$ is dense on $(X, \|.\|)$, then there exists a sequence $(x_n)_n$ such that $\|x_n - x\| \to 0$, which implies that $\|x_n - x\|_F \to 0$, using Remark (2). On the other hand,

$$< f_*, x_n >_F = < x_d - \phi^N x_0, x_n >, \quad \forall n$$

when $n \to +\infty$, we obtain

$$< f_*, x >_F = < x_d - \phi^N x_0, x >, \quad \forall x \in X$$

hence f_* is solution of (11), by uniqueness we deduce that $f_* = f$. Hence $(f_m)_m$ has a subsequence $(f_{m'})_{m'}$ which converges weakly on $(F, \|.\|_F)$ towards f. Suppose that $(f_m)_m$ doesn't converges weakly, on $(F, \|.\|_F)$, towards f, then there exists $v \in F$ such that $< f_m, v >_F$ doesn't converges towards $< f, v >_F$, i.e.,

$$\exists \epsilon, \forall N \ \ \exists n > N \ \ | < f_n, v >_F - < f, v >_F | > \epsilon$$

From this we deduce that, forall $N \in \mathbb{N}$, there exists $\varphi(N) > N$ such that

$$| < f_{\varphi(N)}, v >_F - < f, v >_F | > \epsilon \tag{16}$$

but $(f_{\varphi(N)})_N$ is bounded on F, hence $(f_{\varphi(N)})_N$ has a subsequence $(f_{\varphi(N')})_{N'}$ which converges weakly towards f, hence

$$< f_{\varphi(N')}, v >_F \to < f, v >_F$$

which contradicts (16) thus

$$f_m \rightharpoonup f.$$

To prove that $f_m \to f$ strongly on F, we consider

$$\begin{aligned} &< f_m - f, f_m - f >_F = \\ &\quad < f_m, f_m >_F - < f_m, f >_F - < f, f_m >_F \\ &\quad + < f, f >_F \end{aligned}$$

recall that

$$< f_m, f_m > = < x_d - \phi^N x_0, f_m >$$

hence

$$\lim_{m \to +\infty} < f_m, f_m > = < x_d - \phi^N x_0, f >_{F',F}.$$

On the other hand,

$$\begin{aligned} \lim_{m \to +\infty} < f_m, f > &= < f, f >_F \\ \lim_{m \to +\infty} < f, f_m > &= < f, f >_F \end{aligned}$$

consequently,

$$\begin{aligned}\lim_{m\to+\infty} &< f_m - f, f_m - f > \\ &= < x_d - \phi^N x_0, f >_{F',F} - < f, f >_F \\ &=< x_d - \phi^N x_0, f > - < \Lambda f, f >_{F',F} \\ &=< x_d - \phi^N x_0 - \Lambda f, f >_{F',F} \\ &= 0\end{aligned}$$

thus $f_m \to f$ strongly on F.

Remark 5 *To determine (f_m), we don't need the expression of H_* nor the completion space F.*

Remark 6 *The sequence of inputs $u_n = H^* f_n$ converges strongly, on $L^2(0,T,U)$, towards the optimal control $u^* = H_* f$.*

4.1 Example

Consider the system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i \qquad (17)$$

where $x(t) \in X = L^2(0,1)$, $b_i \in X$, $u_i \in L^2(0,T)$, $A = \frac{\partial^2}{\partial\alpha^2}$ and $D(A) = \{x \in L^2(0,1), \frac{\partial^2 x}{\partial\alpha^2} \in L^2(0,1), x(0) = x(1) = 0\}$. A is self-adjoint and has respectively eigenvalues and eigenvectors given by $\lambda_n = -n^2\pi^2$ and $\Phi_j(t) = \sqrt{2}\sin(j\pi t)$, $t \in [0,1]$ and $j = 1, 2, \ldots$.

We suppose for example that $\int_0^1 b_1(\alpha)\sin(n\pi\alpha)d\alpha \neq 0$, $\forall n \geq 1$, this implies that the system (17) is weakly controllable, (see [4]). If we introduce the operator B

$$\begin{array}{llll} B: & \mathbb{R}^m & \to & X \\ & (u_1, \ldots, u_m) & \hookrightarrow & \sum_{i=1}^{m} b_i u_i \end{array}$$

then the system (17) becomes

$$\dot{x} = Ax + Bu. \qquad (18)$$

Now, consider the discrete version of (18) obtained by a similar way as presented in the introduction of this paper,

$$x_{i+1} = \Phi x_i + \int_{t_i}^{t_{i+1}} B_i(\theta)u(\theta)d\theta \qquad (19)$$

where $t_i = i\Delta, i = 0, \ldots, N$ with Δ is a sampling of $[0,T]$, $x_i = x(t_i)$, $B_i(\theta) = T(t_{i+1} - \theta)B$, $\Phi = T(\Delta)$ where $T(t)$ is the strongly continuous semi group, generated by A, given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < z, \Phi_n > \Phi_n \ , \quad \forall z \in X.$$

Since the system (18) is weakly controllable on $[0,T], \forall T > 0$ we deduce that

$$\forall x_d \in X, \exists u \in L^2(0,T,U) \ : \ ||x(T) - x_d|| < \epsilon$$

which implies that

$$\forall x_d \in X, \exists u \in L^2(0,T,U) \ : \ ||x(t_N) - x_d|| < \epsilon$$

which implies that

$$\forall x_d \in X, \exists u \in L^2(0,T,U) \ : \ ||x_N - x_d|| < \epsilon$$

hence (19) is also weakly controllable on $[0, t_N]$, $\forall N$.

Since X is reflexive, then $T^*(\Delta)$ is generated by $A^* = A$, i.e. $T^*(\Delta) = T(\Delta)$, which gives $\phi^* = \phi$, and $\phi^i = \phi^{*\,i} = T(i\Delta)$.
Let's denote $T^\Delta_{N-j} = T((N-j)\Delta)$, then for any $x \in X$, it follows from equations (3) and (4) that

$$\begin{aligned} &HH^*x = H(H^*x) \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \phi^{N-j}B_{j-1}(\theta)B^*_{j-1}(\theta)\phi^{*\,N-j}x d\theta \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} T^\Delta_{N-j}T(t_j-\theta)BB^*T(t_j-\theta)T^\Delta_{N-j}x d\theta \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} W_j(\theta)BB^*W_j(\theta)x d\theta. \end{aligned}$$

where $W_j(\theta) = T((N-j)\Delta + t_j - \theta)$. On the other hand, the adjoint operator B^* of B is given by

$$\begin{array}{llll} B^*: & X & \to & \mathbb{R}^m \\ & x & \hookrightarrow & (< b_1, x >, \ldots, < b_m, x >). \end{array}$$

If we define

$$\begin{aligned} &\alpha(n,j,\theta) = e^{-n^2\pi^2[t_j - \theta + (N-j)\Delta]} \\ &\Phi^x_j =< x, \Phi_j > \ , \ x \in X \ , \ j \in \mathbb{N} \end{aligned}$$

then

$$\begin{aligned} &B^*T((N-j)\Delta + t_j - \theta)x \\ &= (\sum_{n=1}^{\infty}\alpha(n,j,\theta)\Phi^x_n\Phi^{b_1}_n, \ldots, \sum_{n=1}^{\infty}\alpha(n,j,\theta)\Phi^x_n\Phi^{b_m}_n) \end{aligned}$$

thus

$$BB^*W_j(\theta)x = \sum_{i=1}^{m}\sum_{n=1}^{\infty} e^{-n^2\pi^2[t_j-\theta+(N-j)\Delta]}\Phi^x_n\Phi^{b_i}_n b_i.$$

We have

$$W_j(\theta)BB^*W_j(\theta)x = \sum_{k=1}^{\infty} e^{-k^2\pi^2[t_j-\theta+(N-j)\Delta]} < BB^*W_j(\theta)x, \Phi_k > \Phi_k$$

hence

$$\begin{aligned} HH^*x &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\sum_{k=1}^{\infty}\alpha(k,j,\theta) < h_j(x), \Phi_k > \Phi_k d\theta \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\sum_{k=1}^{\infty}\alpha(k,j,\theta) g_j(x)\Phi_k d\theta. \end{aligned}$$

where

$$h_j(x) = \sum_{i=1}^{m}\sum_{n=1}^{\infty}\alpha(n,j,\theta)\Phi_n^x\Phi_n^{b_i}b_i$$
$$g_j(x) = \sum_{i=1}^{m}\sum_{n=1}^{\infty}\alpha(n,j,\theta)\Phi_n^x\Phi_n^{b_i}\Phi_k^{b_i}$$

Therefore

$$\begin{aligned} &< HH^*\Phi_r, \Phi_s > \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{tj}\alpha(s,j,\theta)\sum_{i=1}^{m}\sum_{n=1}^{\infty}\alpha(n,j,\theta)\Phi_n^{\Phi_r}\Phi_n^{b_i}\Phi_s^{b_i}d\theta \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{tj}\alpha(s,j,\theta)\sum_{i=1}^{m}\alpha(r,j,\theta)\Phi_r^{b_i}\Phi_s^{b_i}d\theta \\ &= (\sum_{j=1}^{N}\int_{t_{j-1}}^{tj}e^{-(s^2+r^2)\pi^2[t_j-\theta+(N-j)\Delta]}d\theta)\sum_{i=1}^{m}\Phi_r^{b_i}\Phi_s^{b_i}. \end{aligned}$$

Let $\gamma_{sr} = \sum_{j=1}^{N}\int_{t_{j-1}}^{tj}e^{-(s^2+r^2)\pi^2[t_j-\theta+(N-j)\Delta]}d\theta$, hence

$$\begin{aligned} \gamma_{sr} &= \sum_{j=1}^{N}\frac{e^{-(s^2+r^2)\pi^2(N-j)\Delta}}{(s^2+r^2)\pi^2}(1-e^{-(s^2+r^2)\pi^2\Delta}) \\ &= (1-e^{-(s^2+r^2)\pi^2\Delta})\tfrac{e^{-(s^2+r^2)\pi^2N\Delta}}{(s^2+r^2)\pi^2}\sum_{j=1}^{N}(e^{(s^2+r^2)\pi^2\Delta})^j \\ &= \frac{(e^{(s^2+r^2)\pi^2\Delta}-1)(e^{-(s^2+r^2)\pi^2N\Delta}-1)}{(s^2+r^2)\pi^2(1-e^{(s^2+r^2)\pi^2\Delta})} \\ &= \frac{1-e^{-(s^2+r^2)\pi^2N\Delta}}{(s^2+r^2)\pi^2}. \end{aligned}$$

It follows from Theorem 3.1 and Remark 6 that the optimal control can be approximated by $u_l = H^*f_l$ where $f_l = \sum_{i=1}^{l} z_i^l\Phi_i$ is the unique solution of the algebraic system

$$< HH^*f_l, \Phi_i > = < x_d - \phi^N x_0, \Phi_i > , \ \forall i = 1,\ldots,l,$$

or equivalently

$$A_l Z_l = X_d$$

where $Z_l = (z_1,\ldots,z_l)^t$, $X_d = (< x_d - \phi^N x_0, \Phi_1 >, \ldots, < x_d - \phi^N x_0, \Phi_l >)^t$ and A_l the matrix

$$\begin{aligned} A_l &= (< HH^*\Phi_s, \Phi_r >)_{1\leq s,r\leq l} \\ &= (\gamma_{sr}\sum_{i=1}^{m} < b_i, \Phi_r >< b_i, \Phi_s >)_{1\leq s,r\leq l}. \end{aligned}$$

On the other hand, from lemma 1, it follows that

$$\begin{aligned} u_l(\theta) &= B_j^*(\theta)(\phi^*)^{N-j}f_l \ , \qquad \forall\theta \in]t_{j-1}, t_j[\\ &= B^*T(t_j-\theta)T((N-j)\Delta)f_l \\ &= B^*T(t_j-\theta+(N-j)\Delta)f_l \\ &= B^*T(N\Delta-\theta)f_l \end{aligned}$$

for simplicity, if we take $m = 1$ then,

$$\begin{aligned} u_l(\theta) &= < b_1, T(N\Delta-\theta)f_l > \\ &= \sum_{n=1}^{\infty}e^{-n^2\pi^2(N\Delta-\theta)} < f_l, \Phi_n >< b_1, \Phi_n > \\ &= \sum_{n=1}^{l}e^{-n^2\pi^2(N\Delta-\theta)} < f_l, \Phi_n >< b_1, \Phi_n > . \end{aligned}$$

hence, the optimal control can be approximated by forall $\theta \in [0, T]$,

$$u_l(\theta) = \sum_{n=1}^{l}e^{-n^2\pi^2(N\Delta-\theta)} < f_l, \Phi_n >< b_1, \Phi_n > . \tag{20}$$

Numerical simulation : We take $m = 1$, $b_1(t) = t^2 + 1$, $N = 10$, $t_i = i\delta$, $\delta = 0.1$, $x_0 = 0$, then $t_N = 1$. To have x_d reachable, we take $x_d = Hu$ where $u(\theta) = 1$, $\forall\theta \in [0, 1]$, then $x_d = (< x_d, \Phi_i >)_{1\leq i\leq l}$ where $< x_d, \Phi_i > = \frac{< b_1, \Phi_i >}{i^2\pi^2}(1 - e^{-i^2\pi^2 N\delta})$.

An approximation of the optimal control is then given by figure 1.

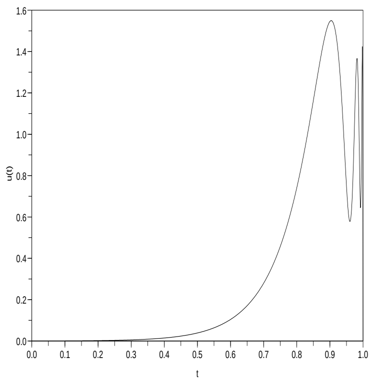

Figure 1: Approximation of the optimal control

5 Finite dimensional case

In this section we take $X = \mathbb{R}^n$ and $U = \mathbb{R}$. Since $Im\, H$ is finite dimensional, the weak controllability of (S) is equivalent to $Im\, H = X$, i.e., the exact controllability of (S). If (S) is controllable, then $Ker\, H^* = \{0\}$ and $\|.\|_F$ is a norm on X equivalent to $\|.\|$, so the completion of X with respect to $\|.\|_F$ is X, i.e., $F = X$.
On the other hand, since $\Lambda = HH^*$ and $Ker \wedge = Ker\, H^* = \{0\}$, then the controllability of (S) implies that Λ is an isomorphism on X.

Proposition 5.1 *If $B_i(\theta)$, $i = 0, \ldots, N-1$, are constant operators, say that $B_i(\theta) = B_i$, then*

$$KerH^* = Ker \begin{bmatrix} B^*_{N-1} \\ B^*_{N-2}\phi^* \\ \vdots \\ B^*_0(\phi^*)^{N-1} \end{bmatrix}$$

proof.
If $x \in Ker\, H^*$, then $H^*x = 0$. From (5) it follows that

$$\sum_{j=1}^{N} B^*_{j-1}(\phi^*)^{N-j}\mathcal{X}_{]t_{j-1},t_j[}(\theta)x = 0, \quad \forall\theta \in [0, T]$$

if we consider respectively $\theta \in]t_0, t_1[, \ldots, \theta \in]t_{N-1}, t_N[$, then

$$B^*_{j-1}(\phi^*)^{N-j}x = 0, \quad \forall j \in 1, 2, \ldots, N$$

if we take respectively j=1, j=2,...j=N, then we obtain

$$B^*_{N-1}x = 0,\ B^*_{N-2}\phi^*x = 0, \ldots, B^*_0(\phi^*)^{N-1}x = 0,$$

which means that

$$x \in Ker \begin{bmatrix} B^*_{N-1} \\ B^*_{N-2}\phi^* \\ \vdots \\ B^*_0(\phi^*)^{N-1} \end{bmatrix}. \tag{21}$$

Conversely, suppose (21), then

$$B^*_{N-1}x = B^*_{N-2}\phi^*x = \ldots = B^*_0(\phi^*)^{N-1}x = 0,$$

which implies that

$$\sum_{j=1}^{N} B^*_{j-1}(\phi^*)^{N-j}\mathcal{X}_{]t_{j-1},t_j[}(\theta)x = 0, \quad \forall\theta \in [0, T]$$

hence $x \in ker\, H^*$. ■

The operator Λ is given by

$$\begin{array}{rccc} \Lambda & : X & \rightarrow & X \\ & x & \mapsto & HH^*x \end{array}$$

from (3) it follows that

$$HH^*x = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \phi^{N-j}B_{j-1}(\theta)H^*x(\theta)d\theta$$

using (4) we deduce that

$$\begin{aligned} \Lambda x &= HH^*x \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \phi^{N-j}B_{j-1}(\theta)B^*_{j-1}(\theta)(\phi^\star)^{N-j}xd\theta. \end{aligned}$$

Finally, from theorem 3.1 we deduce the expression of the optimal control as follows.

Proposition 5.2 *The control $u^* \in L^2(0, T, \mathbb{R}^p)$ given by*

$$u^*(\theta) = B^*_{j-1}(\theta)(\phi^*)^{N-j}f, \quad \forall\theta \in]t_{j-1}, t_j[,\ j = 1, ..., N$$

where $f \in \mathbb{R}^n$ is the unique solution of the algebraic equation

$$\Lambda f = x_d - \phi^N x_0$$

steers the system from the initial state x_0 to the final state x_d at time N with a minimal cost $J(u) = \|u\|$.

6 Conclusion

In this paper, we have studied an optimal control problem for systems having discrete state variables and continuous-time control. We have shown that techniques similar to Hilbert Uniqueness Method can be used to resolve the problem. A numerical approach of the solution have been also developped.

References

[1] H. Bresis , Analyse fonctionnelle : Masson, 1987.

[2] P.G. Ciarlet, Introduction à l'analyse numérique matricielle et à l'optimisation : Masson, 1988.

[3] L. Chraibi, J. Karrakchou, A. Ouansafi and M. Rachik, "Exact controllability and optimal control for distributed systems with a discrete delayed control", Journal of the Franklin institute, Vol. 337, Issue 5, 2000, pp. 499-514.

[4] R.F. Curtain and A.J. Pritchard, Infinite Dimensional Linear Systems Theory : Springer-Verlag Berlin Heidelberg, 1978.

[5] R.G. Faradzhev, Phat Vu Ngoc, and A.V. Shapiro, "Controllability Theory of Discrete Dynamic systems", J.Avtomatika i Telemekhanika, No. 1, January 1986, pp. 5-24.

[6] J. Karrakchou and M. Rachik, "Optimal control of discrete distributed systems with delays in the control: the finite horizon case", Archives of Control Sciences, Vol. 4(XL), No. 1-2, 1995, pp. 37-53.

[7] J. Karrakchou, R. Rabah and M. Rachik, "Optimal control of discrete distributed systems with delays in state and control: State space theory and HUM approaches", SAMS journal, Vol. 30, 1998, pp. 225-245.

[8] K.Y. Lee, S. Chow and R.O. Barr, "On the control of discrete-time distributed parameter systems", SIAM J. Cont., 10, 1972, pp. 361-376.

[9] J.L. Lions, Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles, Paris : Dunod et Gauthiers-Villars, 1968.

[10] J.L. Lions, Contrôlabilité exacte, perturbations et stabilisation des syttèmes distribués, I: Contrôlabilité exacte, II: Perturbations, III: Stabilisation, Paris : RMA 9, Masson, 1988.

[11] V.A. Lun'kov, "Controllability and Stabilization of linear discrete Systems", Differents. Uravn., 16, No. 4, 1980, pp.753-754.

[12] k. Ogata, Discrete-time Control systems : Printice Hall, International Editions, 1995.

[13] R. Rabah and M. Malabre, "Structure at Infinity for Linear Infinite Dimensional Systems", Internal Raport, Institut de recherche Cibernétique de Nantes, March 12, 1999.

[14] M. Rachik, "Quelques éléments sur l'analyse et le contrôle des systèmes distribués", Thèse de doctorat d'état, Université Mohamed V, EMI-Rabat, Morocco, 1995.

[15] M. Rachik, M. Lhous and A. Tridane, "Controllability and optimal control problem for linear time-varying discrete distributed systems", Systems Analysis Modelling Simulation. Vol. 43, N° 2. February 2003, pp. 137-164.

[16] L. Weiss "Controllability, realisation and stability of discrete-time systems", SIAM J. Contr., 10, No 2, 1972, pp. 230-262.