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ON STANDARD FORMS OF 1-DOMINATIONS BETWEEN KNOTS WITH SAME GROMOV VOLUMES

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Let k and k' be two knots in the 3-sphere. Say k 1-dominates k', if there is a proper degree 1 map $f\colon E(k) \to E(k')$.

Theorem: Suppose that any companion of k is prime. If k 1-dominates k' with the same Gromov volume, then k' can be obtained from k by finitely many de-satellizations.

The condition of "same Gromov volume" clearly cannot be removed. We also give a new construction of 1-domination between knots with the same Gromov volume to show that the condition "any companion of k is prime" cannot be removed.

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1. Introduction and Notations

All knots are in the 3-sphere S^3. For basic terminologies in knot theory and in 3–manifold theory, we refer the reader to [12], [7] and [8]. Two knots k_1 and k_2 in the 3-sphere are equivalent if there is a homeomorphism of S^3 sending k_1 to k_2.

We recall the following relation on the set of knots in S^3: let k_1 and k_2 be two knots, say $k_1 \geq k_2$, or equivalently say that k_1 1-dominates k_2, if there is a proper degree one map $f\colon E(k_1) \to E(k_2)$, where $E(k_i)$ is the exterior of the knot k_i. If $k_1 \geq k_2$ but $k_1 \neq k_2$, we often write $k_1 > k_2$, or equivalently say that the 1-domination is non-trivial.

Following the classical results of [20] and [5], it is known that the relation $\geq$ is a partial order on the set of equivalence classes of knots in S^3.

In general, when $k_1 \geq k_2$, the relation between k_1 and k_2 is not known, and there is no fine description of the degree 1 map, up to homotopy, realizing the 1-domination $k_1 \geq k_2$. A simple and common construction of 1-domination $k_1 \geq k_2$ is to build k_1 by a satellization of k_2 around some knot k, then a proper map f realizing a 1-domination is given by a de-satellization which consists in pinching the exterior of the knot k to a solid torus. On the other hand there are many other sophisticated constructions, see [10], [14], [2], [3], [1], [11].

The main result of this paper is to show that under certain conditions a 1-domination between two knot exteriors corresponds to finitely many de-satellizations.

Given a knot k we define its *Gromov volume* as the Gromov volume $||E(k)||$ of its exterior $E(k)$: up to some constant it is equal to the sum of the volume of the hyperbolic pieces of the JSJ-decomposition of $E(k)$, see [6], [17].

Theorem 1.1. *Suppose that any companion of k is prime. If k 1-dominates a knot k' with the same Gromov volume, then k' can be obtained from k by finitely many de-satellizations.*

The condition that the knots k and k' have "the same Gromov volume" clearly cannot be removed, according to the constructions in the papers mentioned above. We will also give a new construction of 1-domination between knots with same Gromov volume to show that the condition "any companion of k is prime" cannot be removed.

The corollary below supports a general opinion that the 1-domination partial order reflects the complexity of knots (see the survey [21]). By results of Schubert [15], see also [16], the bridge number strictly decreases by de-satellization. This follows from the fact that one can isotope any companion solid torus V_0 of a satellite knot k to be taut (in the sense of [16]) with respect to any minimal bridge presentation of k. Hence the bridge number of k is equal to $wb(k_0) + \beta(V, k')$, where w is the wrapping number of k in the solid torus V_0, $b(k_0)$ is the bridge number of the companion knot k_0 and $\beta(V, k')$ is the number of maxima which do not correspond to maxima of k_0 (i.e. saddles of ∂V_0).

Since k_0 is knotted, $b(k) > 2w + \beta(V, k') \geq b(k')$. So we obtain the following corollary:

Corollary 1.1. *Suppose that any companion of k is prime. If $k > k'$ with the same Gromov volume, then $b(k) > b(k')$, where b is the bridge number.*

The paper is organized as follows. After listing some known useful facts, a general study of maps between Seifert pieces and graph pieces in knot complements is given in §2, Theorem 1.1 will be proved in §3, and the new construction of 1-domination will be given in §4. Below we will fix some notions for the remaining sections.

Notation 1.1. On the boundary of each solid torus in S^3, we specify its longitude to be the one which is homologous to zero in the complement. Let k_1 be a geometrically essential knot [12, p110] in an unknotted solid torus $V \subset S^3$ and let k be another knot. Let $h\colon V \to N(k)$ be a longitude preserving homeomorphism, then the new knot $k_2 = h(k_1)$ is called a satellite knot of k, and k is a companion of k_2.

The reversing process of satellization, given by pinching $E(k)$, the exterior of the companion to a solid torus, produces a proper degree one map $f\colon E(k_2) \to E(k_1)$, which will be called a *de-satellization*. Hence $K_2 \geq K_1$.

Notation 1.2. Let $T(p_1, q_1; p_2, q_2; \dots; p_n, q_n)$ be the *iterated torus knot*, which is the (p_1, q_1)–cable of the (p_2, q_2)–cable of ... the (p_n, q_n)–torus knot. (When we say "(p, q)–cable", p denotes the winding number.) The exterior of this knot is denoted by $E = E(p_1, q_1; p_2, q_2; \dots; p_n, q_n)$.

Let $C = C(p_1, q_1; p_2, q_2; \dots; p_n, q_n)$ denote the *iterated cable space*, that is the exterior E of $T(p_1, q_1; p_2, q_2; \dots; p_n, q_n)$ with an open neighborhood of the singular fiber corresponding to (p_n, q_n) removed.

The space E is a graph manifold whose Seifert pieces are the cable spaces $C(p_1, q_1)$, ...,$C(p_{n-1}, q_{n-1})$ and the torus knot exterior $E(p_n, q_n)$. The boundary $\partial E = T_0$, and the JSJ tori of E are denoted by $T_1, \dots, T_{n-1}$, where $\partial C(p_i, q_i) = T_{i-1} \sqcup T_i$.

The iterated cable space C is also a graph manifold whose Seifert pieces are the cable spaces $C(p_1, q_1)$,...,$C(p_n, q_n)$. The boundary $\partial C = T_0 \sqcup T_n$, and the JSJ tori are denoted by $T_1, \dots, T_{n-1}$, where $\partial C(p_i, q_i) = T_{i-1} \sqcup T_i$. Suppose α is a slope on T_n, then $C(\alpha) = C(p_1, q_1; \dots; p_n, q_n; \alpha)$ denotes the manifold obtained from C by Dehn filling along α.

The spaces E and C are submanifolds of S^3 and each torus T_i bounds a solid torus K_i in S^3. We denote by $\mu_i \subset T_i$ the meridian of K_i, and by $\lambda_i \subset T_i$ the longitude of K_i.

Notation 1.3. Suppose F (resp. P) is a properly embedded $(n-1)$-manifold (resp. an embedded n-manifold) in a n-manifold M. We use $M \setminus F$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along F (resp. removing int$P$, the interior of P).

Notation 1.4. Let D_0 be a disc and $D_1, ..., D_n$ be sub-discs in the interior of D_0. For $i = 0, 1, \dots, n$ denote ∂D_i by c_i, and the n-punctured disc $D_0 \setminus \cup_{i=1}^n D_i$ by P_n. Then $\partial P_n = \cup_{i=0}^n c_i$. Note that P_1 is an annulus. Once D_0 is oriented, then P_n and all the curves c_i are oriented.

For $n \geq 2$, we call *n-composite space* a manifold homeomorphic to the product $S^1 \times P_n$. The property that any companion of the knot k is prime is equivalent to the fact that no JSJ piece of $E(k)$ is a composite space.

Notation 1.5. Let $f\colon M \to N$ be a map between two orientable compact connected n–manifolds. We say that f is *proper* if $f^{-1}(\partial N) = \partial M$. We say that f is *allowable* if f is proper and the degree of all possible restrictions $f|\colon F \to S$ have the same sign, where F is a component of ∂M and S is a component of ∂N.

2. Proper Maps Between Seifert Pieces and Graph Pieces in Knot Complements

The following four known facts, see [4], [8], [13] and [18] respectively, will be repeatly used in this paper.

Lemma 2.1. *[4] In* $\mathrm{H}_1(C(p_i, q_i); \mathbb{Z})$*, the following relations holds:*

$$p_i[\mu_{i-1}] = [\mu_i], \quad [\lambda_{i-1}] = p_i[\lambda_i].$$

Moreover the regular Seifert fiber of $C(p_i, q_i)$ *is homologous to* $p_i q_i[\mu_{i-1}] + [\lambda_{i-1}]$ *on* T_{i-1}*, and homologous to* $q_i[\mu_i] + p_i[\lambda_i]$ *on* T_i*.*

Lemma 2.2. *[8] Let* P *be a Seifert piece of the JSJ decomposition of* $E(k)$*. Then* P *is either a torus knot exterior* $E(p, q)$*, or a cable space* $C(p, q)$*, or a composite space* $P_m \times S^1$*. Moreover a composite space* $P_m \times S^1, m > 1$ *appears if and only if some companion of* k *is not prime.*

Lemma 2.3. *[13] Let* $f\colon M \to N$ *be an allowable degree 1 map between aspherical Seifert manifolds. Then* f *is homotopic to a fiber preserving pinch.*

For a compact orientable 3-manifold M with boundary a collection of tori, $H(M)$ denotes the disjoint union of the hyperbolic JSJ pieces of M, and $||M|| = ||H(M)||$ is the Gromov volume of M.

Lemma 2.4. *[18] Let* $f\colon M \to N$ *be a proper map of degree* d *between two Haken 3-manifolds. If the relation* $||M|| = d||N||$ *holds between their Gromov volumes, then* f *can be homotoped to send the hyperbolic pieces* $H(M)$ *to the hyperbolic pieces* $H(N)$ *by a covering map.*

Below we prove some general results about maps between Seifert pieces and graph pieces in knot complements.

Lemma 2.5. *Any proper degree 1 map* $f : E(p, q) \to E(p', q')$ *between torus knot complements is homotopic to a homeomorphism.*

Proof. The lemma follows from the fact that torus knots are minimal (see [2]). It is also a direct corollary of [13]: each manifold involved has only one boundary component, hence f is an allowable degree 1 map. Since each Seifert manifold involved has a unique Seifert fibration, then by [13], f is homotopic to a fiber preserving pinch. Any non-trivial pinch will decrease either the genus of the base, or the number of singular fibers, and since both the genus of the base and the number of singular fibers of $E(p, q)$ and $E(p', q')$ are the same, the pinch must be trivial, therefore the lemma is verified. □

Lemma 2.6. *Suppose M is a Seifert manifold with a π_1–injective boundary component T and $f\colon C(p_1,q_1) \to M$ is a proper map such that $f|\colon T_0 \to T$ is a homeomorphism. Let $t_1 \in \pi_1(C(p_1,q_1))$ and $t \in \pi_1(M)$ represent regular fibers of the corresponding Seifert manifolds. Then the following statements hold.*

(1) $f_*(\pi_1(C(p_1,q_1)))$ is not an abelian group;

(2) $f_*(t_1) = t^{\pm 1}$ if M has a unique Seifert fibration.

Proof. Pick a base point of $C(p_1,q_1)$ in T_0, and a base point of M in T. Then $\pi_1(T_0)$ is naturally a subgroup of $\pi_1(C(p_1,q_1))$, and $\pi_1(T)$ is naturally a subgroup of $\pi_1(M)$.

Assume $f_*(\pi_1(C(p_1,q_1)))$ is an abelian group. Since $f|T_0$ is a homeomorphism, and $\pi_1(T)$ is a maximal abelian subgroup of $\pi_1(M)$, see [9, Chap. VI], $f_*(\pi_1(C(p_1,q_1)))$ must be equal to $\pi_1(T)$. Moreover,

$$f_*\colon \pi_1(C(p_1,q_1)) \to \pi_1(T)$$

factors through $\mathrm{H}_1(C(p_1,q_1);\mathbb{Z})$. The longitutes λ_0, λ_1 represent elements in $\pi_1(C(p_1,q_1))$ and λ_0 is a primitive element in $\pi_1(T_0)$, hence $f_*(\lambda_0)$ is a primitive element in $\pi_1(T)$. Since λ_0 is the p_1–multiple of λ_1 in $\mathrm{H}_1(C(p_1,q_1))$ with $p_1 > 1$, we get a contradicition.

Since t_1 commutes with $\pi_1(C(p_1,q_1))$, and $f_*(\pi_1(C(p_1,q_1)))$ is non-abelian, $f_*(t_1)$ must be a power of t. Moreover $f_*(t_1) = t^{\pm 1}$ because $f\colon T_0 \to T$ is a homeomorphism. □

Lemma 2.7. *Let*

$$f\colon C(\alpha) = C(p_1,q_1;\ldots;p_n,q_n;\alpha) \to E(p,q)$$

be a proper map such that the restriction of f to T_0 is a homeomorphism. Then the restriction of f to T_1 is not π_1–injective.

Proof. Pick a basepoint b of $C(\alpha)$, $b \in T_0$, choose a simple curve γ connecting b to T_{n-1}, such that $\gamma \cap T_i$ consists of a single point. Let $\gamma \cap T_i$ be the base point in T_i and $E(p_{i+1}, q_{i+1})$. Using a path on γ, we can view $\pi_1(T_i)$ and $\pi_1(E(p_{i+1}, q_{i+1}))$ as subgroups of $\pi_1(C(\alpha))$. Let $f_*\colon \pi_1(C(\alpha)) \to \pi_1(E(p,q))$ be the induced homomorphism on the fundamental groups.Let $T_0' = \partial E(p,q)$.

Let $t_i \subset \pi_1(C(p_i,q_i))$ and $t \subset \pi_1(E(p,q))$ represent the regular Seifert fibers in the corresponding Seifert manifolds. By Lemma 2.6, we can assume $f_*(t_1) = t$.

If $n = 1$, then the conclusion trivially holds (since α is in the kernel), so we may assume $n > 1$. The element t_1 is contained in $\pi_1(T_1)$. In fact, t_1 is homologous to $q_1[\mu_1] + p_1[\lambda_1]$ in T_1. Let x denote $f_*(\mu_1)$. Looking for a contradiction, we assume that the restriction of f on T_1 is π_1–injective. Then x, t generate a $\mathbb{Z} \oplus \mathbb{Z}$–subgroup of $\pi_1(E(p,q))$.

The fiber t_2 is homologous to $p_2 q_2[\mu_1] + [\lambda_1]$ on T_1, hence it is not a power of t_1 in $\pi_1(T_1)$. So $f_*(t_2)$ is not a power of t in $\pi_1(E(p,q))$. But t_2 commutes with

$\pi_1(C(p_2,q_2))$, so $f_*(\pi_1(C(p_2,q_2)))$ is in the centralizer of $f_*(t_2)$. Since $f_*(t_2)$ is not a power of the fiber t in the knot group $\pi_1(E(p,q))$, its centralizer must be an abelian group, see [9, Chap. VI]. Hence

$$f_*\colon\ \pi_1(C(p_2,q_2)) \to \pi_1(E(p,q))$$

factors through $\mathrm{H}_1(C(p_2,q_2);\mathbb{Z})$.

In $C(p_2,q_2)$, $p_2(q_1[\mu_1]+p_1[\lambda_1])$ is homologous to $q_1[\mu_2]+p_1p_2^2[\lambda_2]$, hence the corresponding element in $\pi_1(T_2)$ is mapped by f_* to t^{p_2}. By the same reason, $f_*(\mu_2)=x^{p_2}$. So $f|T_2$ is π_1–injective. Moreover, t_3 is homologous to $p_3q_3[\mu_2]+[\lambda_2]$ in T_2, it is linearly independent with $q_1[\mu_2]+p_1p_2^2[\lambda_2]$, since $\gcd(p_1,q_1)=1$. Hence $f_*(t_3)$ is not a power of t in $\pi_1(E(p,q))$. But t_3 commutes with $\pi_1(C(p_3,q_3))$, so $f_*(\pi_1(C(p_3,q_3)))$ is an abelian group.

By the same arguments as above, we find that $f_*(\mu_3)=x^{p_2p_3}$, and the loop corresponding to $q_1[\mu_3]+p_1p_2^2p_3^2[\lambda_3]$ on T_3 is mapped to $t^{p_2p_3}$ by f_*. Hence $f|T_3$ is π_1–injective, and $f_*(t_4)$ is not a power of t in $\pi_1(E(p,q))$.

Go on with such arguments, we finally show that $f|T_{n-1}$ is π_1–injective, and $f_*(t_n)$ is not a power of t, where t_n represents the regular fiber of $C(p_n,q_n)$. Thus $f_*(\pi_1(C(p_n,q_n)))$ is an abelian group, and therefore the group $f_*(\pi_1(C(p_n,q_n;\alpha)))$ is also abelian. Then $f_*|\pi_1(C(p_n,q_n;\alpha))$ factors through $\mathrm{H}_1(C(p_n,q_n,\alpha);\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}_b$ for some positive integer b, which contradicts the fact that $f|T_{n-1}$ is π_1–injective. □

Lemma 2.8. *Let $C(p,q)$ be a cable-space with $\partial C(p,q)=T_0'\sqcup T_1'$, where T_0' corresponds to the boundary of an open neighborhood of the singular fiber of type (p,q) removed from the torus knot exterior $E(p,q)$. Let*

$$f\colon\ C(p_1,q_1;\ldots;p_n,q_n)\to C(p,q)$$

be a proper map.

(1) If $n>1$, then f cannot map T_0 homeomorphically to T_0'.

(2) If $n=1$, and f maps T_0 homeomorphically to T_0', then f is homotopic to a homeomorphism.

Proof. Assume f maps T_0 homeomorphically to T_0'. We claim that $f(T_n)=T_1'$. Otherwise $f(T_n)=T_0'$. Let $f_\#$ be the induced map on homology, then $f_\#([\lambda_n])$ is an integral linear combination of $f_\#([\mu_0])$ and $f_\#([\lambda_0])$. Since $[\lambda_n]$ is equal to $\frac{1}{p}[\lambda_0]$, where $p=p_1p_2\ldots p_n$, we get a contradiction.

Now $f(T_n)=T_1'$. Since $f|T_0$ is a homeomorphism, $\deg f=\deg f|T_0=1=\deg f|T_n$. Thus we can homotope f so that $f|T_n$ is a homeomorphism. Moreover:

$$f_*\colon\ \pi_1(C(p_1,q_1;\ldots;p_n,q_n))\to\pi_1(C(p,q))$$

is an epimorphism. Hence $f_*(\pi_1(C(p_1,q_1;\ldots;p_n,q_n))$ is not abelian.

$$f_{\#}\colon \mathrm{H}_1(C(p_1, q_1; \dots; p_n, q_n); \mathbb{Z}) \to \mathrm{H}_1(C(p, q); \mathbb{Z})$$

is an isomorphism.

By the proof of Lemma 2.6, we can assume that $f_*(t_1) = f_*(t_n) = t$ in $\pi_1(C(p, q)$. In $\mathrm{H}_1(C(p_1, q_1; \dots; p_n, q_n); \mathbb{Z})$, we have

$$[t_1] = p_1 q_1[\mu_0] + [\lambda_0] = p_1 q_1[\mu_0] + p[\lambda_n],$$

$$\text{and } [t_n] = q_n[\mu_n] + p_n[\lambda_n] = q_n p[\mu_0] + p_n[\lambda_n].$$

Since $f_{\#}$ is an isomorphism and $[\mu_0], [\lambda_n]$ generate $\mathrm{H}_1(C(p_1, q_1; \dots; p_n, q_n); \mathbb{Z})$, we must have $p_1 q_1 = q_n p$ and $p = p_n$.

If $n > 1$, it is impossible since $p_1 > 1$.

If $n = 1$, then we have a proper allowable degree map $f\colon C(p_1, q_1) \to C(p, q)$. Applying Rong's result as in the proof of Lemma 2.5, one shows that f is homotopic to a homeomorphism. □

Lemma 2.9. *Let M be either $E(p_1, q_1; \dots; p_n, q_n)$ or $C(p_1, q_1; \dots; p_n, q_n)$, and let P_m denotes the punctured disk with $m \geq 1$ holes. Then there is no proper map $f\colon M \to P_m \times S^1$ such that the restriction of f to a component of ∂M is a homeomorphism.*

Proof. Assume f maps T_0 homeomorphically to T_0', a component of $\partial P_m \times S^1$.

If M is a knot space, then $[T_0'] = f_{\#}([T_0])$ is null homologous in $P_m \times S^1$, which implies $m = 0$, a contradiction.

Now suppose that M is an iterated cable space with boundary T_0 and T_n. Since $[\lambda_0] = p[\lambda_n]$ in $\mathrm{H}_1(M; \mathbb{Z})$, where $p = p_1 ... p_n > 1$, we have $f_{\#}([\lambda_0]) = pf_{\#}([\lambda_n])$ in $\mathrm{H}_1(P_m \times S^1; \mathbb{Z}) = \mathbb{Z}^{m+1}$. There are two subcases:

(a) $f_{\#}([T_n]) = k[T_0']$, $k \in \mathbb{Z}$;

(b) $f_{\#}([T_n]) = k[T_1']$, $k \in \mathbb{Z}$, $T_1' \neq T_0'$, T_1' is a component of $\partial P_m \times S^1$.

In the subcase (a), since $[T_0] + [T_n] = 0$, we have $(k+1)[T_0'] = 0$, which implies that $k = -1$. Now both $f_{\#}([\lambda_0])$ and $f_{\#}([\lambda_n])$ are homologous to closed curves on T_0', and in particular $f_{\#}([\lambda_0])$ is a primitive element in $\mathrm{H}_1(T_0'; \mathbb{Z}) = \mathbb{Z}^2$. Note that $f_{\#}([\lambda_0]) = pf_{\#}([\lambda_n])$ in $\mathrm{H}_1(P_m \times S^1; \mathbb{Z})$, and the homomorphism $\mathrm{H}_1(T_0'; \mathbb{Z}) \to \mathrm{H}_1(P_m \times S^1; \mathbb{Z})$ induced by the inclusion is injective, so $f_{\#}([\lambda_0]) = pf_{\#}([\lambda_n])$ in $\mathrm{H}_1(T_0'; \mathbb{Z})$, which is impossible since $f_{\#}([\lambda_0])$ is primitive.

In the subcase (b), since $[T_0] + [T_n] = 0$, we have $[T_0'] + [T_1'] = 0$, which is impossible if $m > 1$. If $m = 1$, then $P_1 \times S^1 = T_0' \times [0, 1]$, and the homomorphism $\mathrm{H}_1(T_0'; \mathbb{Z}) \to \mathrm{H}_1(T_0' \times [0, 1]; \mathbb{Z})$ induced by the inclusion is an isomorphism. Again $f_{\#}([\lambda_0]) = pf_{\#}([\lambda_n])$ in $\mathrm{H}_1(P_1 \times S^1; \mathbb{Z})$, which is impossible.

In either case we reach a contradiction. □

3. Proof of Theorem 1.1

The dual graph $\Gamma(k)$ to the JSJ decomposition of $E(k)$ is a rooted tree, where the root is corresponding to the unique vertex manifold containing $\partial E(k)$. Let $\Gamma_0 \subset \Gamma(k)$ be the maximal connected subtree which contains the root such that the restriction of f, up to homotopy, to the connected submanifold $M(\Gamma_0) \subset E(k)$ associated to Γ_0 is a homeomorphism onto its image. Moreover the restriction of f to each leaf torus of Γ_0 is π_1-injective.

Since k and k' have the same Gromov volume, by [18], f can be homotoped so that f maps the hyperbolic pieces of $E(k)$ homeomorphically to the hyperbolic pieces of $E(k')$.

If $f\colon E(k) \to E(k')$ is homotopic to a homeomorphism, then Theorem 1.1 is automatically true. So from now we assume that f is not homotopic to a homeomorphism. Thus $M(\Gamma_0) \neq E(k)$.

Let T_0 be the torus corresponding to a leaf of Γ_0, and let X_0 be the JSJ piece of $E(k) \setminus M(\Gamma_0)$ adjacent to T_0. Then by [18] X_0 must be a Seifert piece. Since $f|T_0$ is π_1–injective, $f|X_0$ is non-degenerate, and it follows that we can push $f(X_0)$ into a Seifert piece X'_0 of the JSJ decomposition of $E(k')$. Let $T'_0 = f(T_0)$, then $f| : T_0 \to T'_0$ is a homeomorphism. By the definition of Γ_0, we have a JSJ piece $X \neq X_0$ of $E(k)$ adjacent to T_0 such that $f|\colon X \to X'$ is a homeomorphism, where X' is a JSJ piece of $E(k')$ adjacent to T'_0. Hence $T'_0 \subset \partial X'_0$.

Let U be the maximal connected graph submanifold of $E(k)$ such that $X_0 \subset U$ and $T_0 \subset \partial U$. Since we assume that any companion of k is prime, then U is homeomorphic to either an iterated torus knot exterior $E(p_1, q_1; \ldots; p_n, q_n)$ or an iterated cable knot exterior $C(p_1, q_1; \ldots; p_n, q_n)$.

Lemma 3.1. *U cannot be a torus knot exterior $E(p_1, q_1)$. Hence $T_1 \neq \emptyset$.*

Proof. Otherwise we have $U = X_0 = E(p_1, q_1)$ and then $f(T_0) = T'_0$ is homologous to zero in X'_0, which implies $\partial X'_0 = T'_0$ and therefore $X'_0 = E(p', q')$. Then we have map $f| : E(p, q) \to E(p', q')$ which is degree 1 on the boundary, and therefore degree 1 itself. By Lemma 2.5, $f|$ is homotopic to a homeomorphism, and therefore contradicts the maximality of Γ_0. □

According to the notation 1.2, starting from T_0, we name the JSJ tori in U $T_1, T_2, ..., T_n$ following the order on the JSJ segment.

Lemma 3.2. *$f|T_i$ is not π_1–injective for some i.*

Proof. Otherwise the restriction of f to any Seifert piece in U is non-degenerate. By homotoping f, we can assume that $f^{-1}(X'_0) \cap U$ is the union of some Seifert pieces in $E(k)$.

Let G be a component of $f^{-1}(X'_0)$ containing X_0. Then G is either $E(p_1, q_1; \ldots; p_l, q_l)$ or $C(p_1, q_1; \ldots; p_l, q_l)$.

Claim 1. $X'_0 = E(p', q')$, and $X' \neq X'_0$.

Proof. By Lemma 2.9, X'_0 is not $P_m \times S^1$, $m \geq 1$. Hence either $X'_0 = C(p', q')$ is a cable space or $X'_0 = E(p', q')$ is a torus knot exterior.

Suppose first that $X'_0 = C(p', q')$. A simple homological reason shows that G cannot be $E(p_1, q_1; \dots; p_l, q_l)$. By Lemma 2.8, G cannot be $C(p_1, q_1; \dots; p_l, q_l)$, $l > 1$; moreover if $G = C(p_1, q_1)$, then $f|\colon C(p_1, q_1) \to C(p', q')$ is homotopic to a homeomorphism, which contradicts the maximality of Γ_0.

Hence $X'_0 = E(p', q')$. Since X', which is homeomorphic to X, has at least two boundary components, we have $X'_0 \neq X'$. □

Claim 2. $f^{-1}(X'_0) \cap U = U$.

Proof. Let S' be a Seifert surface of $E(p', q')$. Since $f|\colon T_0 \to T'_0$ is a homeomorphism, up to a homotopy relative to T_0, we may assume that $f^{-1}(S')$ is incompressible, and that there is only one component of $f^{-1}(S')$, denoted by S, with ∂S a simple loop c on T_0. Since $f(X) = X'$ and $X' \neq X'_0$, it follows that $f^{-1}(S') \cap intX = \emptyset$. Since T_0 is separating and S is connected, one must have $S \subset E(k_0)$. Hence $c = \lambda_0$, where $E(k_0)$ is the companion exterior bounded by T_0 and containing U. Since the winding number of each JSJ torus T_i in U is non-zero with respect to T_0, we have $S \cap T_i \neq \emptyset$ for each i, and it follows that $f^{-1}(X'_0) \cap U = U$. □

Claim 3. $U = E(p_1, q_1; \dots; p_n, q_n)$.

Proof. Let us assume that $U = C(p_1, q_1; \dots; p_n, q_n)$. Let Y be the JSJ piece of $E(k) \setminus U$ adjacent to T_n. By the definition of U, Y must be a hyperbolic piece, so $f|Y$ must be a homeomorphism. Since $f(T_n) \subset T'_0$, we must have $f(Y) \subset X'$ which implies that X' is a hyperbolic piece. Since $f\colon X \to X'$ is a homeomorphism by our assumption, it follows that X is a hyperbolic piece. Therefore f send two different hyperbolic JSJ pieces of $E(k)$ to a single hyperbolic JSJ piece of $E(k')$. This contradicts the fact that the restriction of f to the hyperbolic part is a homeomorphism. □

Now we have a proper map $f\colon E(p_1, q_1; \dots; p_n, q_n) \to E(p', q')$ which is a homeomorphism on the boundary. By Lemma 2.7, its restriction $f|T_1$ is not π_1–injective, which contradicts the assumption that we made at the beginning of the proof .

This finishes the proof of Lemma 3.2. □

Lemma 3.3. *$f|T_1$ is not π_1–injective.*

Proof. By Lemma 3.2, some $f|T_i$ is not π_1–injective for some T_i in U. We may assume that the restriction of f is π_1–injective on T_i for $i < k$ and that the restriction of f is not π_1–injective on T_k. We have $f(C(p_1, q_1; ...; p_k, q_k)) \subset X'_0$. Since $f|T_k$ is not π_1–injective, there is a simple loop $\alpha \subset T_k$ in the kernel of f_*. Therefore f

extends to a proper map $\bar{f}\colon C(p_1, q_1; ...; p_k, q_k; \alpha) \to X'_0$ such that $\bar{f}|T_0$ is a homeomorphism. Then a homological argument shows that $X'_0 = E(p, q)$. By Lemma 2.7 $\bar{f}|T_1 = f|T_1$ is not π_1–injective. □

Proof of Theorem 1.1. Let $V = M(\Gamma_0)$, $V' = f(V)$. Then $f|\colon V \to V'$ is a homeomorphism. Denote the knot complement separated by T_i in $E(k)$ by $E(k_i)$, $i = 0, 1$ and $W = E(k) \setminus E(k_0)$. Then we have $E(k_0) = C(p_1, q_1) \cup_{T_1} E(k_1)$ and there is a proper degree one map

$$f\colon E(k) = W \cup_{T_0} C(p_1, q_1) \cup_{T_1} E(k_1) \to E(k')$$

such that $f(C(p_1, q_1)) \subset X'_0$, $f|\colon T_0 \to T'_0$ is a homeomorphism, and a simple closed curve $\alpha \subset T_1$ lies in the kernel of $f|T_1$. Then the proper degree one map $f\colon E(k) \to E(k')$ induces a factorization

$$E(k) \longrightarrow W \cup_{T_0} C(p_1, q_1; \alpha) \cup_{\alpha^*} E(k_1, \alpha) \xrightarrow{\hat{f}} E(k'). \tag{1}$$

Here $C(p_1, q_1; \alpha)$ and $E(k_1, \alpha)$ are 3–manifolds obtained by Dehn filling $C(p_1, q_1)$ and $E(k_1)$ respectively along the simple curve $\alpha \subset T_1$, and $C(p_1, q_1; \alpha) \cup_{\alpha^*} E(k_1, \alpha)$ is obtained by identifying the core α^* of the filling solid tori in $C(p_1, q_1; \alpha)$ and $E(k_1, \alpha)$.

Since $E(k_1, \alpha)$ is a closed 3–manifold, it makes no contribution to the degree of the proper degree one map f and thus

$$\hat{f}|\colon W \cup_{T_0} C(p_1, q_1; \alpha) \to E(k') \tag{2}$$

is a proper degree one map.

Since

$$||E(k')|| = ||E(k)|| \geq ||W \cup_{T_0} C(p_1, q_1)|| \geq ||W \cup_{T_0} C(p_1, q_1; \alpha)|| \geq ||E(k')||,$$

we have $||E(k)|| = ||W \cup_{T_0} C(p_1, q_1)|| = ||W \cup_{T_0} C(p_1, q_1)|| + ||E(k_1)||$. Therefore $||E(k_1)|| = 0$ and $E(k_1)$ is an iterated torus knot exterior $E(p_2, q_2; ...; p_n, q_n)$, since it is a graph manifold in S^3 without any composite space as JSJ piece. It follows that

$$C(p_1, q_1) \cup_{T_1} E(k_1) = C(p_1, q_1) \cup_{T_1} E(p_2, q_2; ...; p_n, q_n) \tag{3}$$

and thus $C(p_1, q_1) \cup_{T_1} E(p_2, q_2; ...; p_n, q_n) = E(p_1, q_1; ...; p_n, q_n)$.

Moreover $\hat{f}(C(p_1, q_1; \alpha)) \subset X'_0$ and $f|\colon T_0 \to T'_0$ is a homeomorphism, hence it follows that $X'_0 = E(p', q')$ and

$$\hat{f}|\colon C(p_1, q_1; \alpha) \to E(p', q') \tag{4}$$

is homotopic to a homeomorphism. Finally we have

$$f\colon W \cup_{T_0} C(p_1, q_1) \cup_{T_1} E(p_2, q_2; ...; p_n, q_n) \to E(k') = W' \cup_{T'_0} E(p', q'). \tag{5}$$

Let S' be a Seifert surface of $E(p', q')$, then up to a homotopy relative to T_0, we may assume that $f^{-1}(S')$ is incompressible, and that there is only one component of $f^{-1}(S')$, denoted by S, with ∂S a simple closed curve on T_0 . Let X be a JSJ piece of $E(k)$ adjacent to X_0 along T_0, and let X' be a JSJ piece of $E(k')$ adjacent to X'_0 along T'_0. By our choice of T_0, $f|X$ is a homeomorphism. Since X has at least two boundary components while $X'_0 = E(p', q')$ has only one boundary component, we must have $f(X) \subset X'$ and therefore $f^{-1}(S') \cap intX = \emptyset$. Since T_0 is separating and S is connected, we must have $S \subset E(p_1, q_1; p_2, q_2; ...; p_n, q_n)$ and therefore it is a Seifert surface of $E(p_1, q_1; p_2, q_2; ...; p_n, q_n)$ which intersects T_1 in parallel copies of λ_1. It follows that $\alpha = \lambda_1$. Now we rewrite (1) as

$$(6) \qquad E(k) \longrightarrow V \cup_{T_0} C(p_1, q_1; \lambda_1) \cup_{\lambda_1^*} E(k_1, \lambda_1) \xrightarrow{\hat{f}} W' \cup_{T'_0} E(p', q').$$

Note that the core λ_1^* of the filling solid torus is a retractor of $E(k_1, \lambda_1)$, and $\hat{f}|\colon C(p_1, q_1; \lambda_1) \to E(p', q')$ is homotopic to a homeomorphism by [13]. Hence we have a further factorization

$$\begin{aligned} E(k) &\to W \cup_{T_0} C(p_1, q_1; \lambda_1) \cup_{\lambda_1^*} E(k_1, \lambda_1) \\ &\to W \cup_{T_0} C(p_1, q_1; \lambda_1) = W \cup_{T_0} E(p', q') \to W' \cup_{T'_0} E(p', q'). \end{aligned}$$

Therefore f factors through the de-satellization:

$$E(k) \to W \cup_{T_0} E(p', q') \to E(k').$$

Clearly $W \cup_{T_0} E(p', q') = E(k'')$ for some knot k'' in S^3. Moreover any companion of k'' is prime, and k'' has the same Gromov volume as k'. So we can apply the above process to the induced degree one map $E(k'') \to E(k')$. Since by [19] any knot admits at most finitely many de-satellization, the process must stop after finitely many steps. This finishes the proof of Theorem 1.1. □

4. New Construction

Example 4.1. We construct a degree one map from a graph knot (i.e., the complement of the knot is a graph manifold) to a torus knot which is not a de-satellization.

Below c_i and P_n are given in Notation 1.4. We use $\bar{T}(3, 2)$ to denote the mirror image of $T(3, 2)$ and $\bar{E}(3, 2)$ to donote the exterior of $\bar{T}(3, 2)$.

Lemma 4.1 (Schubert). *The JSJ decomposition pieces of $E(k_1\#...\#k_n)$ are $E(k_1), ..., E(k_n)$ and $P_n \times S^1$, moreover $E(k_1\#...\#k_n)$ is obtained by identifying $\partial E(k_i)$ and $c_i \times S^1$ such that the meridian m_i of $E(k_i)$ is identified with $x_i \times S^1$, where x_i is a point in c_i, $i = 1, ..., n$.*

To construct our example, we need first to orient knot exteriors and their meridians and Seifert fibers and to take a careful look at Lemma 4.1.

The orientation of each knot exterior below is induced from the 3-sphere with fixed orientation; the torus boundary of each knot exterior has induced orientation; on each torus boundary, the meridian and the Seifert fiber are oriented so that their product gives the orientation of the torus.

Suppose the meridian and the Seifert fiber of $E(3,2)$ have been oriented.

Lemma 4.2. *(i) The meridian and the Seifert fiber of $E(3p,2)$ can be oriented so that there is a proper map*

$$\pi_p\colon\ E(3p,2) \to E(3,2)$$

of degree p for any odd p which sends the Seifert fiber of $E(3p,2)$ to the p times of Seifert fiber of $E(3,2)$ and send the meridian to the meridian.

(ii) The meridian and the Seifert fiber of $\bar{E}(3,2)$ can be oriented so that there is a proper degree -1 map

$$\bar{\pi}\colon\ \bar{E}(3,2) \to E(3,2)$$

which send the meridian to the meridian and reverses the direction of the Seifert fiber.

Proof. (i) Let A be a cyclic group of order p acts freely along the regular Seifert fiber on $E(3p,2)$ which induces the identity on the base space. One can verify directly that the quotient $E(3p,2)/A = E(3,2)$ for odd p. Moreover if we lift the orientations of the meridian and the Seifert fiber of $E(3p,2)$ to those of $E(3p,2)$, then the quotient map $\pi_p : E(3p,2) \to E(3,2)$ meets all the conditions.

(ii) By the definition there is a proper degree -1 map

$$r : \bar{E}(3,2) \to E(3,2)$$

induced by the mirror reflection. Now orient the meridian and the Seifert fiber of $\bar{E}(3,2)$ so that r reverses the direction of meridian and preserves the oriented Seifert fiber. Since the trefoil knot is strongly invertible, there is an orientation preserving involution τ which reverses both the directions of the Seifert fiber and the meridian on $\partial E(3,2)$. Then the composition $\bar{\pi} = \tau \circ r$ meets all the conditions. □

In the next lemma, P_n's are oriented and ∂P_n's have induced orientations. The proof of the lemma is very direct.

Lemma 4.3. *Let $d_1, ..., d_n$ be integers such that $\sum d_i = 1$. There is a proper degree one map $h(d_1, ..., d_n)\colon\ (P_n, c_0, \cup_{i=1}^n c_i) \to (P_1, c_0, c_1)$ such that the restriction $h|\colon\ c_0 \to c_0$ is of degree 1 and $h|\colon\ c_i \to c_1$ is of degree d_i.*

Now we are going to construct a degree one map

$$f\colon\ E(T(9,2)\#\bar{T}(3,2)\#\bar{T}(3,2)) \to E(3,2)$$

which we call "folding". To define the map, we need to present the domain and the target as follows:

$$f\colon\ (P_3 \times S^1) \cup_{\phi_i} \sqcup_{i=1}^3 E_i \to (P_1 \times S^1) \cup_{\phi} E(3,2)$$

where $E_1 = E(9,2), E_2 = \bar{E}(3,2), E_3 = \bar{E}(3,2)$, and take a careful look at ϕ_i and ϕ.

First all the meridians and the Seifert fibers of E_i, $i = 1, 2, 3$, are oriented as in Lemma 4.2 and all c_i are oriented as in Lemma 4.3, and S^1 is also oriented.

Now each ϕ_i exactly sends the meridian of E_i to $x_i \times S^1$. Moreover the product structure of $P_3 \times S^1$ can be chosen so that ϕ_i sends the Seifert fiber of E_i to $c_i \times y$, which is possible since the Seifert fiber and the meridian of E_i meets transversally in one point. The product structure of $P_1 \times S^1$ is also chosen so that ϕ has similar property.

Now our map f is obtained by gluing the following proper maps:

(1) $h(3, -1, -1) \times id : P_3 \times S^1 \to P_1 \times S^1$, where $h(3, -1, -1)$ is defined in Lemma 4.3;

(2) $\pi_3 : E_1 \to E(3, 2)$, where π_3 is given by Lemma 4.2 (i);

(3) $\bar{\pi} : E_i \to E(3, 2)$, where $\bar{\pi}$ is given by Lemma 4.2 (ii), $i = 2, 3$.

Clearly f is a proper map of degree one.

Finally we show that the map f is not a de-satellization. Otherwise there would be an essential embedded torus T such that there is a non-trivial simple closed curve c which stays in the kernel of f_*. Since all E_i involved are small knot exteriors, $T \subset E(k)$ must be a vertical torus in $P_3 \times S^1$, which separates $P_3 \times S^1$ into two copies of $P_2 \times S^1$. We may suppose that c_1 and c_2 are in the same $P_2 \times S^1$. Note that f sends (S^1, c_1, c_2) of $P_2 \times S^1$ to $(S^1, 3c_1, -c_1)$ of $P_1 \times S^1$, and c_1 and S^1 form a basis for $\pi_1(P_1 \times S^1)$, one can verify directly that there is no non-trivial simple closed curve on T which stays in the kernel of $f| : T \to P_1 \times S^1$. Since $P_1 \times S^1$ is π_1-injective in $E(3, 2)$, so there is no non-trivial simple closed curve on T which stays in the kernel of $f| : T \to E(3, 2)$, and we reach a contradiction. The verification of the cases that other c_i and c_j are in the same $P_2 \times S^1$ is similar.

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