Coloring and Guarding Arrangements*

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Abstract

Given an arrangement of lines in the plane, what is the minimum number c of colors required to color the lines so that no cell of the arrangement is monochromatic? In this paper we give bounds on the number c, as well as some of its variations. We cast these problems as characterizing the chromatic and the independence numbers of a new family of geometric hypergraphs.

1 Introduction

While dual transformations may allow converting a combinatorial geometry problem about a *configuration of points* into a problem about an *arrangement of lines*, or reversely, the truth is that most mathematical questions appear to be much cleaner and natural in only one of the settings. In many cases, the dual version is considered solely when, besides making sense, it is additionally useful. Both kinds of geometric objects have inspired many problems and attracted much attention. Concerning arrangements of lines, possibly the most prevalent problems consist of studying the number of cells of each size, say triangles, that appear in every arrangement, but many other issues have been considered (see [3, 6, 7]). There are also problems that combine both kinds of objects, like counting incidences between points and lines, or studying the arrangements of lines spanned by point sets, which includes the celebrated *Sylvester-Gallai problem* on ordinary lines [2].

Many other natural questions can be asked when considering arrangements of colored lines. For example, is it true that every bicolored arrangement of lines has a monochromatic cell? We prove in this paper that the generic answer is no, but that it is yes when the colors are slightly unbalanced. This leads immediately to another simple question: How many colors are always sufficient, and occasionally necessary, to color any set of n lines in such a way that the induced arrangement contains no monochromatic cell? This last question brings manifestly the flavor of *Art Gallery Problems*. While coloring and guarding arrangements of lines may appear at first glance as unrelated problems, there is a clean unifying framework provided by considering appropriate geometric hypergraphs. For example, minimally coloring an arrangement while avoiding monochromatic cells can be reformulated as follows: Let $\mathcal{H}_{line-cell}$ be the geometric hypergraph where vertices are lines and edges represent cells of the arrangement; what is its chromatic number? (Here a proper coloring is one where no hyperedge is monochromatic.)

In this work we consider several questions as the ones described above, which arise as fundamental in terms of coloring and guarding arrangements of lines, and translate consistently into problems on geometric hypergraphs, like maximum independent set, minimum vertex cover, or some coloring parameter.

The terminology for hypergraphs on arrangements is introduced in Section 2, where we also provide a table summarizing our results. Coloring problems are then discussed in Section 3 and guarding problems in Section 4. Due to lack of space, proofs of the results in this paper have been omitted. Details will be given in an upcoming extended version.

2 Definitions and Summary of Results

Let $\mathcal{A}$ be an arrangement of a set of lines L in $\mathbb{R}^2$. This arrangement decomposes the plane into different cells, where a *cell* is a maximal connected component of $\mathbb{R}^2 \setminus L$.

We define $\mathcal{H}_{line-cell} = (L, C)$ as the geometric hypergraph corresponding to the arrangement, where C is the set containing all cells defined by L. Similarly, $\mathcal{H}_{vertex-cell} = (V, C)$ is the hypergraph defined by the vertices of the arrangements and its cells, where $V = \binom{L}{2}$ is the set of intersection of lines in $\mathcal{A}$. Finally, $\mathcal{H}_{cell-zone} = (C, Z)$ is the hypergraph defined by the cells of the arrangement and its zones. The zone of a line ℓ in $\mathcal{A}$ is the set of cells bounded by ℓ. The set Z is defined as the set of subsets of C induced

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by the zones of $\mathcal{A}$. Note that this hypergraph is the dual hypergraph of $\mathcal{H}_{line-cell}$.

An *independent set* of a hypergraph $\mathcal{H} = (V, E)$ is a set $S \subseteq V$ such that $\forall e \in E : e \not\subseteq S$. This definition is the natural extension from the graph variant, and requires that no hyperedge is completely contained in S. Analogously, a *vertex cover* of $\mathcal{H}$ is a set $S \subseteq V$ such that $\forall e \in E : e \cap S \neq \emptyset$. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the minimum number of colors that can be assigned to the vertices $v \in V$ so that $\forall e \in E : \exists v_1, v_2 \in e : col(v_1) \neq col(v_2)$; that is, no hyperedge is monochromatic.

In the forthcoming sections we give upper and lower bounds on the worst-case values for these quantities on the three hypergraphs defined from a line arrangement. Our results are summarized in Table 1. Note that the maximum independent set and minimum vertex cover are complementary problems. As a result, any lower bound on one gives an upper bound on the other and vice versa. This property, along with the facts that $|L| = n$, $|V| = \binom{n}{2}$, and $|C| = \frac{n(n+1)}{2} + 1$, are used to complement many entries of the table.

The definitions of an independent set and a proper coloring of the $\mathcal{H}_{line-cell}$ hypergraph of an arrangement are illustrated in Figures 1(a) and 1(b), respectively. Similarly, the definition of a vertex cover of the $\mathcal{H}_{vertex-cell}$ and $\mathcal{H}_{cell-zone}$ hypergraphs are illustrated in Figures 1(c), and 1(d), respectively.

3 Coloring Lines, and Related Results

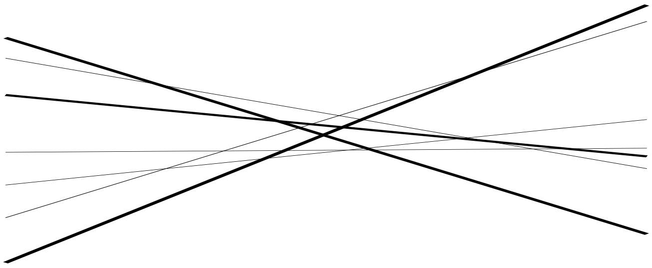
We first consider the chromatic number of the line-cell hypergraph of an arrangement, that is, the number of colors required for coloring the lines so that no cell has a monochromatic boundary. At the end of the section we include some similar results.

3.1 Two-colorability

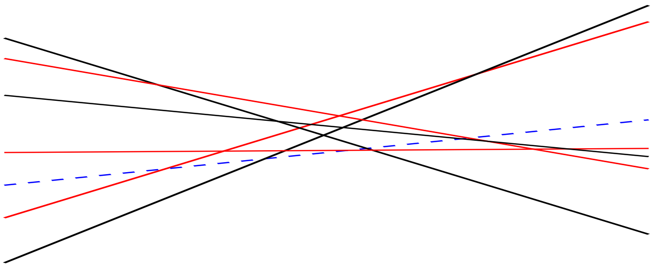
We say that a set of lines L is *k-colorable* if we can color L with k-colors such that no cell is monochromatic (in other words, the corresponding $\mathcal{H}_{line-cell}$ hyper graph has chromatic number k). Any coloring $c : L \to \{0, \dots, k\}$ that satisfies such a property is said to be *proper*. We first tackle the (simple) question of whether the two-colorable $\mathcal{H}_{line-cell}$ hypergraphs have bounded size:

Theorem 1 *There are arbitrarily large two-colorable sets of lines.*

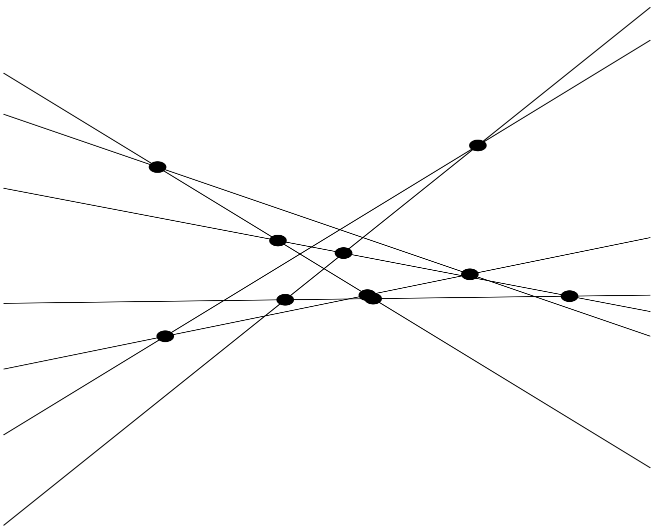
An infinite family of such examples are provided by a set of $2q+1$ lines in convex position (for any $q \in \mathbb{N}$). It is easy to check that, if we color the lines alternatively red and blue by order of slope, no cell will be monochromatic.



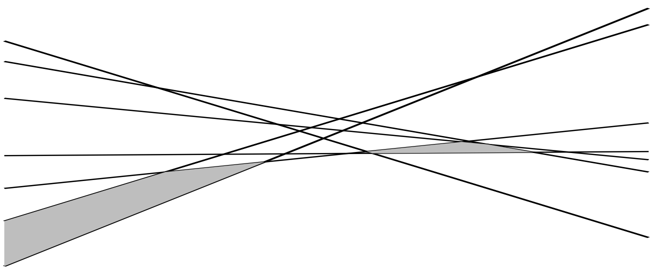
(a) The thick lines form an independent set in the $\mathcal{H}_{line-cell}$ hypergraph: no cell is bounded by those lines only.



(b) A proper 3-coloring of the $\mathcal{H}_{line-cell}$ hypergraph: no cell is monochromatic.



(c) The marked intersections form a vertex cover of the $\mathcal{H}_{vertex-cell}$ hypergraph: every cell has at least one such intersection on its boundary.



(d) The two marked cells form a vertex cover of the $\mathcal{H}_{cell-zone}$ hypergraph: every line has a segment that lies on the boundary of one of those cells.

Figure 1: Illustrations of the definitions.

Hypergraph	Max. Ind. Set	Vertex Cover	Chromatic number
$\mathcal{H}_{line-cell}$	$\geq \frac{\sqrt{n}}{2}$ (Th. 3) $\leq \frac{2n}{3}$ (Th. 4)	$\geq \frac{n}{3}$ (Cor. 14) $\leq n - \frac{\sqrt{n}}{2}$ (Cor. 14)	$\Omega(\log n/\log\log n)$ (Th. 6) $\leq 2\sqrt{n} + O(1)$ (Th. 5)
$\mathcal{H}_{vertex-cell}$	$\geq \frac{n^2}{3} - \frac{5n}{2}$ (Cor. 10) $\leq \frac{n^2}{3} - \frac{n}{2}$ (Cor. 10)	$\geq \frac{n^2}{6}$ (Th. 9) $\leq \frac{n^2}{6} + 2n$ (Th. 9)	2 (Th. 7)
$\mathcal{H}_{cell-zone}$	$\geq \frac{n^2}{2} + \frac{5n}{48} - o(1)$ (Cor. 13) $\leq \frac{n^2}{2} + \frac{5n}{4} + 1$ (Cor. 13)	$\geq \frac{n}{4}$ (Th. 12) $\leq \frac{19n}{48} + o(n)$ (Th. 12)	2 (Th. 8)

Table 1: Worst-case bounds for the different problems studied in this paper.

The coloring used in Theorem 1 uses essentially the same number of lines of each color. This actually holds in general, up to a lower order term.

Theorem 2 *Each color class of a proper 2-coloring $c : L \to \{0, 1\}$ of a set L of n lines has size at most $\frac{n}{2} + \frac{\sqrt{n-1}-1}{2}$.*

3.2 Independent lines in $\mathcal{H}_{line-cell}$

Recall that an independent set of lines in an arrangement is defined as a subset of lines S so that no cell of the arrangement is only adjacent to lines in S.

Theorem 3 *For any set L of n lines, the corresponding $\mathcal{H}_{line-cell}$ hypergraph has an independent set of size $\sqrt{n}/2$.*

Theorem 4 *Given a set L of n lines, an independent set of the corresponding $\mathcal{H}_{line-cell}$ hypergraph has size at most $2n/3$.*

3.3 Chromatic number of $\mathcal{H}_{line-cell}$

In this section, we study the problem of coloring the $\mathcal{H}_{line-cell}$ hypergraph. That is, we want to color the set L so that no cell is monochromatic. We start by giving an upper bound on the required number of colors. This result is proved by iteratively picking a maximal independent set of size $\sqrt{n}/2$.

Theorem 5 *Any arrangement of n lines can be colored with at most $2\sqrt{n} + O(1)$ colors so that no edge of the associated $\mathcal{H}_{line-cell}$ hypergraph is monochromatic.*

We were also able to prove a slightly sublogarithmic lower bound for the chromatic number of $\mathcal{H}_{line-cell}$:

Theorem 6 *There exists an arrangement of n lines whose corresponding hypergraph $\mathcal{H}_{line-cell}$ has chromatic number $\Omega(\log n/\log\log n)$.*

Proof. (Sketch) In order to show the claim, we construct a set of (roughly) k^k lines in which any k-coloring will contain a monochromatic cell (for any $k > 0$). The basic idea behind our construction is the following: consider any coloring with k colors of a set of $k+1$ lines. By the pigeonhole principle there will be two lines with the same assigned color. Moreover, since the two lines must cross, these two lines must be consecutive in the vertical ordering of the lines at some given x coordinate. Our approach is to cross these two lines with a second pair of lines that have the same color assigned, hence obtaining a monochromatic quadrilateral. The main difficulty of the proof is that the line set must satisfy this property for any k-coloring of L. In particular, we do not know neither which color will be repeated, nor at which x-coordinate. $\square$

3.4 Other coloring results

For the sake of completeness, we end this section by stating two easy results on coloring vertices or cells instead of lines.

Theorem 7 *The chromatic number of $\mathcal{H}_{vertex-cell}$ is 2.*

(Remark that cells of size two only have one vertex, hence cannot be polychromatic. Therefore, we only consider cells of size at least 3.)

The following claim is equivalent to the fact that the dual graph of the arrangement is bipartite

Theorem 8 (Folk.) *The chromatic number of $\mathcal{H}_{cell-zone}$ is 2.*

4 Guarding Arrangements

We now consider the vertex cover problem of the above hypergraphs. That is, we would like to select the minimum number of vertices so that any hyperedge is adjacent to the selected subset. Geometrically speaking, we would like to select the minimum number of vertices (or cells or lines), so that each cell (or line or cell, respectively) contains at least one of the selected items.

4.1 Guarding cells with vertices

We first consider the following problem: given an arrangement of lines $\mathcal{A}$, how many vertices do we need to pick in order to guard the whole arrangement when lines act as obstacles blocking visibility? This can be rephrased as finding the smallest subset of vertices V so that each cell contains a vertex in V, and thus we are looking for bounds on the size of a vertex cover for $\mathcal{H}_{vertex-cell}$.

Theorem 9 *For any set L of n lines, a vertex cover of the corresponding $\mathcal{H}_{vertex-cell}$ hypergraph has size at most $n^2/6 + 2n$. Furthermore, $n^2/6$ vertices might be necessary.*

Recall that the hypergraph $\mathcal{H}_{vertex-cell}$ has $\binom{n}{2} = \frac{n^2}{2} - \frac{n}{2}$ vertices. Combining this fact with the previous bounds on the size of a vertex cover allow us to get similar bounds for the independent set problem.

Corollary 10 *For any set L of n lines, a maximum independent set of the corresponding $\mathcal{H}_{vertex-cell}$ hypergraph has size at least $n^2/3 - O(n)$. Furthermore, there exists sets of lines whose largest independent set has size at most $\frac{n^2}{3} - \frac{n}{2}$.*

4.2 Guarding lines with cells

We now consider the problem of touching all lines of L with a smallest subset of cells, i.e., we look for bounds on the size of a vertex cover for $\mathcal{H}_{cell-zone}$.

Theorem 11 *Given a set L of n lines, a minimal vertex cover of the corresponding $\mathcal{H}_{cell-zone}$ hypergraph has size at most $\lceil \frac{n}{2} \rceil$.*

We next provide a lower bound, and improve as well on the upper bound, for large values of n.

Theorem 12 *Given any set L of n lines, a minimal vertex cover of the corresponding $\mathcal{H}_{cell-zone}$ hypergraph has size at most $\frac{19n}{48} + o(n)$. Moreover, there exists a set L of n lines, such that every vertex cover of the corresponding $\mathcal{H}_{cell-zone}$ hypergraph has size at least $\frac{n}{4}$.*

Corollary 13 *For any set L of n lines, a maximum independent set of the corresponding $\mathcal{H}_{cell-zone}$ hypergraph has size at least $\frac{n^2}{2} + \frac{5n}{48} - o(1)$ and at most $\frac{n^2}{2} + \frac{n}{4} + 1$.*

4.3 Guarding cells with lines

For the sake of completeness, we also give bounds on the number of lines needed to guard (touch) all cells.

Corollary 14 *For any set L of n lines, its minimal vertex cover of the corresponding $\mathcal{H}_{line-cell}$ hypergraph has size at least $n/3$ and at most $n - \frac{\sqrt{n}}{2}$.*

5 Concluding Remarks

Clearly, the main open problems arising from our work consist of closing gaps (when they exist) between lower and upper bounds; this is especially interesting in our opinion for the problem of coloring lines without producing any monochromatic cell.

However, it is worth noticing that there are several computational issues that are interesting as well. For example, it is unclear to us which is the complexity of deciding whether a given arrangement of lines admits a two-coloring in which no cell is monochromatic.

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