Int. Journal of Math. Analysis, Vol. 7, 2013, no. 44, 2145 - 2160
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2013.36136

Approximate Solution of Second-Order Integrodifferential Equation of Volterra Type in RKHS Method

Zuhier Altawallbeh, Mohammed H. AL-Smadi*
and Radwan Abu-Gdairi

Department of Mathematics and Computer Science
Tafila Technical University, Tafila 66110, P.O. Box 179, Tafila-Jordan
*mhm.smadi@yahoo.com

Abstract

In this paper, an application of reproducing kernel Hilbert space (RKHS) method is applied to solve second-order integrodifferential equation of Volterra type. The analytical solution is represented in the form of series in the reproducing kernel space. The $n-$truncation approximation $u_n(x)$ is obtained and proved to converge to the analytical solution $u(x)$. Moreover, the presented method has an advantages that it is possible to pick any point in the interval domain and as well the approximate solution and its derivatives will be applicable Numerical experiments are displayed to illustrate the validity, accuracy, efficiency and applicability of the proposed method. Results indicates that our technique is simple, straightforward and effective.

Mathematics Subject Classification: 47B32, 45J05, 34K28

Keywords: Volterra, Integro-differential equation, Reproducing kernel Hilbert space, Boundary value problems

1 Introduction

Integro-differential equations play a crucial role in modeling of much physical phenomena such as particle vibrations in lattices, currents in electrical networks and pulses in biological chains and much more applications. Recently, there have been lots efforts in giving exact or approximate solution relating different kinds of problems in integro-differential equations.

In many branches of physics, mathematics, and engineering, solving a problem means solving a set of ordinary, partial, integral or either integro-differential equations. In fact, integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate or numerical solution. Nowadays, the technique that used more in integro-differential equations is based on applying reproducing kernel. In recent years, there has been a growing interest to solve the operator equation using the reproducing kernel.

The theory of reproducing kernel was used for the first time at the beginning of the 20^{th} century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions. In 1943, N. Aronszajn developed the theory of reproducing kernels which contains the Bergman kernel functions. However, the original idea of Zaremba to apply the kernels to the solution of BVPs was developed by S. Bergman and M. Schiffer (1948).

Actually, this theory has been implemented in several operator, differential, integral, and integro-differential equations such as nonlinear partial differential equations [24], nonlinear operator equations [3], nonlinear second-order singular BVPs [5,22], nonlinear Fredholm-Volterra integral equation [14], nonlinear system of Fredholm integro-differntial equations [13], nonlinear fourth order integro-differential equations [11,12], nonlinear Fredholm-Volterra integro differential equations [15], nonlinear Fredholm integro-differential equations [16], and others.

The purpose of this paper is to extend the application of the reproducing kernel Hilbert space method to provide approximate solution for the second-order boundary value problems of integro-differential equation of the following Volterra type

$$u''(x) + g(u(x), u'(x)) + \int_0^x k(x,s) f(u(s), u'(s)) ds = h(x), 0 \leq x, s \leq 1, \tag{1.1}$$

subject to the boundary conditions

$$u(0) = \alpha_1, \ u(1) = \alpha_2, \tag{1.2}$$

where α_1, α_2 are real finite constants, $u(x) \in W_2^3[0,1]$ is an unknown function to be determined, $k(x,s)$ and $h(x)$ are continuous functions on $[0,1] \times [0,1]$ and $[0,1]$, respectively, and $f(u,u')$ and $g(u,u')$ are linear or nonlinear function of u, u' depend on the problem discussed.

For details about the existence and uniqueness of the solution for such problems, see [19,20,25]. However, we assume that Eq. (1.1) subject to the boundary conditions Eq. (1.2) has a unique analytic solution on the given interval.

The numerical solvability of second-order integro-differential equations with boundary conditions of the Fredholm and Volterra types and other related equations has been pursued by several authors. To mention a few, the authors in [2] have discussed the Legendre polynomials method for solving second-order Fredholm integro-differential equation, the compact finite difference method and the monotone iterative sequences method are carried out for solving second-order Volterra integro-differential equation in [9,18]. Further, series solution of second-order integro-differential equations with boundary conditions of the Fredholm and Volterra types by means of the homotopy analysis method is considered in [21].

The rest of the paper is organized as follows. In the next section, two reproducing kernel spaces and a linear operator are described. In section 3, a complete normal orthogonal basis and some essential results are introduced. The algorithm for solving second-order boundary value problems of integro-differential equation of Volterra type based on reproducing kernel space is proposed in section 4. In section 5, numerical examples are presented to demonstrate the computation efficiency of the presented method. The conclusions of this paper are introduced in the last section.

2 Constructive Method for the Reproducing Kernel Spaces

Definition 2.1 *Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H iff*

1. $\forall t \in E$, $K(\cdot, t) \in H$.
2. $\forall t \in E$, $\forall \varphi \in H$, $(\varphi, K(\cdot, t)) = \varphi(t)$.

Definition 2.2 *A Hilbert spaces H of functions on a set Ω is called a reproducing kernel Hilbert spaces if there exists a reproducing kernel K of H.*

It is known that the reproducing kernel of a Hilbert space is unique, and that existence of a reproducing kernel is due to the Riesz Representation Theorem. The reproducing kernel K of a Hilbert space H completely determines the space H. Every sequence of functions $\{h_n\}_{n=1}^{\infty}$ which converges strongly to a function h in H, converges also in the pointwise sense. Indeed, this convergence is uniform on every subset on Ω on which $x \to K(x,x)$ is bounded.

2.1 The reproducing kernel Hilbert space $W_2^3[0,1]$

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) : u, u', u''$ are absolutely continuous real valued functions, $u''' \in L^2[0,1]$, $u(0) = u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$\langle u(x), v(x)\rangle_{W_2^3} = \sum_{i=1}^{2} u^{(i)}(0)\, v^{(i)}(0) + \int_0^1 u'''(x)\, v'''(x)\, dx \tag{2.1}$$

and the norm $\|u\|_{W_2^3}$ is denoted by $\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Theorem 2.1 *The space $W_2^3[0,1]$ is a complete reproducing kernel space. That is, for each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2^3[0,1]$ such that $\langle u(y), R_x(y)\rangle_{W_2^3} = u(x)$ for any $u(y) \in W_2^3[0,1]$ and $y \in [0,1]$. The reproducing kernel function $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} \sum\limits_{i=1}^{6} c_i(x) y^{i-1}, & y \le x, \\ \sum\limits_{i=1}^{6} d_i(x) y^{i-1}, & y > x. \end{cases} \tag{2.2}$$

Proof. The proof of the completeness and reproducing property of $W_2^3[0,1]$ is similar to the proof in [9]. Now, let us find out the expression form of the reproducing kernel function $R_x(y)$ in the space $W_2^3[0,1]$. Through several integration by parts, we obtain

$$\int_0^1 u'''(y)\partial_y^3 R_x(y)dy = \sum_{i=0}^{2} (-1)^i u^{(i)}(y)\, \partial_y^{5-i} R_x(y) \,\Big|_{y=0}^{y=1} - \int_0^1 u(y)\, \partial_y^6 R_x(y)\, dy.$$

Thus, from Eq. (2.1) we can write

$$\begin{aligned} \langle u(y), R_x(y)\rangle_{W_2^3} &= \sum_{i=0}^{2} u^{(i)}(0)\left[\partial_y^i R_x(0) + (-1)^{i+1}\partial_y^{5-i} R_x(0)\right] \\ &\quad + \sum_{i=0}^{2} (-1)^i u^{(i)}(1)\, \partial_y^{5-i} R_x(1) \\ &\quad - \int_0^1 u(y)\, \partial_y^6 R_x(y)\, dy. \end{aligned} \tag{2.3}$$

Since $R_x(y) \in W_2^3[0,1]$, it follows that

$$R_x(0) = R_x(1) = 0.$$

Since $u(x) \in W_2^3[0,1]$, it follows that $u(0) = u(1) = 0$. Hence, if $\partial_y^4 R_x(1) = 0$, $\partial_y^3 R_x(1) = 0$, $\partial_y^2 R_x(0) - \partial_y^3 R_x(0) = 0$, and $\partial_y^1 R_x(0) + \partial_y^4 R_x(0) = 0$, then Eq. (2.3) implies that

$$\langle u(y), R_x(y)\rangle_{W_2^3} = \int_0^1 u(y)\,(-\partial_y^6 R_x(y))dy$$

Now, for any $x \in [0,1]$, if $R_x(y)$ satisfies

$$\partial_y^6 R_x(y) = -\delta(x-y),\ \delta \text{ dirac-delta function,} \tag{2.4}$$

then $\langle u(y), R_x(y)\rangle_{W_2^3} = u(x)$. Obviously, $R_x(y)$ is the reproducing kernel function of the space $W_2^3[0,1]$.
Here, we will give the expression of the reproducing kernel function $R_x(y)$. The auxiliary equation of Eq. (2.4) is given by $\lambda^6 = 0$, and their auxiliary values are $\lambda = 0$ with multiplicity 6. So, let the expression of the reproducing kernel function $R_x(y)$ be as defined in Eq. (2.2).

But on the other aspect as well, for Eq. (2.4), let $R_x(y)$ satisfy the equation

$$\partial_y^i R_x(x+0) = \partial_y^i R_x(x-0),\ i = 0,1,...,4$$

Integrating $\partial_y^6 R_x(y) = -\delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $\partial_y^5 R_x(y)$ at $y = x$ given by

$$\partial_y^5 R_x(x+0) - \partial_y^5 R_x(x-0) = -1.$$

Through the last descriptions, the unknown coefficients $c_i(x)$ and $d_i(x)$, $i = 1,2,...,6$ of Eq. (2.2) can be obtained. However, by using Mathematica 7.0, the representation form of the reproducing kernel function $R_x(y)$ is provided by

$$R_x(y) = \begin{cases} \frac{-1}{18720}(x-1)y(156y^4+6x^2(120+30y+10y^2-5y^3+y^4) \\ -4x^3(120+30y+10y^2-5y^3+y^4)+x^4(120+30y & y \le x, \\ +10y^2-5y^3+y^4)+12x(360-300y-100y^2-15y^3+3y^4)), \\ \frac{-1}{18720}(y-1)x(30xy(-120+6y-4y^2+y^3)+10x^2y(-120 \\ +6y-4y^2+y^3)+120y(36+6y-4y^2+y^3)-5x^3y(36+ & y > x, \\ 6y-4y^2+y^3)+x^4(156+36y+6y^2-4y^3+y^4)), \end{cases}$$

This completes the proof.

2.2 The reproducing kernel Hilbert space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined as $W_2^1[0,1] = \{u(x) : u$ is absolutely continuous real valued function, $u' \in L^2[0,1]\}$. The inner product in $W_2^1[0,1]$ is given by

$$\langle u, v\rangle_{W_2^1} = u\,(t)\,v\,(t) + \int_0^1 u'\,(t)\,v'\,(t)\,dt.$$

and the norm $\|u\|_{W_2^1}$ is denoted by $\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}$, where $u, v \in W_2^1[0,1]$.

In [24], it has been proved that $W_2^1[0,1]$ is also a complete reproducing kernel space and its reproducing kernel is

$$K_x\,(y) = \begin{cases} 1+y & y \le x, \\ 1+x & y > x. \end{cases}.$$

From the definitions of the reproducing kernel spaces $W_2^3\,[0,1]$ and $W_2^1\,[0,1]$, clearly that $W_2^1\,[0,1] \supset W_2^3\,[0,1]$ for any $u(x) \in W_2^3\,[0,1]$ and $||u||_{W_2^1} \le ||u||_{W_2^3}$.

3.2 Introduction into a linear operator

Let $Lu = u''$, $L : W_2^3[0,1] \to W_2^1[0,1]$. After homogenization of the initial conditions, then Eq. (1.1) and Eq. (1.2) can be converted into the following form

$$\begin{cases} Lu\,(x) = F(x, u(x), u'(x), Tu(x)),\ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{2.5}$$

where $u(x) \in W_2^3[0,1]$, $Tu(x) = \int\limits_0^x k(x,s) f(u\,(s)\,, u'\,(s))ds$, and $F(x, y_1, y_2, y_3) \in W_2^1[0,1]$ for $y_1 = y_1(x), y_2 = y_2(x), y_3 = y_3(x) \in W_2^3[0,1]$.

Theorem 2.2 *The operator $L : W_2^3[0,1] \longrightarrow W_2^1[0,1]$ is bounded linear operator.*

Proof. We need to prove $\|Lu(x)\|^2_{W_2^1} \le M\,\|Lu(x)\|^2_{W_2^3}$, where M is positive constant. From the definition of the inner product and the norm of $W_2^1[0,1]$, we have $\|Lu\|^2_{W_2^1} = \langle Lu, Lu\rangle_{W_2^1} = [(Lu)(0)]^2 + \int\limits_0^1 [(Lu)'(x)]^2 dx$. By reproducing property of $R_x(y)$, we have

$$u(x) = \langle u, R_x\rangle_{W_2^3}, (Lu)(x) = \langle u, (LR_x)\rangle_{W_2^3}, (Lu)'(x) = \langle u, (LR_x)'\rangle_{W_2^3}.$$

By Schwarz inequality, we get

$$\begin{aligned} |(Lu)(x)| &= \left|\langle u, (LR_x)\rangle_{W_2^3}\right| \\ &\le \|LR_x\|_{W_2^3}\,\|u\|_{W_2^3} = M_1\,\|u\|_{W_2^3}, \\ |(Lu)'(x)| &= \left|\langle u, (LR_x)'\rangle_{W_2^3}\right| \\ &\le \|(LR_x)'\|_{W_2^3}\,\|u\|_{W_2^3} = M_2\,\|u\|_{W_2^3}, \end{aligned}$$

where $M_1, M_2 > 0$ are positive constants.
Thus $\|(Lu)(x)\|^2_{W_2^1} = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx \leq (M_1^2 + M_2^2)\|u\|_{W_2^3} = M\|u\|_{W_2^3}$, where $M = (M_1^2 + M_2^2) > 0$ is positive constant.

3 An Orthogonal Basis

Now, we construct an orthogonal system of functions. Let $\varphi_i(x) = K_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the conjugate operator of L and $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$. In terms of the properties of $K_x(y)$, one obtains $\langle u(x), \psi_i(x)\rangle_{W_2^3} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i), i = 1, 2, ...$

The normal orthogonal system $\{\bar{\psi}_i(x)\}_{i=0}^{\infty}$ in $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=0}^{\infty}$ as follows

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \ (\beta_{ii} > 0, \ i \in N). \tag{3.1}$$

where β_{ik} are orthogonalization coefficients and are given by the following subroutine

$$\begin{aligned}
\beta_{ij} &= \frac{1}{\|\psi_1\|}, \text{ for } i = j = 1, \\
\beta_{ij} &= \frac{1}{d_{ik}}, \text{ for } i = j \neq 1, \\
\beta_{ij} &= -\frac{1}{d_{ik}} \sum_{k=j}^{i-1} c_{ik}\beta_{kj}, \text{ for } i > j,
\end{aligned}$$

such that $d_{ik} = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} (c_{ik})^2}$, $c_{ik} = \langle \psi_i, \overline{\psi}_k\rangle_{W_2^3}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is the orthogonal system in $W_2^3[0,1]$.

Through the next theorem the subscript y by the operator L indicates that the operator L applies to the function of y.

Theorem 3.1 *If* $\{x_i\}_{i=0}^{\infty}$ *is dense on* $[0,1]$, *then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is a complete function system of* $W_2^3[0,1]$ *and* $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof. Notice that

$$\begin{aligned}
\psi_i(x) &= L^*\varphi_i(x) \\
&= \langle L^*\varphi_i(y), R_x(y)\rangle \\
&= \langle \varphi_i(y), L_y R_x(y)\rangle = L_y R_x(y)|_{y=x_i}.
\end{aligned}$$

Clearly, $\psi_i(x) \in W_2^3[0,1]$. Now, let $\langle u(x), \psi_i(x)\rangle = 0, i = 1, 2, ...,$ for each fixed $u(x) \in W_2^3[0,1]$. That is, $\langle u(x), L^*\varphi_i(x)\rangle = \langle Lu(\cdot), \varphi_i(\cdot)\rangle = Lu(x_i) =$

0. Note that $\{x_i\}_{i=0}^{\infty}$ is dense on $[0,1]$, therefore $Lu(x)=0$. It follows that $u(x)=0$ from the existence of L^{-1}. So, the proof of the Theorem is complete.

Lemma 3.1 *If $u(x)\in W_2^3[0,1]$, then there exists $M_1>0$ such that*

$$||u(x)||_{C^2[0,1]} \le M_1 ||u(x)||_{W_2^3},$$

where $||u(x)||_{C^2} = \max_{x\in[0,1]}|u(x)| + \max_{x\in[0,1]}|u'(x)| + \max_{x\in[0,1]}|u''(x)|$.

Proof. For any $x,y\in[0,1]$, we have $u(x)=\langle u(y),\partial_x R_x(y)\rangle_{W_2^3}$. By the expression form of $R_x(y)$, it follows that $\|\partial_x R_x(y)\|_{W_2^3}\le M_1$.
Thus, $|u(x)| = \left|\langle u(x),\partial_x R_x(x)\rangle_{W_2^3}\right| \le \|\partial_x R_x(x)\|_{W_2^3}\|u(x)\|_{W_2^3} \le M_1\|u(x)\|_{W_2^3}$.
Hence, $||u(x)||_{C^2}\le M_1\|u(x)\|_{W_2^3}$.

Lemma 3.2 *If $\|u_n-u\|_{W_2^3}\to 0$, $x_n\to x$, $(n\to\infty)$ and $F(x,y,z,w)$ for $x\in[0,1]$, $y,z,w\in(-\infty,+\infty)$ is continuous with respect to x,y,z,w, then $F(x_n,u_{n-1}(x_n),u'_{n-1}(x_n),Tu_{n-1}(x_n))\to F(x,u(x),u'(x),Tu(x))$ as $n\to\infty$.*

Proof. Since $|u_n(x_n)-u(x)| = |u_n(x_n)-u_n(x)+u_n(x)-u(x)| \le |u_n(x_n)-u_n(x)|+|u_n(x)-u(x)|$. By reproducing property of $R_x(y)$, we have $u_n(x_n)=\langle u_n,R_{x_n}(\cdot)\rangle$ and $u_n(x)=\langle u_n,R_x(\cdot)\rangle$. Thus, $|u_n(x_n)-u_n(x)|=|\langle u_n,R_{x_n}(\cdot)-R_x(\cdot)\rangle_{W_2^3}|\le\|u_n\|_{W_2^3}\|R_{x_n}(\cdot)-R_x(\cdot)\|_{W_2^3}\to 0$ as soon as $x_n\to x, n\to\infty$.
On the other hand, by Lemma 3.1, we know that $u_n(x)$ is convergent uniformly to $u(x)$. Thus, for any $x\in[0,1]$, it holds that $|u_n(x)-u(x)|_{C^2}\to 0$ as soon as $||u_n(x)-u(x)||_{W_2^3}\to 0$ as $n\to\infty$. Therefore, $u_n(x_n)\longrightarrow u(x)$ in the sense of $\|\cdot\|_{W_2^3}$ as $x_n\to x$ and $n\to\infty$. Thus, by means of the continuation of $Tu(\cdot)$, it is obtained that $Tu_n(x_n)\to Tu(x)$ as $n\to\infty$. Hence, by the continuity of F, we have $F(x_n,u_{n-1}(x_n),u'_{n-1}(x_n),Tu_{n-1}(x_n))\to F(x,u(x),u'(x),Tu(x))$ as $n\to\infty$.

4 Numerical Algorithm

In this section, it is explained how to deduce the exact solution from the orthogonal basis $\{\bar\psi_i(x)\}_{i=0}^{\infty}$ in $W_2^3[0,1]$.

Theorem 4.1 *The exact solution of* (2.5) *can be expressed by*

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),u'(x_k),Tu(x_k))\,\bar\psi_i(x). \tag{4.1}$$

Proof. The exact solution $u(x)$ can be expanded to a Fourier series in terms of orthonormal basis $\{\bar\psi_i(x)\}_{i=0}^{\infty}$ in $W_2^3[0,1]$ as $u(x)=\sum_{i=1}^{\infty}\langle u(x),\bar\psi_i(x)\rangle\bar\psi_i(x)$.

Since the space $W_2^3[0,1]$ is Hilbert space so the series $\sum_{i=1}^{\infty}\left\langle u(x),\bar{\psi}_i(x)\right\rangle\bar{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2^3}$. Also, note that $\langle w(x),\varphi_i(x)\rangle = w(x_i)$ for each $w(x)\in W_2^1[0,1]$. Hence, we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x),\bar{\psi}_i(x)\right\rangle_{W_2^3}\bar{\psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),\psi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),L^*\varphi_k(x)\right\rangle_{W_2^3}\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x),\varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F(x,u(x),u'(x),Tu(x)),\varphi_k(x)\right\rangle_{W_2^1}\bar{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),u'(x_k),Tu(x_k))\,\bar{\psi}_i(x). \qquad (4.2)
\end{aligned}$$

The $n-$truncation approximate solution $u_n(x)$ of (2.5) can be obtained by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),u'(x_k),Tu(x_k))\,\bar{\psi}_i(x), \qquad (4.3)$$

which is $n-$truncation Fourier series of the exact solution $u(x)$ in (2.5).

Remark If Eq. (2.5) is nonlinear, then the $n-$truncation approximate solution $u_n(x)$ of Eq. (2.5) can be obtained using the following iteration formula.

We construct the iterative sequences $u_n(x)$, putting

$$\begin{cases}\forall \text{ fixed } u_0(x)\in W_2^3[0,1],\\ u_n(x) = \sum_{i=1}^{n}A_i\bar{\psi}_i(x),\end{cases} \qquad (4.4)$$

where the coefficients A_i of $\bar{\psi}_i(x)$, $i = 1,2,...,n$ are given as

$$\begin{cases}A_1 = \beta_{11}F(x_1,u_0(x_1),u_0'(x_1),Tu_0(x_1)),\\ A_2 = \sum_{k=1}^{2}\beta_{2k}F\left(x_k,u_{k-1}(x_k),u_{k-1}'(x_k),Tu_{k-1}(x_k)\right),\\ \ldots\\ A_n = \sum_{k=1}^{n}\beta_{nk}F\left(x_k,u_{k-1}(x_k),u_{k-1}'(x_k),Tu_{k-1}(x_k)\right),\end{cases} \qquad (4.5)$$

Lemma 4.1 *$\{u_n\}_{n=1}^{\infty}$ in Eq. (4.4) is monotone increasing i the sense of the norm of $W_2^3[0,1]$.*

Proof. By Theorem 3.1, $\{\overline{\psi}_i\}_{i=1}^{\infty}$ is the complete orthonormal system in the space $W_2^3[0,1]$. Hence, we have

$$\|u_n\|^2_{W_2^3} = \langle u_n(x), u_n(x)\rangle_{W_2^3} = \langle \sum_{i=1}^{n} A_i\bar{\psi}_i(x), \sum_{i=1}^{n} A_i\bar{\psi}_i(x)\rangle_{W_2^3} = \sum_{i=1}^{n}(A_i)^2.$$

Therefore, $\|u_n\|_{W_2^3}$ is monotone increasing.

Lemma 4.2 *$Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j), Tu_{j-1}(x_j)\right)$, $j \le n$ holds.*

Proof. The proof will be obtained by induction as follows. If $j \le n$, then

$$\begin{aligned} Lu_n(x_j) &= \sum_{i=1}^{n} A_i L\bar{\psi}_i(x_j) = \sum_{i=1}^{n} A_i \left\langle L\bar{\psi}_i(x), \phi_j(x)\right\rangle_{W_2^1} \\ &= \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), L^*\phi_j(x)\right\rangle_{W_2^3} = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^3}. \end{aligned}$$

The orthogonality of $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$ yields that

$$\begin{aligned} \sum_{l=1}^{j} \beta_{jl} Lu_n(x_l) &= \sum_{i=1}^{n} A_i \langle \bar{\psi}_i(x), \sum_{l=1}^{j}\beta_{jl}\psi_l(x)\rangle_{W_2^3} = \sum_{i=1}^{n} A_i \left\langle \bar{\psi}_i(x), \bar{\psi}_j(x)\right\rangle_{W_2^3} \\ &= A_j = \sum_{l=1}^{j}\beta_{jl} F\left(x_l, u_{l-1}(x_l), u'_{l-1}(x_l), Tu_{l-1}(x_l)\right). \end{aligned}$$

Now, if $j=1$, then $Lu_n(x_1) = F(x_1, u_0(x_1), u'_0(x_1), Tu_0(x_1))$.
If $j=2$, then $\beta_{21}Lu_n(x_1) + \beta_{22}Lu_n(x_2) = \beta_{21}F(x_1, u_0(x_1), u'_0(x_1), Tu_0(x_1))$ $+\beta_{22}F(x_2, u_1(x_2), u'_1(x_2), Tu_1(x_2))$.
So, $Lu_n(x_2) = F(x_2, u_1(x_2), u'_1(x_2), Tu_1(x_2))$. Hence, by induction, we have that $Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j), Tu_{j-1}(x_j)\right)$.

Theorem 4.1 *If $\{x_i\}_{i=0}^{\infty}$ is dense on $[0,1]$ and $||u_n||_{W_2^3}$ is bounded, then $u_n(x)$ in the iterative formula (4.4) is convergent to the exact solution $u(x)$ of Eq. (2.5) in $W_2^3[0,1]$ and*

$$u(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i(x),$$

where A_i is given by Eq. (4.5).

Proof. (1) First, we will prove the convergence of $u_n(x)$. By Eq. (4.4), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1}\bar{\psi}_{n+1}(x). \tag{4.6}$$

From the orthogonality of $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned} ||u_{n+1}||_{W_2^3}^2 &= ||u_n||_{W_2^3}^2 + (A_{n+1})^2 = ||u_{n-1}||_{W_2^3}^2 + (A_n)^2 + (A_{n+1})^2 \\ &\quad ... \\ &= ||u_0||_{W_2^3}^2 + \sum_{i=1}^{n+1}(A_i)^2. \end{aligned} \tag{4.7}$$

From Lemma 4.1, the sequence $||u_n||_{W_2^3}$ is monotone increasing. Due to the condition that $||u_n||_{W_2^3}$ is bounded, $||u_n||_{W_2^3}$ is convergent as $n \to \infty$. Then, there exists a constant c such that $\sum_{i=1}^{\infty}(A_i)^2 = c$. It implies that

$$A_i = \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), Tu_{k-1}(x_k)\right) \in l^2, i = 1, 2, ...$$

Let $m > n$, for $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp ... \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} ||u_m(x) - u_n(x)||_{W_2^3}^2 &= ||u_m(x) - u_{m-1}(x) + u_{m-1}(x) - ... + u_{n+1}(x) - u_n(x)||_{W_2^3}^2 \\ &\leq ||u_m(x) - u_{m-1}(x)||_{W_2^3}^2 + ... + ||u_{n+1}(x) - u_n(x)||_{W_2^3}^2 \\ &= \sum_{i=n+1}^{m} (A_i)^2. \end{aligned}$$

Consequently, as $n, m \to \infty$, we have $||u_m(x) - u_n(x)||_{W_2^3}^2 \to 0$ as $\sum_{i=n+1}^{m}(A_i)^2 \to 0$. Considering the completeness of $W_2^3[0,1]$, there exists a $u(x) \in W_2^3[0,1]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in the sense of $||\cdot||_{W_2^3}$.

(2) Second, we will prove that $u(x)$ is the solutions of Eq. (2.5). From Lemma 4.2, since $\{x_i\}_{i=0}^{\infty}$ is dense on $[0,1]$, for any $x \in [0,1]$, there exists subsequence $\left\{x_{n_j}\right\}$, such that $x_{n_j} \to x$ as $j \to \infty$. It is clear that $Lu\left(x_{n_j}\right) = F\left(x_{n_j}, u_{n_j-1}(x_k), u'_{n_j-1}(x_k), Tu_{n_j-1}(x_k)\right)$. Hence, let $j \to \infty$, by the continuity of F, we have $Lu(x) = F(x, u(x), u'(x), Tu(x))$. That is, $u(x)$ is the solution of Eq. (2.5), where $u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i(x)$ and A_i is given by Eq. (4.4).

Theorem 4.2 *Assume that $u(x) \in W_2^3[0,1]$ is the solution of Eq. (2.5) and $r_n(x)$ is the difference between the approximate solution $u_n(x)$ and the exact solution $u(x)$. Then, $r_n(x)$ is monotone decreasing in the sense of the norm of $W_2^3[0,1]$. i.e.* $r_n \to 0$ as $n \to \infty$.

Proof. From Eqs. (4.1) and (4.3), it is obvious that

$$\begin{aligned} \|r_n(x)\|^2_{W_2^3} &= \|u(x)-u_n(x)\|^2_{W_2^3} \\ &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{n} \beta_{nk} F\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k), Tu_{k-1}(x_k)\right) \bar{\psi}_i(x) \right\|^2_{W_2^3} \\ &= \left\| \sum_{i=n+1}^{\infty} A_i \bar{\psi}_i(x) \right\|^2_{W_2^3} = \sum_{i=n+1}^{\infty} (A_i)^2, \end{aligned}$$

and $\|r_{n-1}(x)\|^2_{W_2^3} = \sum_{i=n}^{\infty} (A_i)^2$. Thus, $\|r_n(x)\|_{W_2^3} \leq \|r_{n-1}(x)\|_{W_2^3}$. Consequently, the difference $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$. So, the proof of the theorem is complete.

5 Numerical Example

In this section, numerical examples are studied to demonstrate the accuracy of the present algorithm. Results obtained by the algorithm are compared with the analytical solution and are found to be in good agreement. The examples are computed using Mathematica 7.0.

Example 5.1 [1] Consider the following linear Volterra IDE

$$\begin{cases} u''(x) - \int_0^x e^{-s} \sin x u'(s) ds + u(x) = \left(\frac{1}{2} e^{-x} \sin 2x - \sin x\right), \ 0 \leq x \leq 1, \\ u(0) = -1, u(1) = \sin 1 - \cos 1. \end{cases}$$

The exact solution is $u(x) = \sin x - \cos x$. Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$. The numerical results at some selected nods for $n = 101$ are displayed in Table 1.

Table 1. Numerical results for Example 5.1.

x	Exact Sol.	Approximate Sol.	Absolute error	Relative error
0.16	−0.827909	−0.8279086004757638	4.76286×10^{-7}	5.75287×10^{-7}
0.32	−0.634669	−0.6346681574629389	7.00003×10^{-7}	1.10294×10^{-6}
0.48	−0.425216	−0.4252150356389058	7.11599×10^{-7}	1.67350×10^{-6}
0.64	−0.204900	−0.2048997511869910	5.65335×10^{-7}	2.75907×10^{-6}
0.80	0.0206494	0.02064970692970153	3.25377×10^{-7}	1.57572×10^{-5}
0.96	0.2456716	0.24567164364805905	6.14195×10^{-8}	2.50007×10^{-7}

Example 5.2 [1] Consider the following nonlinear Volterra IDE

$$\begin{cases} u''(x) + \int_0^x (u(s))^2 ds + \left(\frac{x}{2} - \sinh x - \frac{1}{4} \sinh 2x\right) = 0, \ 0 \leq x \leq 1, \\ u(0) = 0, u(1) = \sinh(1). \end{cases}$$

The exact solution is $u(x) = \sinh x$. Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$. The numerical results at some selected nods for $n = 51$ are displayed in Table 2.

Table 2. Numerical results for Example 5.2.

x	Exact Sol.	Approximate Sol.	Absolute error	Relative error
0.16	0.160684	0.1606828700813321	6.70931×10^{-7}	4.17548×10^{-6}
0.32	0.325489	0.3254880335933262	1.33004×10^{-6}	4.08627×10^{-6}
0.48	0.498646	0.4986436927330889	1.81246×10^{-6}	3.63477×10^{-6}
0.64	0.684594	0.6845922814317368	1.94620×10^{-6}	2.84285×10^{-6}
0.80	0.888106	0.8881044338725901	1.54832×10^{-6}	1.74339×10^{-6}
0.96	1.114400	1.1144013728752382	4.20849×10^{-7}	3.77645×10^{-7}

As we mentioned earlier, it is possible to pick any point in $[0, 1]$ and as well the approximate solutions and all their derivatives up to order two will be applicable. However, Table 3 has new numerical results for Example 5.2 which include the absolute error at some selected gird nodes in $[0, 1]$.

Table 3. Absolute error of Example 5.2.

	$x = 0.16$	$x = 0.48$	$x = 0.64$	$x = 0.96$
$u'_{51}(x)$	4.32658×10^{-6}	2.1107×10^{-6}	6.26721×10^{-7}	9.76649×10^{-6}
$u''_{51}(x)$	1.19349×10^{-5}	1.33227×10^{-5}	2.66454×10^{-5}	1.55431×10^{-5}

6 Conclusion

In this paper,we construct a reproducing kernel space in which each function satisfies boundary value conditions of considered problems. In this space, a numerical algorithm is presented for solving second-order IDEs of Volterra type. The analytical solution is given with series form in $W_2^3[0, 1]$. The approximate solution obtained by present algorithm converges to analytical solution uniformly. The numerical results are displayed to demonstrate the validity of the present algorithm.

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Received: June 15, 2013