Entanglement of a two-particle Gaussian state interacting with a heat bath

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Abstract

The effect of a thermal reservoir is investigated on a bipartite Gaussian state. We derive a pre-Lindblad master equation in the non-rotating wave approximation for the system. We then solve the master equation for a bipartite harmonic oscillator Hamiltonian with entangled initial state. We show that for strong damping the loss of entanglement is the same as for freely evolving particles. However, if the damping is small, the entanglement is shown to oscillate and eventually tend to a constant nonzero value.

Keywords: entanglement, Gaussian state, master equation

Motivation

Entanglement is one of quantum mechanics' most fascinating features. It was first described in a celebrated paper by Einstein, Podolsky and Rosen [Einstein et al.(1935)Einstein, Podolsky, and Rosen] but owes its name to Schrödinger [Schrödinger(1935)], who investigated its broader significance for the measurement question. It has taken on enhanced significance in quantum information. In this regard, the fragility of entanglement when the system is subjected to "outside" influence is of even greater importance. In the current work, we study a bipartite system with a Gaussian wave function. The state is prepared such that it is entangled, then shared between two parties who let their respective particle evolve either freely or interacting via a harmonic potential, but interacting with its own environment or

heat bath. We study the resulting loss of entanglement between the particles. To do so, we use the pre-Lindblad non-rotating-wave master equation, [Munro and Gardiner(1996), Gardiner and Zoller(2000)], for which we outline a simple perturbative derivation starting with the Quantum Langevin Equation as derived in [Ford et al.(1988)Ford, Lewis, and O'Connell] and using a simple perturbation method as in [Ghesquière(2009)].

The loss of entanglement in a system interacting with an environment is a well-known phenomenon. It has been studied in various systems, see e.g. [Yu and Eberly(2003), Yu and Eberly(2004), Yu and Eberly(2006), Pratt and Eberly(2001), Diósi(2003), Roszak and Machnikowski(2006)], where it was found that there is often a sharp loss of entanglement when compared to a decoherence time scale, which has been termed entanglement sudden-death (E.S.D.). These studies are mainly in the context of qubits and the Rotating Wave Approximation (R.W.A.). The R.W.A. is obtained by discarding the fast oscillating terms in the equations of motion. This appproximation works well for weak coupling and systems with well-spaced energy levels. However, we wish to consider a more general setting and as such this work presents a study of E.S.D. in a continuous-variables setting using the Non-Rotating-Wave (N.R.W) approximation. Note that the master equation obtained in the N.R.W approximation is not of the Lindblad form [Lindblad(1976)], hence does not in general satisfy the complete positivity condition. Yet, because the physical limits of the validity of this property are not well-understood [Vacchini(2000)], complete positivity alone does not ensure physicality of the result and one can easily check for the validity of the density matrix by checking its positive semi-definiteness. At the same time, the N.R.W master equation often works better for systems which are strongly coupled to the environment [Munro and Gardiner(1996)]. Moreover, the unphysical behaviour occurs for low temperatures only. Caldeira and Leggett [Caldeira and Leggett(1983)] have derived a pre-Lindblad equation using a path-integrals method which is presumably not perturbative. We present a simple perturbative derivation of the N.R.W. master equation in **??**. Diósi [Diòsi(1993a), Diòsi(1993b)] has generalised the Caldeira-Leggett derivation to obtain a more complicated equation which is valid for a range of low temperatures.

The choice of a continuous variables setting allows for a more realistic study of the evolution of the state of the chosen system. Gaussian states form a class of continuous variable states which is becoming more and more essential to the field of quantum optics. Indeed, their ease of ex-

perimental manipulation makes them very attractive for quantum information processing [Ferraro et al.(2005)Ferraro, Olivares, and Paris]. Gaussian states have also been widely studied analytically in the context of a system coupled to a heat bath, see e.g. [Xiang et al.(2008)Xiang, Shao, and Song, Serafini et al.(2004)Serafini, Illuminati, Paris, and Siena, Prauzner-BechcickiJ(2004), Serafini et al.(2005)Serafini, Paris, Illuminati, and Siena, Vasile et al.(2009)Vasile, Olivares, Pa to cite but a few. In [Vasile et al.(2009)Vasile, Olivares, Paris, and Maniscalco] in particular, Vasile et al. study two non interacting quantum harmonic oscillators, coupled to two independent structured reservoir, examining various spectral densities for the bath. In [Ficek and Tanás(2006)], Ficek and Tanás study a system of two qubits coupled to a radiation field where they allow spontaneous decay of the atoms. They show that the entanglement vanishes but then is revived twice. In [Ficek and Tanás(2008)], the authors study the emergence of entanglement between two initially non-entangled qubits due to spontaneous emission, provided both atoms are initially excited and in the asymmetric state. Their results suggest that an interaction between two particles which are initially entangled can delay the vanishing of the entanglement and even revive it, or create entanglement between two initially non-entangled particles. We introduce a harmonic potential with frequency ω_0 as the interaction between the particles in our system and examine the dynamics of the entanglement. We show that entanglement revival can occur depending on the strength of the damping, i.e. how strong the coupling γ is with respect to the oscillator's frequency. We show that if the damping is small ($\gamma < 2\sqrt{2}\omega_0$), the entanglement eventually tends towards a limiting value and does not vanish asymptotically.

In Section 1 we recall the Langevin equation and present the main steps in the derivation of the master equation. We then recall, in Section 2, the formalism used to describe Gaussian states and the particular measure for entanglement we use. Section 3 considers free evolution, illustrating E.S.D. while Section 4 considers a harmonic interaction. Section 5 contains some concluding remarks.

1 Framework

In the following we outline very briefly a perturbative derivation of the NRW Master Equation used here. Further details are in **??**. The derivation is given for one particle but generalises easily to the case of two particles, each coupled

to its own environment. We consider a heat bath modelled by independent oscillators coupled harmonically to the particle [Ford et al.(1988)Ford, Lewis, and O'Connell]. The corresponding Hamiltonian has the form

$$H = \frac{p^2}{2m} + V(x) + \frac{1}{2} \sum_j \left\{ \frac{p_j^2}{m_j} + m_j \omega_j^2 (q_j - x)^2 \right\} \tag{1}$$

Solving the Heisenberg equations of motion for q_j yields the Quantum Langevin Equation (see [Ford et al.(1988)Ford, Lewis, and O'Connell])

$$m\ddot{x} + \int_{-\infty}^{t} \mu(t - t')\dot{x}(t')\,dt' + V'(x) \;=\; \xi(t) \tag{2}$$

where the dot denotes the derivative with respect to time and the prime that with respect to x. $\mu(t)$ and $\xi(t)$ describe the influence of the bath on the system and are known as the *memory function* and the operator-valued *random force* respectively and are expressed explicitly in **??**. In the case of a Ohmic heat bath, $\mu(t)$ effectively reduces to a constant γ. The Quantum Langevin Equation for a general observable Y of the small system (particle) then reads

$$\dot{Y} = \frac{i}{\hbar}\left[H_s, Y\right] - \frac{i}{2\hbar}\left[\left[x, Y\right], \xi(t)\right]_+ + \frac{i\gamma}{2\hbar}\left[\left[x, Y\right], \dot{x}(t)\right]_+. \tag{3}$$

This equation is an equation for the system operators (Heisenberg representation), whereas a Master Equation is an (approximate) equation acting on the density operator of the quantum system under study (Schrödinger picture). The adjoint equation provides a link between the two formalisms:

$$\mathrm{Tr}\left\{Y(t)\rho\right\} = \mathrm{Tr}\left\{Y\rho(t)\right\}, \tag{4}$$

where Tr denotes the trace. Inserting (40), we obtain

$$\begin{aligned}
\dot{\rho}(t) \;&=\; -\frac{i}{\hbar}\left[H_s, \rho(t)\right] - \frac{i}{2\hbar}\left[\left[\xi(t), \rho(t)\right]_+, x\right] \\
&\quad + \frac{i\gamma}{2\hbar}\left[\left[\dot{x}, \rho(t)\right]_+, x\right]
\end{aligned} \tag{5}$$

In order to derive the Master Equation from this adjoint equation, we assume that the bath is large and hence stays at thermal equilibrium, and that for $t \to -\infty$, the system and the bath are decoupled so that $\rho(t) \sim \rho_s(t)\rho_B$.

This assumption is critical to the derivation of any Master Equation. Finally, assuming that the noise is small we write $\xi(t) \to \epsilon\xi(t)$, where ϵ is a small parameter. (This assumption is in fact not essential to the result but allows for a simpler derivation.) Applying a perturbation method and tracing over the bath yields the Non-Rotating-Wave Master Equation for $\rho_s(t)$ (See **??**)

$$\dot{\rho}_s(t) = -\frac{i}{\hbar} [H_s, \rho(t)] + \frac{i\gamma}{2\hbar} \left[[\dot{x}, \rho_s(t)]_+ , x \right]$$
$$- \frac{kT\gamma}{\hbar^2} \left[[\rho_s(t), x] , x \right].$$

(In position space this equation agrees with (5.10) in [Caldeira and Leggett(1983)].) This equation generalises in an obvious way to the case of two particles, each in its own heat bath:

$$\dot{\rho}(t) = -\frac{i}{\hbar} [H_s, \rho(t)]$$
$$+ \frac{i\gamma_1}{2\hbar} \left[[\dot{x}_1, \rho(t)]_+ , x_1 \right] + \frac{i\gamma_2}{2\hbar} \left[[\dot{x}_2, \rho(t)]_+ , x_2 \right]$$
$$- \frac{kT_1\gamma_1}{\hbar^2} \left[[\rho(t), x_1] , x_1 \right] - \frac{kT_2\gamma_2}{\hbar^2} \left[[\rho(t), x_2] , x_2 \right].$$
$$(6)$$

(Here we have omitted the subscript s. γ_1 and γ_2 are the coupling parameters for the individual heat baths and T_1 and T_2 are the temperatures of the baths.)

2 Gaussian states and the logarithmic negativity

Since the states we will study are Gaussian, we now briefly recall the formalism for Gaussian states [Anders(2003), Eisert and Plenio(2003), Plenio et al.(2004)Plenio, Hartl

Gaussian states can be completely specified in terms of their first and second moments, described respectively by the displacement vector

$$d_j = \langle R_j \rangle_\rho = \text{Tr}[R_j \rho]$$

and the covariance matrix

$$\Gamma_{j,k} = 2 \, \text{Re} \, \text{Tr} \left[\rho(R_j - \langle R_j \rangle_\rho)(R_k - \langle R_k \rangle_\rho) \right]$$

where R is the vector $R^T = (q_1, p_1; ....; q_n, p_n)$; q_j and p_j are the canonical variables of a system of n oscillators with the usual canonical relations written as $[R_j, R_k] = i\hbar\sigma_{jk}$ and σ a real skew-symmetric $2n \times 2n$ block matrix given by

$$\sigma = \bigoplus_{k=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The displacement vector is irrelevant in the study of entanglement and is taken to be zero in our examples. The covariance matrix thus reduces to

$$\Gamma_{j,k} = 2 \operatorname{Re} \operatorname{Tr} [\rho R_j R_k] \tag{7}$$

Any real symmetric positive-definite matrix A can be brought to its Williamson normal form [Williamson(1936)] via symplectic transformations, i.e. transformations that preserve the canonical commutation relations, $A_{WF} = SAS^T = \operatorname{diag}(a_1, a_1, ...a_n, a_n)$ where the a_i's are the symplectic eigenvalues of A. One can calculate them as the positive eigenvalues of $i\sigma A$ or more simply as the positive square root of the eigenvalues of $-\sigma A \sigma A$.

A particularly suitable measure of the entanglement of mixed Gaussian states is the logarithmic negativity [Vidal and Werner(2002), Anders(2003), Eisert and Plenio(2003), Plenio et al.(2004)Plenio, Hartley, and Eisert]. It vanishes for separable states, does not increase under LOCC (local operations and classical communication), and stays invariant under local unitary transformations. It is defined as

$$\mathcal{E}_\mathcal{N}(\rho) = -\sum_{i=1}^{2n} \log_2 \left(\min\left(1, |\lambda_i|\right)\right), \tag{8}$$

where the λ_i are the symplectic eigenvalues of the partially transposed covariance $\gamma^{(T_1)}$, which is obtained from $\gamma = \Gamma/\hbar$ by reversing the time in all variables of one of the subsystems. Choosing to transpose with respect to particle 1, we replace $x_1 \to x_1$ and $p_1 \to -p_1$. The λ_i's are thus the square roots of the eigenvalues of $-\sigma\gamma^{T_1}\sigma\gamma^{T_1}$.

3 Free evolution of an entangled initial state

Let us consider a bipartite initial state with the Gaussian wavefunction, as suggested by Ford and O'Connell [Ford and O'Connell(2008), Ford et al.(2010)Ford, Gao, and O

$$\Psi(x_1, x_2) = \Omega^{1/2} e^{-\frac{(x_1-x_2)^2}{4s^2}} e^{-\frac{(x_1+x_2)^2}{16d^2}} \tag{9}$$

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where s is the mean distance between the particles and d is the width of the center-of-mass system and $\Omega = \frac{1}{2\pi sd}$ is a normalisation factor.

We first consider the case of a free-particle Hamitonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} \tag{10}$$

with $p|x\rangle = -i\hbar\frac{\partial}{\partial x}|x\rangle$. This will allow us to examine the dynamics of the entanglement when an entangled bipartite Gaussian state is left to evolve, each particle coupled to its own heat bath. We highlight here the main steps in solving (6). The full solution may be found in B. In position-space, (6) becomes

$$\begin{aligned}
\frac{\partial}{\partial t}\langle x| \rho |y\rangle &= \frac{i\hbar}{2m}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2}\right)\rho \\
&\quad - \frac{\gamma_1}{2m}(x_1 - y_1)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}\right)\rho \\
&\quad - \frac{\gamma_2}{2m}(x_2 - y_2)\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}\right)\rho \\
&\quad - \frac{\gamma_1 kT_1}{\hbar^2}(x_1 - y_1)^2\rho - \frac{\gamma_2 kT_2}{\hbar^2}(x_2 - y_2)^2\rho
\end{aligned} \tag{11}$$

After performing the change of variables

$$\mathbf{x} = \mathbf{u} + \hbar\mathbf{z} \text{ and } \mathbf{y} = \mathbf{u} - \hbar\mathbf{z}, \tag{12}$$

and replacing $\rho(\mathbf{x}, \mathbf{y}, 0) \to P(\mathbf{u}, \mathbf{z}, 0)$, we apply a Fourier transformation with respect to $\mathbf{u}$:

$$\tilde{P}(\mathbf{q}, \mathbf{z}, t) = \int P(\mathbf{u}, \mathbf{z}, t)e^{-iq_1u_1 - iq_2u_2}du_1\,du_2 \tag{13}$$

to obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{P}(\mathbf{q}, \mathbf{z}, t) &= -\left[\left(\frac{\gamma_1}{m}z_1 + \frac{q_1}{2m}\right)\frac{\partial}{\partial z_1} + 4\gamma_1 kT_1 z_1^2\right]\tilde{P}(\mathbf{q}, \mathbf{z}, t) \\
&\quad - \left[\left(\frac{\gamma_2}{m}z_2 + \frac{q_2}{2m}\right)\frac{\partial}{\partial z_2} + 4\gamma_2 kT_2 z_2^2\right]\tilde{P}(\mathbf{q}, \mathbf{z}, t).
\end{aligned} \tag{14}$$

This first order equation can be solved with the method of characteristics. The solution is

$$\tilde{P}(\mathbf{q}, \mathbf{z}, t) = \tilde{P}(\mathbf{q}, \mathbf{z}_0(t), 0) \exp\left[-\tau_1\, q_1^2\, t - \tau_2\, q_2^2\, t\right]$$

$$\times \exp\left[-\lambda_1(t)\left(z_1 + \frac{q_1}{2\gamma_1}\right)^2 - \lambda_2(t)\left(z_2 + \frac{q_2}{2\gamma_2}\right)^2\right.$$

$$\left. +\alpha_1(t)\left(z_1 + \frac{q_1}{2\gamma_1}\right) + \alpha_2(t)\left(z_1 + \frac{q_2}{2\gamma_2}\right)\right] \tag{15}$$

where

$$\tau_i = \frac{kT_i}{\gamma_i}, \quad \lambda_i(t) = 2mkT_i(1 - e^{-2\gamma_i t/m}), \quad \alpha_i(t) = \frac{4mkT_i}{\gamma_i}(1 - e^{-\gamma_i t/m})$$

$$z_{0,i}(t) = z_i\, e^{-\gamma_i t/m} - \frac{q_i}{2\gamma_i}\left(1 - e^{-\gamma_i t/m}\right) \tag{16}$$

We compute the time-evolved state by inserting the Fourier transform of the density operator corresponding to (12) (projection onto Ψ) $\rho_0 = |\Psi\rangle\langle\Psi|$ into (18):

$$\tilde{P}(\mathbf{q}, \mathbf{z}_0; 0)$$
$$= \exp\left[-\epsilon_+\hbar^2 z_{01}^2 - \epsilon_+\hbar^2 z_{02}^2 + 2\epsilon_-\hbar^2 z_{01}z_{02}\right]$$
$$\times \exp\left[-\frac{\epsilon_+(q_1^2 + q_2^2)}{4(\epsilon_+^2 - \epsilon_-^2)} - \frac{\epsilon_- q_1 q_2}{2(\epsilon_+^2 - \epsilon_-^2)}\right] \tag{17}$$

This eventually yields

$$\tilde{P}(\mathbf{q}, \mathbf{z}, t) = e^{-A_1 q_1^2 - A_2 q_2^2 - B_1 z_1^2 - B_2 z_2^2 - Dz_1 z_2 - E q_1 q_2}$$
$$\times e^{-C_{11}z_1 q_1 - C_{22}z_2 q_2 - C_{12}z_1 q_2 - C_{21}z_2 q_1}, \tag{18}$$

where the coefficients are given by

$$
\begin{aligned}
A_j &= \frac{d^2}{2} + \frac{s^2}{8} + \tau_j t - \frac{\alpha_j}{2\gamma_j} + \frac{\lambda_j}{4\gamma_j{}^2} + \frac{\hbar^2 \epsilon_+}{2\gamma_j{}^2}\left(1 - e^{-\frac{\gamma_j t}{m}}\right)^2 \\
B_j &= \hbar^2 \epsilon_+ \, e^{-\frac{2\gamma_j t}{m}} + \lambda_j \\
C_{jj} &= \frac{\lambda_j}{\gamma_j} - \alpha_j - \frac{\hbar^2 \epsilon_+}{\gamma_j}\, e^{-\frac{\gamma_j t}{m}}\left(1 - e^{-\frac{\gamma_j t}{m}}\right) \\
D &= -2\hbar^2 \epsilon_- \, e^{-\gamma_1 t/m}\, e^{-\gamma_2 t/m} \\
E &= d^2 - \frac{s^2}{4} - \frac{\hbar^2 \epsilon_-}{2\gamma_1 \gamma_2}(1 - e^{-\gamma_1 t/m})(1 - e^{-\gamma_2 t/m}) \\
C_{jk} &= \frac{\hbar^2 \epsilon_-}{\gamma_j}\, e^{-\frac{\gamma_j t}{m}}\left(1 - e^{-\frac{\gamma_k t}{m}}\right)
\end{aligned}
\tag{19}
$$

with

$$
\epsilon_\pm = \frac{1}{2s^2} \pm \frac{1}{8d^2}.
\tag{20}
$$

The entries of the covariance matrix can be calculated directly from (21) taking into account the change of variables (15):

$$
\begin{aligned}
2\,\mathrm{Re}\langle X_i X_j \rangle &= -2\left(\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}\tilde{P}(\mathbf{q}, \mathbf{z} = 0, t)\right)\big|_{\mathbf{q}=0} \\
2\,\mathrm{Re}\langle X_i P_j \rangle &= \frac{\partial}{\partial q_i}\frac{\partial}{\partial z_j}\tilde{P}(\mathbf{q}, \mathbf{z}, t)\big|_{\mathbf{q}=0, \mathbf{z}=0} \\
2\,\mathrm{Re}\langle P_i X_j \rangle &= \frac{\partial}{\partial z_i}\frac{\partial}{\partial q_j}\tilde{P}(\mathbf{q}, \mathbf{z}, t)\big|_{\mathbf{q}=0, \mathbf{z}=0} \\
2\,\mathrm{Re}\langle P_i P_j \rangle &= -\frac{1}{2}\left(\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_j}\tilde{P}(\mathbf{q} = 0, \mathbf{z}, t)\right)\big|_{\mathbf{z}=0}
\end{aligned}
$$

The covariance matrix is then

$$
\Gamma_t = \begin{pmatrix}
4A_1 & -C_{11} & E & -C_{21} \\
-C_{11} & B_1 & -C_{12} & D \\
E & -C_{12} & 4A_2 & -C_{22} \\
-C_{21} & D & -C_{22} & B_2
\end{pmatrix}
\tag{21}
$$

To determine the entanglement (logarithmic negativity) we now perform

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a partial transposition with respect to particle 1:

$$\Gamma_t^{T_1} = \begin{pmatrix} 4A_1 & C_{11} & E & -C_{21} \\ C_{11} & B_1 & C_{12} & -D \\ E & C_{12} & 4A_2 & -C_{22} \\ -C_{21} & -D & -C_{22} & B_2 \end{pmatrix} \tag{22}$$

This matrix is real and symmetric. The squares of its symplectic eigenvalues are given by

$$\lambda_\pm^2 = \frac{\varepsilon_{11} + \varepsilon_{33}}{2} \pm \frac{1}{2} \sqrt{(\varepsilon_{11} - \varepsilon_{33})^2 + 4\varepsilon_{13}\varepsilon_{24} - 4\varepsilon_{14}\varepsilon_{23}} \tag{23}$$

where $\varepsilon_{12} = \varepsilon_{21} = \varepsilon_{34} = \varepsilon_{43} = 0$ and

$$
\begin{aligned}
\varepsilon_{11} = \varepsilon_{22} = & \quad 4A_1B_1 - DE + C_{12}C_{21} - C_{11}^2 \\
\varepsilon_{33} = \varepsilon_{44} = & \quad 4A_2B_2 - C_{22}^2 - DE + C_{12}C_{21} \\
\varepsilon_{13} = \varepsilon_{42} = & \quad EB_1 - 4A_2D - C_{11}C_{12} + C_{12}C_{22} \\
\varepsilon_{14} = -\varepsilon_{32} = & \quad -C_{12}B_2 - C_{21}B_1 + C_{11}D + C_{22}D \\
\varepsilon_{23} = -\varepsilon_{41} = & \quad -EC_{11} + 4A_1C_{12} + 4A_2C_{21} - EC_{22} \\
\varepsilon_{24} = \varepsilon_{31} = & \quad EB_2 - C_{22}C_{21} + C_{11}C_{21} - 4A_1D
\end{aligned}
\tag{24}
$$

The logarithmic negativity is thus

$$\mathcal{E}_\mathcal{N}(\rho) = -2 \left(\log_2 \left(\min(1, |\lambda_+|/\hbar) \right) + \log_2 \left(\min(1, |\lambda_-|/\hbar) \right) \right). \tag{25}$$

Figure 1 shows the logarithmic negativity as a function of time for three values of s. We have chosen $\gamma_1 = \gamma_2 = \gamma$ and $T_1 = T_2 = T$ for simplicity and set the units by taking $\hbar = 1$ and $m = 1$. We can observe that there is complete disentanglement between the particles from a sharp cut-off time onwards, which obviously depends on s, and hence on the initial degree of entanglement. The sharp cut-off time characterizes entanglement sudden death (ESD). Note that this follows from the linear behaviour in time of the eigenvalues which results from the terms $\tau_j t$ in the expressions for A_j. This linear increase means that the eigenvalues increase beyond 1 for large t. Figure 2 shows the logarithmic negativity as a function of s for three different times. One can see that at $t = 0$, the entanglement is present everywhere except where $s = 2d$ and that the range of s around $2d$ for which there is no
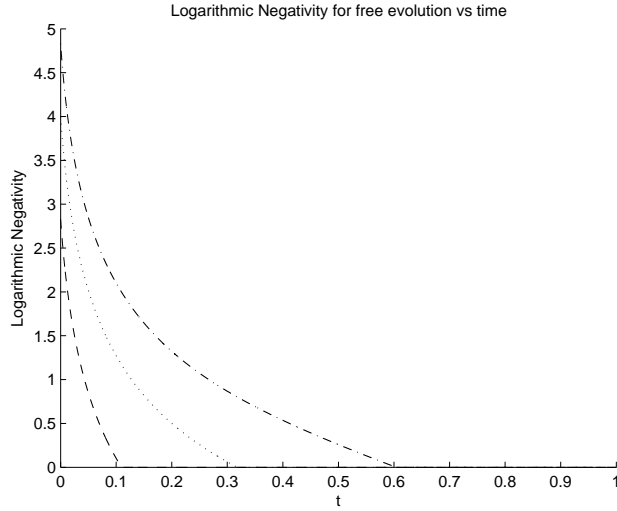
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Figure 1: Logarithmic negativity as a function of time for three values of s and $d = 2$, $\gamma = 1$, $T = 1$, $m = 1$. The values of s are : dashed $s = 0.75$, dotted $s = 1$, dash-dotted $s = 2$.

entanglement increases as as t increases. (For the initial state, the eigenvalues $\lambda_\pm$ are obviously easy to compute:

$$
\begin{aligned}
|\lambda_+|^2 &= \frac{1}{2}\left|\frac{9s^2}{16d^2} - \frac{d^2}{s^2}\right| \text{ and} \\
|\lambda_-|^2 &= \frac{1}{2}\left|2 + \frac{3d^2}{s^2} - \frac{3s^2}{16d^2}\right|.
\end{aligned}
$$

In particular, for $s = 2d$, $\lambda_\pm = 1$ and the log-negativity vanishes.)

4 Evolution with a harmonic potential interaction

We recall the initial state

$$
\Psi(x_1, x_2) = \Omega^{1/2} e^{-\frac{(x_1 - x_2)^2}{4s^2}} e^{-\frac{(x_1 + x_2)^2}{16d^2}} . \tag{26}
$$

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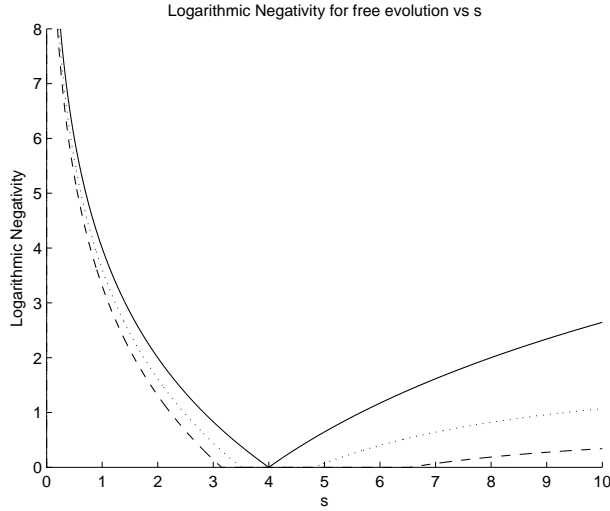Figure 2: Logarithmic negativity vs s for three values of t, $d = 2$, $\gamma = 1$, $T = 1$, $m = 1$. The values of t are : dashed $t = 0.01$, dotted $t = 0.005$, full $t = 0$.

If we introduce a harmonic potential interaction into the Hamiltonian, (13) generalises to

$$H_s = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega_0^2}{2}(x_1 - x_2)^2 \tag{27}$$

We can include this into (6) and solve the resulting differential equation following the method described in B. Again we choose $\gamma_1 = \gamma_2 = \gamma$, $T_1 = T_2 = T$ for simplicity. The solution is of the same form

$$
\begin{aligned}
\tilde{P} = \quad & \exp\big[-Aq_1^2 - Aq_2^2 - Eq_1q_2 - Bz_1^2 - Bz_2^2 - Dz_1z_2 \\
& -C_1z_1q_1 - C_1z_2q_2 - C_2z_1q_2 - C_2z_2q_1\big],
\end{aligned}
\tag{28}
$$

but the coefficients A, B, C_1, C_2, D and E are considerably more complicated: see Appendix B.

Figure 3 illustrates that, in the presence of a harmonic interaction between the particles, there is a marked difference in behaviour between two damping regimes. In the over-damped case ($\gamma > 2\sqrt{2}\omega_0$), the difference between the
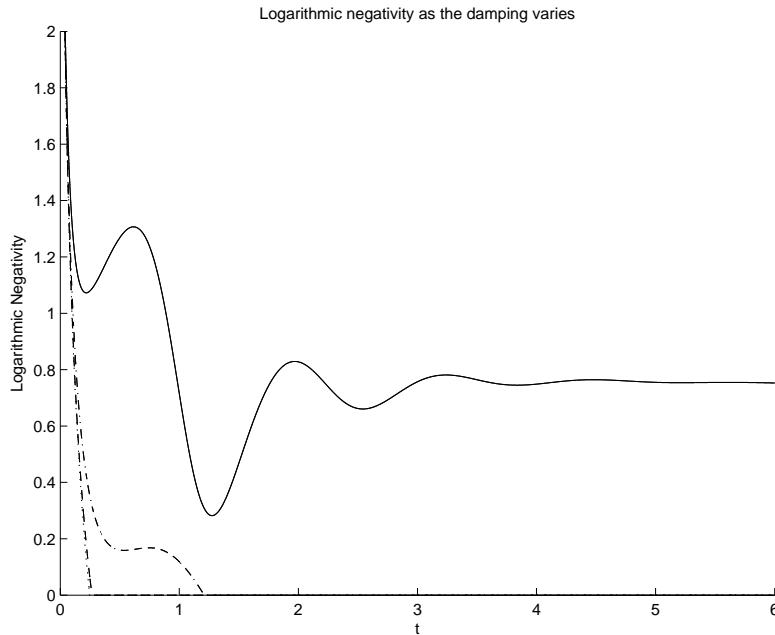
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Figure 3: $\mathcal{E}$ as we vary ω_0

The plots are obtained with $T = 1$, $s = 1$, $d = 2$, $\gamma = 1.3$ and : full $\omega_0 = 1.8$, dashed $\omega_0 = 1.3$, dash-dotted $\omega_0 = 1$ and dotted $\omega_0 = 0.5$

graphs is so small as to be almost invisible. This shows that if the coupling is much stronger than the harmonic potential, the decay of the entanglement is unaffected by the potential. On the other hand, if the damping is small $(\gamma < 2\sqrt{2}\omega_0)$, the entanglement can reappear several times. Furthermore, it can be easily seen that as the harmonic potential becomes stronger, the entanglement does not disappear. Instead it decreases sharply before being "restored". It then tends towards a non-zero constant value for large times. This suggests that allowing the particles to interact harmonically effectively can save the entanglement. Figure 4 allows us to determine at which point the system becomes under-damped enough that the entanglement survives. We can see that as the coupling with the environment becomes smaller, the entanglement is restored.

One may want to note that our choice of temperatures $T_1 = T_2 = T$, means that we are studying a system at equilibrium. The oscillations in
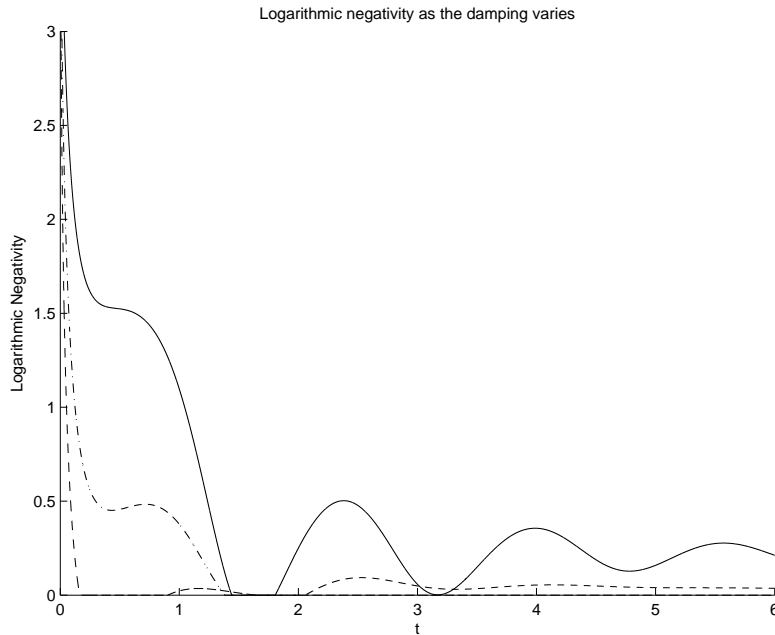
13

Figure 4: $\mathcal{E}$ as we vary γ

The plots are obtained with $T = 1$, $s = 1$, $d = 2$, $\omega_0 = 1.4$ and : full $\gamma = 0.5$, dash-dotted $\gamma = 1.2$ and dashed $\gamma = 2.5$

the evolution of the entanglement are therefore, a direct consequence of the interaction between the two particles. Figure 5 illustrate that for a given d, the width of the center-of-mass system, the entanglement evolves towards the same value, regardless of s, the distance between the particles. Also, one can easily see that the amplitude of the oscillations increases as s increases. This is a consequence of the harmonic term in the Hamiltonian. However, this increase in amplitude also result in the entanglement vanishing for some time before being restored. This suggests that to maintain entanglement at all times, the distance between the particles must be small.

5   Concluding observations

The evolution of the entanglement within a bipartite system, coupled to two heat baths ate equilibrium, was studied. We found that when the system
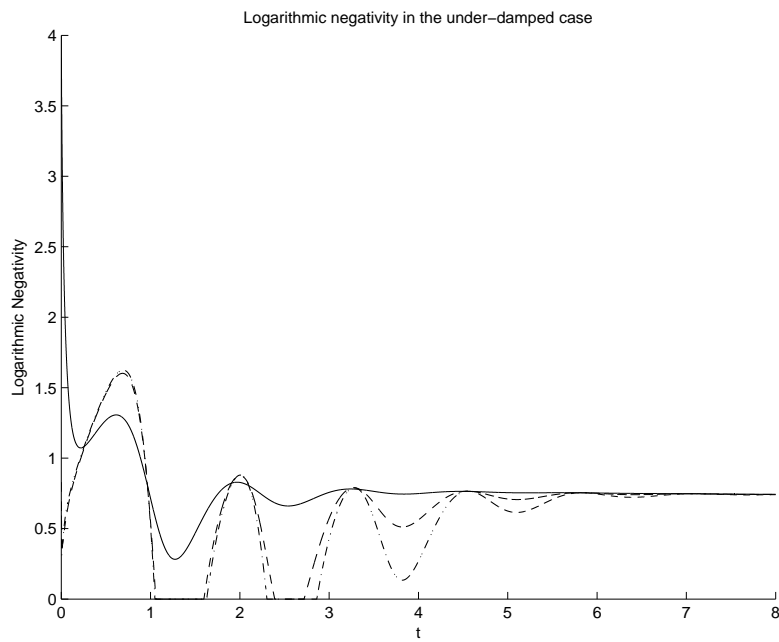
14

Figure 5: $\mathcal{E}$ as we vary s

The plots are obtained with $T = 1$, $s = 1$, $d = 2$, $\omega_0 = 1.4$ and : full $\gamma = 0.5$, dash-dotted $\gamma = 1.2$ and dashed $\gamma = 2.5$

is left to evolve freely, the coupling with the reservoirs destroys the entanglement in a Entanglement-Sudden-Death fashion. To counteract this effect, we introduced a harmonic potential between the particles. Two very distinct behaviours were observed. In the over-damped case, the entanglement vanishes following an ESD curve. However, in the under-damped situation, the entanglement is revived and tends towards a constant valu..

Acknowledgments

A   The Master equation

In this section, we will describe the derivation of the master equation in more details. A simple derivation of the Quantum Langevin Equation, starting from a heat bath modelled by independent oscillators coupled harmonically to a system of one particle, was given in [Ford et al.(1988)Ford, Lewis, and O'Connell]. The Hamiltonian has the form

$$H = \frac{p^2}{2m} + V(x) + \frac{1}{2} \sum_j \left\{ \frac{p_j^2}{m_j} + m_j \omega_j^2 (q_j - x)^2 \right\} \tag{29}$$

Solving the Heisenberg equations of motion yields

$$
\begin{aligned}
q_j(t) &= q_j^h(t) + x(t) - \int_{-\infty}^{t} \cos\left[\omega_j(t - t')\right] \dot{x}(t') \, dt' \\
q_j^h(t) &= q_j \cos(\omega_j t) + \frac{p_j}{\omega_j m_j} \sin(\omega_j t).
\end{aligned}
\tag{30}
$$

Introducing the quantities

$$
\begin{aligned}
\mu(t) &= \sum_j m_j \omega_j^2 \cos(\omega_j t) \Theta(t) \\
\xi(t) &= \sum_j m_j \omega_j^2 q_j^h(t)
\end{aligned}
\tag{31}
$$

($\Theta(t)$ is the Heaviside function) we obtain the Quantum Langevin Equation

$$m\ddot{x}_k + \int_{-\infty}^{t} \mu_k(t - t')\dot{x}_k(t')\, dt' + V'(x_k) \;\; = \;\; \xi_k(t) \tag{32}$$

where the dot denotes the derivative with respect to time and the prime that with respect to x. $\mu(t)$ and $\xi(t)$ describe the influence of the bath on the system and are known as the *memory function* and the *operator-valued random force* respectively. We also introduce the *spectral distribution*

$$\begin{aligned} G(\omega) \;\; &= \;\; \mathfrak{Re}\left[\tilde{\mu}(\omega + \mathrm{i}0^{+})\right] \\ &= \;\; \frac{\pi}{2}\sum_j m_j\omega_j^2\left[\delta(\omega - \omega_j) + \delta(\omega + \omega_j)\right], \end{aligned} \tag{33}$$

in terms of which the autocorrelation of $\xi(t)$ is given by

$$\frac{1}{2}\langle[\xi(t), \xi(t')]_{+}\rangle = \\ \frac{1}{\pi}\int_{0}^{\infty} G(\omega)\hbar\omega \coth\left(\frac{\hbar\omega}{2kT}\right)\cos\left[\omega(t - t')\right]\, d\omega \tag{34}$$

where $[\,,\,]_{+}$ denotes the anticommutator. For a general observable Y of the small system (particle), one can write

$$\begin{aligned} \dot{Y} \;\; &= \;\; \frac{\mathrm{i}}{\hbar}[H, Y] \\ &= \;\; \frac{\mathrm{i}}{\hbar}[H_s, Y] - \frac{\mathrm{i}}{2\hbar}[[x, Y], \xi(t)]_{+} \\ &\quad + \frac{\mathrm{i}}{2\hbar}\left[[x, Y], \int_{-\infty}^{t} dt'\, \mu(t - t')\dot{x}(t')\right]_{+} \end{aligned} \tag{35}$$

In the case of an Ohmic heat bath, we can replace

$$\int_{-\infty}^{t} \mu(t')\dot{x}(t')\, dt' \to \gamma\dot{x}(t) \qquad \text{and} \qquad G(\omega) \to \gamma \tag{36}$$

so that the Quantum Langevin Equation reads

$$\dot{Y} = \frac{\mathrm{i}}{\hbar}[H_s, Y] - \frac{\mathrm{i}}{2\hbar}[[x, Y], \xi(t)]_{+} + \frac{\mathrm{i}\gamma}{2\hbar}[[x, Y], \dot{x}(t)]_{+}. \tag{37}$$

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If we define

$$\text{Tr}_s \{Y(t)\rho\} = \text{Tr}_s \{Y\rho(t)\} \tag{38}$$

where Tr_s is the trace over the system. Let us introduce $\rho(t) = \rho_s(t) \otimes \rho_B$ where ρ_s is the density matrix of the system and ρ_B that of the bath. It follows easily from (40) that $\rho(t)$ satisfies the adjoint equation

$$
\begin{aligned}
\dot{\rho}(t) &= -\frac{i}{\hbar}[H_s, \rho(t)] - \frac{i}{2\hbar}\left[[\xi(t), \rho(t)]_+, x\right] \\
&\quad + \frac{i\gamma}{2\hbar}\left[[\dot{x}, \rho(t)]_+, x\right]
\end{aligned}
\tag{39}
$$

To derive the master equation (that is, an effective equation for $\rho_s(t)$) from the adjoint equation, we assume that the noise is small and temporarily introduce a small parameter ϵ, replacing $\xi(t)$ by $\epsilon\xi(t)$. This allows us to derive the master equation in a perturbative manner, which has the advantage of being simpler than that of Caldeira and Leggett [Caldeira and Leggett(1983)]. We can write $\nu(t)$ to second order in ϵ as

$$\nu(t) = \nu_0(t) + \epsilon\nu_1(t) + \epsilon^2\nu_2(t)$$

We also assume the baths and the system are decoupled at $t = -\infty$, so that $\rho_0(t) = \rho_0(t)\rho_B$. Inserting this expansion into (42) yields equations for ρ_0, ρ_1 and ρ_2 which can be solved successively. The equation for ρ_0 reads

$$\dot{\rho}_0(t) = -\frac{i}{\hbar}[H_s, \rho_0(t)] + \frac{i\gamma}{2\hbar}\left[[\dot{x}, \rho_0(t)]_+, x\right]. \tag{40}$$

The equation for ρ_1 can be written as

$$
\begin{aligned}
\dot{\rho}_1(t) &= -\frac{i}{\hbar}[H_s, \rho_1(t)] + \frac{i\gamma}{2\hbar}\left[[\dot{x}, \rho_1(t)]_+, x\right] \\
&\quad - \frac{i}{2\hbar}[\xi(t), \rho_B]_+[\rho_0(t), x].
\end{aligned}
\tag{41}
$$

The solution for the first-order term can be written as

$$\rho_1(t) = -\frac{i}{2\hbar}\int_{-\infty}^{t} e^{A_s(t-t')}\left\{[\rho_0(t'), x] \otimes [\xi(t'), \rho_B]_+\right\} dt'$$

where A_s is a super operator which, applied to $\rho_k(t)$ yields

$$A_s\rho_k(t) = -\frac{\mathrm{i}}{\hbar}\left[H_s, \rho_k(t)\right] + \frac{\mathrm{i}\gamma}{2\hbar}\left[\left[\dot{x}, \rho_k(t)\right]_+, x\right].$$
(42)

Finally, we insert this solution into the equation for ρ_2:

$$\dot{\rho}_2(t) = -\frac{\mathrm{i}}{\hbar}[H_s, \rho_2(t)] + \frac{\mathrm{i}\gamma}{2\hbar}\left[\left[\dot{x}, \rho_2(t)\right]_+, x\right] - \frac{\mathrm{i}}{2\hbar}\left[\left[\xi(t), \rho_1(t)\right]_+, x\right]$$
(43)

Taking the trace over the bath variables and using the autocorrelation (37) in the Ohmic limit which becomes at high temperatures,

$$\frac{1}{2}\langle\left[\xi(t), \xi(t')\right]_+\rangle \to 2kT\gamma\delta(t - t'),$$
(44)

we obtain the Non-Rotating-Wave Master Equation (upon removal of ϵ)

$$\dot{\rho}_s(t) = -\frac{\mathrm{i}}{\hbar}\left[H_s, \rho_s(t)\right]s + \frac{\mathrm{i}\gamma}{2\hbar}\left[\left[\dot{x}, \rho_s(t)\right]_+, x\right] - \frac{kT\gamma}{\hbar^2}\left[\left[\rho_s(t), x\right], x\right].$$
(45)

The equation for two particles, where each particle is coupled to its own heat bath, is an obvious generalisation: see (6).

B   Solution of the Master Equation

The derivation of the solution to the master equation will here be given in a general way. The method will be described for a harmonic potential Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega_0^2}{2}(x_1 - x_2)^2$$
(46)

with $p|x\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial x}|x\rangle$, but is easily generalised to other types of Hamiltonians. Note that we assume that the particles have the same mass. In

position-space, (6) becomes

$$
\begin{aligned}
\frac{\partial}{\partial t} \langle x | \rho | y \rangle \;=\;& \frac{\mathrm{i}\hbar}{2m} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \rho \\
& - \frac{\gamma_1}{2m} (x_1 - y_1) \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1} \right) \rho \\
& - \frac{\gamma_2}{2m} (x_2 - y_2) \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2} \right) \rho \\
& - \frac{\mathrm{i} m \omega_0^2}{2\hbar} \left((x_1 - x_2)^2 - (y_1 - y_2)^2 \right) \rho \\
& - \frac{\gamma_1 k T_1}{\hbar^2} (x_1 - y_1)^2 \rho - \frac{\gamma_2 k T_2}{\hbar^2} (x_2 - y_2)^2 \rho
\end{aligned}
\tag{47}
$$

Using the change of variables (15) and replacing $\rho(\mathbf{x}, \mathbf{x'}, 0) \to P(\mathbf{u}, \mathbf{z}, 0)$, we apply a Fourier transformation with respect to $\mathbf{u}$:

$$
\tilde{P}(\mathbf{q}, \mathbf{z}, t) = \frac{1}{4\pi^2} \int P(\mathbf{u}, \mathbf{z}, t) e^{-\mathrm{i} q_1 u_1 - \mathrm{i} q_2 u_2} du_1 \, du_2
\tag{48}
$$

obtaining an equation for $\tilde{P}(\mathbf{q}, \mathbf{z}, t)$:

$$
\begin{aligned}
\frac{\partial}{\partial t} \tilde{P}(\mathbf{q}, \mathbf{z}, t) \;=\;& - \left[\left(\frac{\gamma_1}{m} z_1 + \frac{q_1}{2m} \right) \frac{\partial}{\partial z_1} + \left(\frac{\gamma_2}{m} z_2 + \frac{q_2}{2m} \right) \frac{\partial}{\partial z_2} \right] \tilde{P}(\mathbf{q}, \mathbf{z}, t) \\
& + \left[2 m \omega_0^2 \left(\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \right) (z_1 - z_2) \right] \tilde{P}(\mathbf{q}, \mathbf{z}, t) \\
& - \left[4 \gamma_1 k T_1 z_1^2 + 4 \gamma_2 k T_2 z_2^2 \right] \tilde{P}(\mathbf{q}, \mathbf{z}, t).
\end{aligned}
\tag{49}
$$

This equation can in principle again be solved using the method of characteristics. The characteristic equation is

$$
\frac{\partial \mathbf{v}}{\partial t} = \frac{M}{2m} \mathbf{v}
\tag{50}
$$

with $\mathbf{v} = (z_1, z_2, q_1, q_2)^T$ and

$$
M = \begin{pmatrix} 2\gamma_1 & 0 & 1 & 0 \\ 0 & 2\gamma_2 & 0 & 1 \\ -4m^2\omega_0^2 & 4m^2\omega_0^2 & 0 & 0 \\ 4m^2\omega_0^2 & -4m^2\omega_0^2 & 0 & 0 \end{pmatrix}
\tag{51}
$$

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On a characteristic,

$$\frac{d}{dt}\tilde{P}(\mathbf{q}, \mathbf{z}(t), t) = -[4\gamma_1 k T_1 z_1^2(t) + 4\gamma_2 k T_2 z_2^2(t)]\,\tilde{P}(\mathbf{q}, \mathbf{z}(t), t)$$

The eigenvalues and eigenvectors of M can be computed to be (we take $\gamma_1 = \gamma_2 = \gamma$ and $T_1 = T_2 = T$ for simplicity)

$$
\begin{aligned}
\boldsymbol{\lambda}^T &= (0, 2\gamma, \gamma + \sqrt{\gamma^2 - 8m^2\omega_0^2}, \gamma - \sqrt{\gamma^2 - 8m^2\omega_0^2}) \\
&= (\lambda_1, \lambda_2, \lambda_+, \lambda_-)
\end{aligned}
\tag{52}
$$

and

$$
Q = \begin{pmatrix}
-\frac{1}{2\gamma} & 1 & \frac{1}{\lambda_-} & \frac{1}{\lambda_+} \\
-\frac{1}{2\gamma} & 1 & -\frac{1}{\lambda_-} & -\frac{1}{\lambda_+} \\
1 & 0 & -1 & -1 \\
1 & 0 & 1 & 1
\end{pmatrix}
\tag{53}
$$

Since $Q^{-1}MQ = D$ where D is the diagonal matrix, we need Q^{-1} as

$$
Q^{-1} =
$$
$$
\begin{pmatrix}
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{4\gamma} & \frac{1}{4\gamma} \\
\frac{\lambda_+\lambda_-}{2(\lambda_+-\lambda_-)} & -\frac{\lambda_+\lambda_-}{2(\lambda_+-\lambda_-)} & \frac{\lambda_-}{2(\lambda_+-\lambda_-)} & -\frac{\lambda_-}{2(\lambda_+-\lambda_-)} \\
-\frac{\lambda_+\lambda_-}{2(\lambda_+-\lambda_-)} & \frac{\lambda_+\lambda_-}{2(\lambda_+-\lambda_-)} & -\frac{\lambda_+}{2(\lambda_+-\lambda_-)} & \frac{\lambda_+}{2(\lambda_+-\lambda_-)}
\end{pmatrix}
\tag{54}
$$

Then we can write $2m\frac{\partial \mathbf{w}}{\partial t} = D\mathbf{w}$ with $\mathbf{w} = Q^{-1}\mathbf{v}$ which is easily solved so that $\mathbf{v}(t) = Qe^{Dt/2m}Q^{-1}\mathbf{v}_0$ with

$$
e^{Dt/2m} = \begin{pmatrix}
1 & 0 & 0 & \\
0 & e^{\gamma t/m} & 0 & 0 \\
0 & 0 & e^{\lambda_+ t/2m} & 0 \\
0 & 0 & 0 & e^{\lambda_- t/2m}
\end{pmatrix}
\tag{55}
$$

Some more algebra yields the solution

$$
\begin{aligned}
\tilde{P}(\mathbf{q}, \mathbf{z}, t) = \exp\big[&- A q_1^2 - A q_2^2 - E q_1 q_2 - B z_1^2 - B z_2^2 - D z_1 z_2 \\
&- C_1 z_1 q_1 - C_1 z_2 q_2 - C_2 z_1 q_2 - C_2 z_2 q_1 \big]
\end{aligned}
\tag{56}
$$

which is of the same form as (21), with $A_1 = A_2 = A$, $B_1 = B_2 = B$, $C_{11} = C_{22} = C_1$ and $C_{12} = C_{21} = C_2$ except that the explicit expressions for A, B, etc. are more complicated. The coefficients have explicit expressions

$$
\begin{aligned}
\mathcal{A} =& (\delta_1{}^2 + \delta_2{}^2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - (\delta_1{}^2 - \delta_2{}^2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&+ (\frac{1}{4} + \nu^2)(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) + (\frac{1}{4} - \nu^2)(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 4\gamma kT \left((\delta_1 + 2\delta_2\nu)\Lambda_1 + (\delta_1 - 2\delta_2\nu)\Lambda_2 \right)
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
\mathcal{B} =& (\alpha_1{}^2 + \alpha_2{}^2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - (\alpha_1{}^2 - \alpha_2{}^2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&+ 2\beta^2(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) - \beta^2(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 8\gamma kT\beta\alpha_2 \left(\Lambda_2 - \Lambda_1 \right)
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
\mathcal{D} =& 4(\alpha_1{}^2 - \alpha_2{}^2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - (\alpha_1{}^2 + \alpha_2{}^2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&- 4\beta^2(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) + 2\beta^2(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 16\gamma kT\beta\alpha_2 \left(\Lambda_1 - \Lambda_2 \right)
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\mathcal{E} =& 4(\delta_1{}^2 - \delta_2{}^2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - (\delta_1{}^2 + \delta_2{}^2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&+ 4(\frac{1}{4} - \nu^2)(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) + (\frac{1}{4} + \nu^2)(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 8\gamma kT \left((\delta_1 - 2\delta_2\nu)\Lambda_1 + (\delta_1 + 2\delta_2\nu)\Lambda_2 \right)
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\mathcal{C}_1 =& 4(\alpha_1\delta_1 + \alpha_2\delta_2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - 2(\alpha_1\delta_1 - \alpha_2\delta_2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&+ 4\beta\nu)(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) - 2\beta\nu(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 4\gamma kT(\alpha_1 + 2\alpha_2\nu - 2\beta\delta_2)\Lambda_1 + 4\gamma kT(\alpha_1 - 2\alpha_2\nu + 2\beta\delta_2)\Lambda_2
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\mathcal{C}_2 =& 4(\alpha_1\delta_1 - \alpha_2\delta_2)(\epsilon_+ \hbar^2 + 4\gamma kT\chi_1) - 2(\alpha_1\delta_1 + \alpha_2\delta_2)(2\epsilon_- \hbar^2 - 4\gamma kT\theta_1) \\
&- 4\beta\nu(\tilde{\epsilon}_+ + 4\gamma kT\chi_2) + 2\beta\nu(2\tilde{\epsilon}_- + 4\gamma kT\theta_2) \\
&+ 4\gamma kT(\alpha_1 - 2\alpha_2\nu + 2\beta\delta_2)\Lambda_1 + 4\gamma kT(\alpha_1 + 2\alpha_2\nu - 2\beta\delta_2)\Lambda_2
\end{aligned} \tag{62}
$$

with $\tilde{\epsilon}_\pm = \frac{\epsilon_\pm}{4(\epsilon_+^2 - \epsilon_-^2)}$

$$\chi_1 = \frac{m}{4\gamma}(e^{2\gamma t/m} - 1) + \frac{m\lambda_+}{2(\lambda_+ - \lambda_-)^2}(e^{\lambda_+ t/m} - 1) + \frac{m\lambda_-}{2(\lambda_+ - \lambda_-)^2}(e^{\lambda_- t/m} - 1)$$
$$- \frac{8m^3\omega_0^2}{\gamma(\lambda_+ - \lambda_-)^2}(e^{\gamma t/m} - 1)$$

$$\theta_1 = \frac{m}{2\gamma}(e^{2\gamma t/m} - 1) - \frac{m\lambda_+}{(\lambda_+ - \lambda_-)^2}(e^{\lambda_+ t/m} - 1) - \frac{m\lambda_-}{(\lambda_+ - \lambda_-)^2}(e^{\lambda_- t/m} - 1)$$
$$+ \frac{16m^3\omega_0^2}{\gamma(\lambda_+ - \lambda_-)^2}(e^{\gamma t/m} - 1)$$

$$\chi_2 = \frac{t}{8\gamma^2} + \frac{m}{16\gamma^3}(e^{2\gamma t/m} - 1) - \frac{m}{4\gamma^3}(e^{\gamma t/m} - 1) + \frac{m(e^{\lambda_+ t/m} - 1)}{2(\lambda_+ - \lambda_-)^2\lambda_+}$$
$$+ \frac{m(e^{\lambda_- t/m} - 1)}{2(\lambda_+ - \lambda_-)^2\lambda_-} - \frac{m(e^{\gamma t/m} - 1)}{\gamma(\lambda_+ - \lambda_-)^2}$$

$$\theta_2 = \frac{t}{4\gamma^2} + \frac{m}{8\gamma^3}(e^{2\gamma t/m} - 1) - \frac{m}{2\gamma^3}(e^{\gamma t/m} - 1) - \frac{m(e^{\lambda_+ t/m} - 1)}{(\lambda_+ - \lambda_-)^2\lambda_+}$$
$$- \frac{m(e^{\lambda_- t/m} - 1)}{(\lambda_+ - \lambda_-)^2\lambda_-} + \frac{2m(e^{\gamma t/m} - 1)}{\gamma(\lambda_+ - \lambda_-)^2}$$

$$\Lambda_1 = \frac{m}{4\gamma^2}(e^{2\gamma t/m} - 1) - \frac{m}{2\gamma^2}(e^{\gamma t/m} - 1) + \frac{m(e^{\lambda_+ t/m} - 1)}{(\lambda_+ - \lambda_-)^2} + \frac{m(e^{\lambda_- t/m} - 1)}{(\lambda_+ - \lambda_-)^2}$$
$$- \frac{2m(e^{\gamma t/m} - 1)}{(\lambda_+ - \lambda_-)^2}$$

$$\Lambda_2 = \frac{m}{4\gamma^2}(e^{2\gamma t/m} - 1) - \frac{m}{2\gamma^2}(e^{\gamma t/m} - 1) - \frac{m(e^{\lambda_+ t/m} - 1)}{(\lambda_+ - \lambda_-)^2} - \frac{m(e^{\lambda_- t/m} - 1)}{(\lambda_+ - \lambda_-)^2}$$
$$+ \frac{2m(e^{\gamma t/m} - 1)}{(\lambda_+ - \lambda_-)^2} \tag{63}$$

and

$$\alpha_\pm = \frac{e^{-\gamma t/m}}{2} \pm \frac{\lambda_+ \, e^{-\lambda_+ t/2m} - \lambda_- \, e^{-\lambda_- t/2m}}{2(\lambda_+ - \lambda_-)} = \alpha_1 \pm \alpha_2$$

$$\beta = \lambda_+ \lambda_- \frac{e^{-\lambda_+ t/2m} - e^{-\lambda_- t/2m}}{2(\lambda_+ - \lambda_-)}$$

$$\delta_\pm = -\frac{1}{4\gamma} + \frac{e^{-\gamma t/m}}{4\gamma} \pm \frac{e^{-\lambda_+ t/2m} - e^{-\lambda_- t/2m}}{2(\lambda_+ - \lambda_-)} = \delta_1 \pm \delta_2$$

$$\nu_\pm = \frac{1}{2} \pm \frac{\lambda_+ \, e^{-\lambda_- t/2m} - \lambda_- \, e^{-\lambda_+ t/2m}}{2(\lambda_+ - \lambda_-)} = \frac{1}{2} \pm \nu \tag{64}$$

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