Block-compatible metaplectic cocycles

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Let $\mathbb{F}$ be a local field such that the group $\mu_r(\mathbb{F})$ of r-th roots of unity in $\mathbb{F}^\times$ has cardinality $r \geq 1$. Let $\mathbb{G}$ be the $\mathbb{F}$-rational points of a simple Chevalley group defined over $\mathbb{F}$. In his thesis, Matsumoto [5] gave a beautiful construction for the *metaplectic cover* $\widetilde{\mathbb{G}}$ of $\mathbb{G}$, a central extension of $\mathbb{G}$ by $\mu_r(\mathbb{F})$ whose existence is intimately connected with the deep properties of the r-th order Hilbert symbol $(\cdot,\cdot)_\mathbb{F} : \mathbb{F}^\times \times \mathbb{F}^\times \to \mu_r(\mathbb{F})$. Metaplectic groups figure prominently in the study of number theory, representation theory, and physics, arising naturally in the theory of theta functions, dual pair correspondences, Weil representations, and spin geometry. In this paper we study the class of central extensions of a simple Chevalley group over an arbitrary infinite field, of which the metaplectic groups form an important subclass.

Metaplectic groups were constructed quite explicitly in Weil's memoir [10] in the case that $\mathbb{G}$ is symplectic. In [3] and [4], Kubota gave the construction of the r-fold metaplectic cover of $GL_2(\mathbb{F})$. Moreover, he described an explicit 2-cocycle σ_K on $GL_2(\mathbb{F})$ that represents the second cohomology class of the extension (cf. §3 Corollary 8), which makes it possible to deal quite rapidly with many concrete problems in this setting. Steinberg [9] and Moore [7] considered the algebraic problem of determining the central extensions of a simple Chevelley group over an arbitrary field; they were also led to the metaplectic groups. This line of investigation was completed by Matsumoto [5], whose work forms the foundation of the present paper.

To summarize our results, let $\mathbb{F}$ be an infinite field, $\mathbb{G}$ the $\mathbb{F}$-rational points of a simple Chevalley group defined over $\mathbb{F}$, $\mathcal{A}$ an abelian group, and $c : \mathbb{F}^\times \times \mathbb{F}^\times \to \mathcal{A}$ a Steinberg symbol that is bilinear if $\mathbb{G}$ is not symplectic (cf. §1). In this paper we describe an explicit 2-cocycle $\sigma_\mathbb{G}$ in $Z^2(\mathbb{G};\mathcal{A})$ that represents the cohomology class in $H^2(\mathbb{G};\mathcal{A})$ of the central extension $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ by $\mathcal{A}$ constructed by Matsumoto [5]

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using the Steinberg symbol c. In particular, if $\mathbb{F}$ is a local field such that $\mu_r(\mathbb{F})$ has cardinality r, and one takes $\mathcal{A} := \mu_r(\mathbb{F})$ and $c := (\cdot,\cdot)_{\mathbb{F}}^{-1}$, then $\widetilde{\mathbb{G}}$ is the metaplectic cover of $\mathbb{G}$ discussed above. In this case $\sigma_{\mathbb{G}}$ is a *metaplectic 2-cocycle*.

By pulling back $\sigma_{\mathbb{G}}$ from $\mathbb{G} := SL_{n+1}(\mathbb{F})$ to $G := GL_n(\mathbb{F})$ under a particular embedding of G into $\mathbb{G}$, we obtain for every $n \geq 1$ an explicit 2-cocycle σ_n in $Z^2(G;\mathcal{A})$ that represents the second cohomology class of the central extension $\widetilde{G}$ of G by $\mathcal{A}$, where $\widetilde{G}$ denotes the preimage of G in $\widetilde{\mathbb{G}}$. We show that the 2-cocycles $\{\sigma_n \mid n \geq 1\}$ are well-behaved with respect to restriction, and they satisfy a nice block formula on all standard Levi subgroups of G, i.e., that they are *block-compatible.* We also show that σ_2 is the Kubota 2-cocycle σ_K on $GL_2(\mathbb{F})$.

The paper is organized as follows. In §1 we review Matsumoto's construction of the central extension $\widetilde{\mathbb{G}}$ of a simple Chevalley group $\mathbb{G}$ over an infinite field. Our main result in this section (Theorem 3) is a presentation of $\widetilde{\mathbb{G}}$ in terms of generators and relations. We also describe the natural projection $\mathfrak{p} : \widetilde{\mathbb{G}} \to \mathbb{G}$.

In §2 we define the 2-cocycle $\sigma_{\mathbb{G}}$ by constructing an explicit section $\mathfrak{s}_{\mathbb{G}} : \mathbb{G} \to \widetilde{\mathbb{G}}$ with respect to $\mathfrak{p}$. The basic properties that $\sigma_{\mathbb{G}}$ satisfies are listed in Proposition 4. In this section we define the notion of a *standard subgroup* of $\mathbb{G}$ and also show that for every standard subgroup $\mathbb{G}^\sharp$, $\sigma_{\mathbb{G}}\big|_{\mathbb{G}^\sharp\times\mathbb{G}^\sharp} = \sigma_{\mathbb{G}^\sharp}$ (Lemma 5). In other words, our 2-cocycles are well-behaved with respect to restriction to standard subgroups. In Lemma 6 we prove that if $\{\mathbb{G}^i \mid 1 \leq i \leq p\}$ is any collection mutually commuting standard subgroups of $\mathbb{G}$, then the preimages $\{\widetilde{\mathbb{G}}^i \mid 1 \leq i \leq p\}$ in $\widetilde{\mathbb{G}}$ are also mutually commuting. Moreover, in Theorem 7 we establish the following *block formula*:

$$\sigma_{\mathbb{G}}(g_1 \dots g_p, g_1' \dots g_p') = \prod_{i=1}^{p} \sigma_{\mathbb{G}^i}(g_i, g_i')$$

for all $g_i, g_i' \in \mathbb{G}^i$, $1 \leq i \leq p$.

In §3 we pull back the 2-cocycle $\sigma_{\mathbb{G}}$ from $\mathbb{G} := SL_{n+1}(\mathbb{F})$ to $G := GL_n(\mathbb{F})$ and define the 2-cocycle $\sigma_n \in Z^2(G;\mathcal{A})$ for every $n \geq 1$. The basic properties that σ_n satisfies are listed in Theorem 7, and the nine properties listed there actually *characterize* the 2-cocycle. Using this characterization we show that σ_2 is the

Kubota 2-cocycle σ_K (Corollary 8), and the restriction of σ_n to any copy of $GL_m(\mathbb{F})$ embedded along the diagonal in G agrees with the 2-cocycle σ_m (Corollary 9). Finally, in Theorem 11 we show that for all standard Levi subgroups of G, the following block formula holds:

$$\sigma_n\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_p \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_p' \end{pmatrix}\right) = \prod_{i=1}^{p} \sigma_{n_i}(g_i, g_i') \prod_{i<j} c\big(\det(g_i), \det(g_j')\big)^{-1},$$

where $n = n_1 + \ldots + n_p$ and $g_i, g_i' \in GL_{n_i}(\mathbb{F})$ for $1 \leq i \leq p$. Note that although our 2-cocycle σ_n agrees on the torus with the 2-cocycle introduced in the foundational work of Kazhdan and Patterson [2], their 2-cocycle does not satisfy the block formula.

In §4 we describe a method of calculating $\sigma_n(g, g')$ for an arbitrary pair of elements $g, g' \in GL_n(\mathbb{F})$. Using Lemma 1 and some results from §3, the calculation of $\sigma_n(g, g')$ is achieved by performing $\ell + 1$ 2-cocycle calculations on the *torus*, where ℓ is the length of g. The method is straightforward and easily implemented on a computer.

In §5 we describe a different method of calculating the 2-cocycle for pairs of elements in the Weyl group W of permutation matrices. We introduce the notion of the *canonical expression* of an element $w \in W$. To compute $\sigma_n(w, w')$ for arbitrary $w, w' \in W$, one first determines the canonical expressions for w, w', and ww'. Since the section $\mathfrak{s}_n$ corresponding to σ_n is easily described on W once the canonical expressions are known (Corollary 3), one simply applies the relations of Lemma 4 to determine $\sigma_n(w, w')$.

The metaplectic groups are fascinating and important objects of study that arise in a number of disciplines. Unfortunately, the subject has had a long history of errors, perhaps due to the deep and subtle nature of the underlying ideas. For this reason, we have referred quite extensively to the remarkable thesis of Matsumoto [5] as an (apparently) error-free foundation for our work. Moreover, we have included many details of our calculations in order to convince the skeptical reader of the veracity of our proofs.

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§1. Central Extensions of a Simple Chevalley Group

Let $\mathbb{F}$ be an infinite field, and let $\mathbb{G}$ be the $\mathbb{F}$-rational points of a simply-connected almost simple linear algebraic group $\mathbf{G}$ that is defined and split over $\mathbb{F}$. Let $\mathbf{H}$ be a maximal split torus of $\mathbf{G}$ also defined over $\mathbb{F}$, let Φ be the set of roots of $\mathbf{G}$ with respect to $\mathbf{H}$, and let $\Delta \subset \Phi$ be a base of simple roots. These choices determine a Borel subgroup of $\mathbf{G}$ whose unipotent radical we denote by $\mathbf{N}$. Let $\mathbb{H}$ and $\mathbb{N}$ be the $\mathbb{F}$-rational points of $\mathbf{H}$ and $\mathbf{N}$, respectively. Let $\Phi^+ \subset \Phi$ be the set of positive roots determined by Δ. The set Φ can be embedded in a Euclidean space in a standard way $\big($cf. [1] p. 63$\big)$, and we define:

$$\langle \alpha, \beta \rangle := 2\,\frac{(\alpha, \beta)}{(\beta, \beta)}, \qquad \alpha, \beta \in \Phi,$$

where $(\cdot, \cdot)$ is the Euclidean norm. For every $\alpha \in \Phi$, let $\mathbb{N}_\alpha$ be the standard unipotent subgroup of $\mathbb{G}$ corresponding to α:

$$\mathbb{N}_\alpha := \exp \big\{ X \in \mathrm{Lie}(\mathbb{G}) \,\big|\, \mathrm{Ad}(h)X = \alpha(h)X \text{ for all } h \in \mathbb{H} \big\}.$$

We fix isomorphisms $\big\{ n_\alpha : \mathbb{F} \to \mathbb{N}_\alpha \,\big|\, \alpha \in \Phi \big\}$ based on an explicit decomposition:

$$\mathfrak{g}_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} + \sum_{\alpha \in \Phi} \mathbb{Z}\mathbf{e}_\alpha$$

of the Chevalley algebra corresponding to $\mathbb{G}$ $\big($cf. [5] pp. 8,12$\big)$. It is well-known that the subgroups $\big\{ \mathbb{N}_\alpha \,\big|\, \alpha \in \Phi \big\}$ generate $\mathbb{G}$ as an abstract group, and Steinberg [8] has given the following presentation.

Proposition 1. *The group* $\mathbb{G}$ *has the presentation* $\langle \mathcal{G}_{\mathrm{S}} \,|\, \mathcal{R}_{\mathrm{S}} \rangle$*, where the set of generators is:*

$$\mathcal{G}_{\mathrm{S}} := \{ n_\alpha(x) \,|\, \alpha \in \Phi,\ x \in \mathbb{F} \},$$

and the list $\mathcal{R}_{\mathrm{S}}$ *of relations consists of the following:*

$$\mathcal{R}1: \qquad n_\alpha(x)\, n_\alpha(y) = n_\alpha(x+y), \qquad \alpha \in \Phi,\ x, y \in \mathbb{F},$$

$$\mathcal{R}2: \quad (i) \quad n_\alpha(x)\, n_\beta(y) = \Big[\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} n_\gamma(\mathfrak{m}_{\alpha,\beta;i,j}\, x^i y^j) \Big]\, n_\beta(y)\, n_\alpha(x),$$

$$\alpha, \beta \in \Phi,\ \alpha + \beta \neq 0,\ x, y \in \mathbb{F},$$

where the $\mathfrak{m}_{\alpha,\beta;i,j}$*'s are certain rational integers independent of* $x, y \in \mathbb{F}$*,*

$$(ii) \quad w_\alpha(x)\, n_\alpha(y)\, w_\alpha(x)^{-1} = n_{-\alpha}(-x^{-2} y), \qquad \alpha \in \Phi,\ x \in \mathbb{F}^\times,\ y \in \mathbb{F},$$

where $w_\alpha(x) := n_\alpha(x)\, n_{-\alpha}(-x^{-1})\, n_\alpha(x)$*,*

$$\mathcal{R}3: \qquad h_\alpha(x)\, h_\alpha(y) = h_\alpha(xy), \qquad \alpha \in \Phi,\ x, y \in \mathbb{F}^\times,$$

where $h_\alpha(x) := w_\alpha(x)\, w_\alpha(1)^{-1}$*.*

□

Note that the expression in brackets in $\mathcal{R}2(i)$ is a product of commuting terms. To see this, simply apply $\mathcal{R}2(i)$ to the terms appearing in that expression. One of the goals of this section is to give a different presentation of the group $\mathbb{G}$.

Let $\mathcal{A}$ be an abelian group (written multiplicatively) with identity element $1_{\mathcal{A}}$. Suppose that we are given a central extension E of $\mathbb{G}$ by $\mathcal{A}$, i.e., an exact sequence:

$$1 \to \mathcal{A} \hookrightarrow E \xrightarrow{\pi} \mathbb{G} \to 1.$$

Steinberg [8] showed that there exist unique lifts $\{ n^*_\alpha : \mathbb{F} \to E \,|\, \alpha \in \Phi \}$ of the maps $\{ n_\alpha \,|\, \alpha \in \Phi \}$ that also satisfy the relations $\mathcal{R}1$ and $\mathcal{R}2$ of Proposition 1. In particular, for every $\alpha \in \Phi$, the extension E splits over $\mathbb{N}_\alpha$, and $\mathbb{N}^*_\alpha := \mathrm{Im}(n^*_\alpha)$ is the image in E of $\mathbb{N}_\alpha$ under the splitting. It is known that $\mathbb{N}$ is the product in $\mathbb{G}$ of the groups $\{ \mathbb{N}_\alpha \,|\, \alpha \in \Phi^+ \}$, and Steinberg has shown (cf. [5] Lemme 5.1(a)) that the projection π induces an isomorphism from the subgroup $\mathbb{N}^* := \prod_{\alpha\in\Phi^+} \mathbb{N}^*_\alpha$ of E to the subgroup $\mathbb{N}$ of $\mathbb{G}$.

By [5] Lemme 5.4 and Proposition 5.5, if we define for all $\alpha \in \Phi$, $x \in \mathbb{F}^\times$:

$$\tilde{w}_\alpha(x) := n^*_\alpha(x)\, n^*_{-\alpha}(-x^{-1})\, n^*_\alpha(x), \qquad \tilde{h}_\alpha(x) := \tilde{w}_\alpha(x)\, \tilde{w}_\alpha(1)^{-1},$$

then for every $\alpha \in \Delta$, the map $c_\alpha : \mathbb{F}^\times \times \mathbb{F}^\times \to \mathcal{A}$ given by:

$$c_\alpha(x,y) := \tilde{h}_\alpha(x)\, \tilde{h}_\alpha(y)\, \tilde{h}_\alpha(xy)^{-1}$$

satisfies the following relations for all $x, y, z \in \mathbb{F}^\times$:

$\mathcal{S}1$: $$c_\alpha(x,y)\, c_\alpha(xy,z) = c_\alpha(x,yz)\, c_\alpha(y,z),$$

$\mathcal{S}2$: $$c_\alpha(1,1) = 1_{\mathcal{A}}, \qquad c_\alpha(x,y) = c_\alpha(x^{-1}, y^{-1}),$$

$\mathcal{S}3$: $$c_\alpha(x,y) = c_\alpha\big(x, (1-x)y\big), \qquad x \neq 1.$$

Moreover, the relations:

$\mathcal{S}4$: $$c_\alpha(xy,z) = c_\alpha(x,z)\, c_\alpha(y,z), \qquad x,y,z \in \mathbb{F}^\times,$$

also hold unless $\mathbb{G}$ is symplectic and α is its long simple root. Note that the maps $\big\{c_\alpha \,\big|\, \alpha \in \Delta\big\}$ are determined from one another by the relations $\big($cf. [5] p. 38$\big)$:

(1) $$c_\alpha(x, y^{\langle\alpha,\beta\rangle}) = c_\beta(y, x^{\langle\beta,\alpha\rangle})^{-1}, \qquad \alpha,\beta \in \Delta,\ x,y \in \mathbb{F}^\times.$$

We call any map $c : \mathbb{F}^\times \times \mathbb{F}^\times \to \mathcal{A}$ satisfying $\mathcal{S}1$, $\mathcal{S}2$ and $\mathcal{S}3$ a *Steinberg symbol.* A Steinberg symbol is said to be *bilinear* if it also satisfies $\mathcal{S}4$. Since every Steinberg symbol c satisfies $c(x,y) = c(y^{-1},x)$ $\big($cf. [5] Proposition 5.7(a)$\big)$, and a symplectic group has only one long simple root, it follows from $\mathcal{S}4$ and (1) that the Steinberg symbol c_α is independent of the choice of a long simple root α. Accordingly, we define the *Steinberg symbol of E* $\big($with respect to our choices of $\mathbb{H}$, Δ and $\big\{n_\alpha \,\big|\, \alpha \in \Phi\big\}\big)$ to be the Steinberg symbol c_α for any long simple root α.

Lemma 2. *Let E be a central extension of $\mathbb{G}$ by $\mathcal{A}$ with Steinberg symbol c. Define $\{c_\alpha \,|\, \alpha \in \Delta\}$ using (1) above. For every $\alpha \in \Phi^+$, let $\{n^*_\alpha(x) \,|\, x \in \mathbb{F}\}$, $\{\tilde{h}_\alpha(x) \,|\, x \in \mathbb{F}^\times\}$ and $\{\tilde{w}_\alpha(x) \,|\, x \in \mathbb{F}^\times\}$ be defined as above, and let $\tilde{w}_\alpha := \tilde{w}_\alpha(-1)$ for all $\alpha \in \Delta$. Then the following relations hold in the extension E:*

$\mathcal{R}_{\mathcal{A}}$: $\quad a\,b = (ab), \qquad a, b \in \mathcal{A},$

$\mathcal{R}^2_{\widetilde{\mathbb{H}}}$: $\quad \tilde{h}_\alpha(x)\,\tilde{h}_\alpha(y) = c_\alpha(x,y)\,\tilde{h}_\alpha(xy), \qquad \alpha \in \Delta,\ x, y \in \mathbb{F}^\times,$

$\mathcal{R}^*_{\widetilde{\mathbb{H}}}$: $\quad \tilde{h}_\alpha(x)\,\tilde{h}_\beta(y) = c_\alpha\big(x, y^{\langle\alpha,\beta\rangle}\big)\,\tilde{h}_\beta(y)\,\tilde{h}_\alpha(x), \qquad \alpha, \beta \in \Delta,\ x, y \in \mathbb{F}^\times,$

$\mathcal{R}_{\widetilde{\mathbb{H}},\mathcal{A}}$: $\quad \tilde{h}_\alpha(x)\,a = a\,\tilde{h}_\alpha(x), \qquad \alpha \in \Delta,\ x \in \mathbb{F}^\times,\ a \in \mathcal{A},$

$\mathcal{R}^0_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad \tilde{w}_\alpha\,\tilde{w}_\beta = \tilde{w}_\beta\,\tilde{w}_\alpha, \qquad \alpha, \beta \in \Delta,\ \langle\alpha,\beta\rangle = 0,$

$\mathcal{R}^{-1}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad \tilde{w}_\alpha\,\tilde{w}_\beta\,\tilde{w}_\alpha = \tilde{w}_\beta\,\tilde{w}_\alpha\,\tilde{w}_\beta, \qquad \alpha, \beta \in \Delta,\ \langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = -1,$

$\mathcal{R}^{-2}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad (\tilde{w}_\alpha\,\tilde{w}_\beta)^2 = (\tilde{w}_\beta\,\tilde{w}_\alpha)^2, \qquad \alpha, \beta \in \Delta,\ \langle\alpha,\beta\rangle = -2,$

$\mathcal{R}^{-3}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad (\tilde{w}_\alpha\,\tilde{w}_\beta)^3 = (\tilde{w}_\beta\,\tilde{w}_\alpha)^3, \qquad \alpha, \beta \in \Delta,\ \langle\alpha,\beta\rangle = -3,$

$\mathcal{R}_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\mathcal{A}}$: $\quad \tilde{w}_\alpha\,a = a\,\tilde{w}_\alpha, \qquad \alpha \in \Delta,\ a \in \mathcal{A},$

$\mathcal{R}^2_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$: $\quad \tilde{w}^2_\alpha = \tilde{h}_\alpha(-1), \qquad \alpha \in \Delta,$

$\mathcal{R}^*_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$: $\quad \tilde{w}_\alpha\,\tilde{h}_\beta(x) = \tilde{h}_\alpha\big(x^{-\langle\alpha,\beta\rangle}\big)\,\tilde{h}_\beta(x)\,\tilde{w}_\alpha, \qquad \alpha, \beta \in \Delta,\ x \in \mathbb{F}^\times,$

$\mathcal{R}^2_{\mathbb{N}^*}$: $\quad n^*_\alpha(x)\,n^*_\alpha(y) = n^*_\alpha(x+y), \qquad \alpha \in \Phi^+,\ x, y \in \mathbb{F},$

$\mathcal{R}^*_{\mathbb{N}^*}$: $\quad n^*_\alpha(x)\,n^*_\beta(y) = \Big[\prod_{\substack{i,j\in\mathbb{Z}^+\\ i\alpha+j\beta=\gamma\in\Phi}} n^*_\gamma\big(\mathfrak{m}_{\alpha,\beta;i,j}\,x^i y^j\big)\Big]\,n^*_\beta(y)\,n^*_\alpha(x), \qquad \alpha, \beta \in \Phi^+,\ x, y \in \mathbb{F},$

$\mathcal{R}_{\mathbb{N}^*,\mathcal{A}}$: $\quad n^*_\alpha(x)\,a = a\,n^*_\alpha(x), \qquad \alpha \in \Phi^+,\ x \in \mathbb{F},\ a \in \mathcal{A},$

$\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$: $\quad n^*_\alpha(x)\,\tilde{h}_\beta(y) = \tilde{h}_\beta(y)\,n^*_\alpha\big(xy^{-\langle\alpha,\beta\rangle}\big), \qquad \alpha \in \Phi^+,\ \beta \in \Delta,\ x \in \mathbb{F},\ y \in \mathbb{F}^\times,$

$\mathcal{R}^2_{\mathbb{N}^*,\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad \tilde{w}_\alpha\,n^*_\alpha(x)\,\tilde{w}_\alpha = n^*_\alpha(-x^{-1})\,\tilde{h}_\alpha(x^{-1})\,\tilde{w}_\alpha\,n^*_\alpha(-x^{-1}), \qquad \alpha \in \Delta,\ x \in \mathbb{F}^\times,$

$\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{M}}_{\mathbb{Z}}}$: $\quad n^*_\alpha(x)\,\tilde{w}_\beta = \tilde{w}_\beta\,n^*_{s_\beta\alpha}\big(\mathfrak{n}_{\beta,\alpha}\,x\big), \qquad \alpha \in \Phi^+,\ \beta \in \Delta,\ \alpha \neq \beta,\ x \in \mathbb{F},$

where $\{s_\alpha \,|\, \alpha \in \Delta\}$ is the set of simple reflections in the Weyl group of $(\mathbf{G}, \mathbf{H})$, and the $\mathfrak{n}_{\alpha,\beta}$'s are constants (equal to ± 1) independent of $\mathcal{A}$ and c.

Proof : The constants $\{\mathfrak{n}_{\alpha,\beta}\}$ are partially described in [5] Lemme 5.1(c). Our proof of the lemma relies heavily on results of Matsumoto [5]. Note that Steinberg also gave a list of relations for the extension E.

The relations involving $\mathcal{A}$ are clear since E is a central extension of $\mathbb{G}$ by $\mathcal{A}$. The relations $\mathcal{R}^2_{\widetilde{\mathbb{H}}}$ were used to define the Steinberg symbols $\{c_\alpha \,|\, \alpha \in \Delta\}$, and the relations $\mathcal{R}^2_{\widetilde{\mathbb{M}_\mathbb{Z}},\widetilde{\mathbb{H}}}$ follow from the definition of $\tilde{h}_\alpha$. To see this, observe that since the maps $\{n^*_\alpha \,|\, \alpha \in \Phi\}$ are homomorphisms, the definition of $\tilde{w}_\alpha(x)$ implies that $\tilde{w}_\alpha = \tilde{w}_\alpha(1)^{-1}$ for all $\alpha \in \Delta$. The relations $\mathcal{R}^2_{\mathbb{N}^*}$ and $\mathcal{R}^*_{\mathbb{N}^*}$ all appear in $\mathcal{R}1$ and $\mathcal{R}2$ of Proposition 1.

For the remaining relations, we refer to [5]. The relations $\mathcal{R}^*_{\widetilde{\mathbb{H}}}$ are given in [5] Lemme 5.4(c). To establish the relations $\mathcal{R}^*_{\widetilde{\mathbb{M}_\mathbb{Z}},\widetilde{\mathbb{H}}}$, we use [5] Lemme 5.2(f),(g) and the relations $\mathcal{R}^2_{\widetilde{\mathbb{H}}}$, $\mathcal{R}^*_{\widetilde{\mathbb{H}}}$ and $\mathcal{S}2$ as follows:

$$\begin{aligned}
\tilde{w}_\alpha^{-1}\tilde{h}_\alpha(x^{-\langle\alpha,\beta\rangle})\tilde{h}_\beta(x)\tilde{w}_\alpha &= \tilde{h}_\alpha(x^{\langle\alpha,\beta\rangle})\tilde{h}_\beta(x)\tilde{h}_\alpha(x^{-\langle\alpha,\beta\rangle}) \\
&= c_\alpha(x^{-\langle\alpha,\beta\rangle}, x^{\langle\alpha,\beta\rangle})^{-1}\tilde{h}_\alpha(x^{\langle\alpha,\beta\rangle})\tilde{h}_\alpha(x^{-\langle\alpha,\beta\rangle})\tilde{h}_\beta(x) \\
&= c_\alpha(x^{-\langle\alpha,\beta\rangle}, x^{\langle\alpha,\beta\rangle})^{-1}c_\alpha(x^{\langle\alpha,\beta\rangle}, x^{-\langle\alpha,\beta\rangle})\tilde{h}_\beta(x) \\
&= \tilde{h}_\beta(x).
\end{aligned}$$

The elements $\{\tilde{w}_\alpha \,|\, \alpha \in \Delta\}$ satisfy the relations in (W2) of [5] Lemme 6.1; this follows from [5] Lemme 5.2(a), the second and third lines of [5] Lemme 5.1(c), and the elementary observation that if $\alpha, \beta \in \Delta$ are orthogonal, then neither $\alpha - \beta$ nor $\alpha + \beta = s_\beta(\alpha - \beta)$ is a root. Matsumoto notes (cf. [5], proof of Théorème 6.3) that given the relations $\mathcal{R}^*_{\widetilde{\mathbb{M}_\mathbb{Z}},\widetilde{\mathbb{H}}}$ and $\mathcal{R}^2_{\widetilde{\mathbb{M}_\mathbb{Z}},\widetilde{\mathbb{H}}}$, the relations (W2) are equivalent to those in $(W2')$ of [5] Théorème 6.3, which are precisely the relations $\mathcal{R}^0_{\widetilde{\mathbb{M}_\mathbb{Z}}}$, $\mathcal{R}^{-1}_{\widetilde{\mathbb{M}_\mathbb{Z}}}$, $\mathcal{R}^{-2}_{\widetilde{\mathbb{M}_\mathbb{Z}}}$ and $\mathcal{R}^{-3}_{\widetilde{\mathbb{M}_\mathbb{Z}}}$. The relations $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$ follow immediately from [5] Lemme 5.2(c), the relations $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{M}_\mathbb{Z}}}$ follow from [5] Lemme 5.1(b), and the relations $\mathcal{R}^2_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$ follow from [5] Lemme

5.2(h) and the relations $\mathcal{R}^2_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$ and $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$ as follows:

$$\begin{aligned}\tilde{w}_\alpha n^*_\alpha(x)\tilde{w}_\alpha &= \tilde{w}_\alpha n^*_\alpha(x)\tilde{w}_\alpha^{-1}\tilde{h}_\alpha(-1)\\ &= n^*_\alpha(-x^{-1})\tilde{h}_\alpha(x^{-1})\tilde{w}_\alpha^{-1}n^*_\alpha(-x^{-1})\tilde{h}_\alpha(-1)\\ &= n^*_\alpha(-x^{-1})\tilde{h}_\alpha(x^{-1})\tilde{w}_\alpha n^*_\alpha(-x^{-1}).\end{aligned}$$

This completes the proof. □

The relations given in Lemma 2 actually form a *complete* list of relations for the central extension E by the following theorem.

Theorem 3. *Suppose that there exists a central extension E of $\mathbb{G}$ by $\mathcal{A}$ with Steinberg symbol c. Then E is isomorphic to the group $\widetilde{\mathbb{G}}$ that is given by the presentation $\langle \mathcal{G}_{\widetilde{\mathbb{G}}} \,|\, \mathcal{R}_{\widetilde{\mathbb{G}}}\rangle$, where the set $\mathcal{G}_{\widetilde{\mathbb{G}}}$ of generators is the union of the sets:*

$$\begin{aligned}\mathcal{G}_{\mathcal{A}} &:= \mathcal{A},\\ \mathcal{G}_{\widetilde{\mathbb{H}}} &:= \{\tilde{h}_\alpha(x) \,|\, \alpha\in\Delta,\ x\in\mathbb{F}^\times\},\\ \mathcal{G}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}} &:= \{\tilde{w}_\alpha \,|\, \alpha\in\Delta\},\\ \mathcal{G}_{\mathbb{N}^*} &:= \{n^*_\alpha(x) \,|\, \alpha\in\Phi^+,\ x\in\mathbb{F}\},\end{aligned}$$

and the list $\mathcal{R}_{\widetilde{\mathbb{G}}}$ of relations consists precisely of the relations given in Lemma 2.

Proof : By Lemma 2, there is a natural homomorphism $\phi : \widetilde{\mathbb{G}} \to E$, and we must show that ϕ is an isomorphism.

To see that ϕ is surjective, it suffices by Proposition 1 to show that the elements $\{n^*_\alpha(x)\in E \,|\, \alpha\in\Phi,\ x\in\mathbb{F}\}$ all lie in the image of ϕ. But this follows from the relations in $\mathcal{R}2(ii)$ of Proposition 1 for E and the relations $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{M}}_{\mathbb{Z}}}$ for E and $\widetilde{\mathbb{G}}$, since the simple reflections $\{s_\alpha \,|\, \alpha\in\Delta\}$ generate the full Weyl group $\mathbf{W}$ of $(\mathbf{G},\mathbf{H})$.

Since the relations in $\mathcal{R}_{\mathcal{A}}$ hold in $\widetilde{\mathbb{G}}$, the subgroup of $\widetilde{\mathbb{G}}$ generated by $\mathcal{G}_{\mathcal{A}}$ is a quotient of $\mathcal{A}$. On the other hand, $\phi(a) = a$ for all $a\in\mathcal{G}_{\mathcal{A}} = \mathcal{A}$, hence the map ϕ provides an isomorphism from this subgroup to $\mathcal{A}$ (here we use our hypothesis that E is a central extension of $\mathbb{G}$ *by* $\mathcal{A}$). To prove that ϕ is injective, it therefore suffices to show that the kernel of the composition $\mathfrak{p} : \widetilde{\mathbb{G}} \xrightarrow{\phi} E \xrightarrow{\pi} \mathbb{G}$ is this copy of $\mathcal{A}$ in $\widetilde{\mathbb{G}}$. This will be our approach, and we will not need to mention E again in this proof.

The natural homomorphism $\mathfrak{p} : \widetilde{\mathbb{G}} \to \mathbb{G}$ described above is determined by its values on the generators of $\widetilde{\mathbb{G}}$. Clearly:

$$\mathfrak{p}(a) = 1_{\mathbb{G}}, \qquad a \in \mathcal{A},$$

$$\mathfrak{p}\big(\tilde{h}_\alpha(x)\big) = h_\alpha(x), \qquad \alpha \in \Delta,\ x \in \mathbb{F}^\times,$$

$$\mathfrak{p}(\tilde{w}_\alpha) = w_\alpha, \qquad \alpha \in \Delta,$$

$$\mathfrak{p}\big(n^*_\alpha(x)\big) = n_\alpha(x), \qquad \alpha \in \Phi^+,\ x \in \mathbb{F},$$

where $1_{\mathbb{G}}$ is the identity element in $\mathbb{G}$, and $w_\alpha := w_\alpha(-1)$ for all $\alpha \in \Delta$.

It is well-known that the (finite) subgroup $\mathbb{M}_{\mathbb{Z}}$ of $\mathbb{G}$ generated by the elements $\big\{w_\alpha \,\big|\, \alpha \in \Delta\big\}$ is a central extension of $\mathbf{W}$ by the group $\mathbb{H} \cap \mathbb{M}_{\mathbb{Z}}$ that is generated by $\big\{h_\alpha(-1) \,\big|\, \alpha \in \Delta\big\}$. Let $\widetilde{\mathfrak{N}}$ be the abstract group defined by formal generators $\big\{\tilde{w}_\alpha,\ \tilde{h}_\alpha(-1) \,\big|\, \alpha \in \Delta\big\}$ subject only to the relations in $\mathcal{R}^0_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$, $\mathcal{R}^{-1}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$, $\mathcal{R}^{-2}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$ and $\mathcal{R}^{-3}_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$, those in $\mathcal{R}^2_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$, and the relations in $\mathcal{R}^*_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$ with $x = -1$. Then Matsumoto proved (cf. [5] Théorème 6.3) that $\widetilde{\mathfrak{N}}$ is a cover of $\mathbb{M}_{\mathbb{Z}}$ by a cyclic group whose order is infinite [resp. two] if the group $\mathbb{G}$ is [resp. is not] symplectic. Because of the additional relations $\tilde{h}_\alpha(1)^2 = \tilde{h}_\alpha(1)$ and $\tilde{h}_\alpha(-1)^2 = c_\alpha(-1,-1)\,\tilde{h}_\alpha(1)$ in $\mathcal{R}^2_{\widetilde{\mathbb{H}}}$, it follows that the subgroup $\widetilde{\mathbb{M}}_{\mathbb{Z}}$ of $\widetilde{\mathbb{G}}$ generated by $\big\{\tilde{w}_\alpha \,\big|\, \alpha \in \Delta\big\}$ is a central extension of $\mathbb{M}_{\mathbb{Z}}$ by the subgroup of $\mathcal{A}$ generated by $c(-1,-1)$. Thus, the composed map $\widetilde{\mathbb{M}}_{\mathbb{Z}} \to \mathbb{M}_{\mathbb{Z}} \to \mathbf{W}$ is surjective, and using the information above we conclude that if $\widetilde{\mathfrak{M}} = \big\{\tilde{\eta}_w \,\big|\, w \in \mathbf{W}\big\}$ is a complete set of representatives in $\widetilde{\mathbb{M}}_{\mathbb{Z}}$ for $\mathbf{W}$ under this surjection, then for all $\alpha \in \Delta$, $\tilde{w}_\alpha \widetilde{\mathfrak{M}} \subset \widetilde{\mathbb{H}}\,\widetilde{\mathfrak{M}}$, where $\widetilde{\mathbb{H}}$ is the subgroup of $\widetilde{\mathbb{G}}$ generated by the elements in $\mathcal{G}_{\mathcal{A}}$ and $\mathcal{G}_{\widetilde{\mathbb{H}}}$.

Fix a set $\widetilde{\mathfrak{M}}$ of representatives as in the previous paragraph, and for each $\alpha \in \Phi^+$, let $\mathbb{N}^*_\alpha$ now denote the subgroup $n^*_\alpha(\mathbb{F})$ of $\widetilde{\mathbb{G}}$. Choose an ordering of the simple roots Δ, and extend it to an order on Φ^+ by expressing each root as a non-increasing sum of simple roots, then comparing pairs of roots lexicographically. We will prove the following Bruhat decomposition for $\widetilde{\mathbb{G}}$:

$$\widetilde{\mathbb{G}} = \coprod_{w \in \mathbf{W}} \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde{\eta}_w \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha, \tag{2}$$

where $\prod'_{\alpha \in \Phi^+} \mathbb{N}^*_\alpha$ denotes the set of products of the form $n^*_{\alpha_1}(x_1) \dots n^*_{\alpha_k}(x_k)$ with the α_i's increasing. Observe that this decomposition implies the theorem.

To establish the Bruhat decomposition, it suffices to show that the right hand side of (2) is preserved under left multiplication by the generators of $\widetilde{\mathbb{G}}$ (disjointness of the union follows from the projection to $\mathbb{G}$). The union is clearly preserved under left multiplication by the (central) elements in $\mathcal{G}_\mathcal{A}$. Using the relations $\mathcal{R}^2_{\mathbb{N}^*}$ and $\mathcal{R}^*_{\mathbb{N}^*}$, it can also be shown that the union is preserved under left multiplication by elements of $\mathcal{G}_{\mathbb{N}^*}$, and the relations in $\mathcal{R}^*_{\mathbb{N}^*, \widetilde{\mathbb{H}}}$ imply that the union is preserved under left multiplication by elements of $\mathcal{G}_{\widetilde{\mathbb{H}}}$. This leaves only the elements in $\mathcal{G}_{\widetilde{\mathbb{M}}_\mathbb{Z}}$. Fix $\beta \in \Delta$ and $w \in \mathbf{W}$. Using $\mathcal{R}^*_{\mathbb{N}^*, \widetilde{\mathbb{M}}_\mathbb{Z}}$, $\mathcal{R}^*_{\mathbb{N}^*, \widetilde{\mathbb{H}}}$ and $\mathcal{R}^*_{\widetilde{\mathbb{M}}_\mathbb{Z}, \widetilde{\mathbb{H}}}$, it follows that:

$$\begin{aligned}
\tilde{w}_\beta \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde{\eta}_w \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha &= \Big(\prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \beta}}{}' \mathbb{N}^*_\alpha \Big) \tilde{w}_\beta \mathbb{N}^*_\beta \widetilde{\mathbb{H}} \tilde{\eta}_w \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \\
&= \Big(\prod_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \beta}}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde{w}_\beta \mathbb{N}^*_\beta \tilde{\eta}_w \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha.
\end{aligned}$$

We consider two cases. First suppose that the root $w^{-1}\beta$ is positive. Write w as a product of minimal length of simple reflections: $w = s_{\alpha_1} \dots s_{\alpha_\ell}$. Then $\tilde{\eta}_w$ can be expressed in the form $\tilde{\eta}_w = \tilde{h} \tilde{w}_{\alpha_1} \dots \tilde{w}_{\alpha_\ell}$ for some $\tilde{h} \in \widetilde{\mathbb{H}}$, and for each $i = 1, \dots, \ell$, we have $s_{\alpha_{i-1}} \dots s_{\alpha_1} \beta \neq \alpha_i$. Consequently, the relations in $\mathcal{R}^*_{\mathbb{N}^*, \widetilde{\mathbb{M}}_\mathbb{Z}}$, $\mathcal{R}_{\mathbb{N}^*, \mathcal{A}}$ and $\mathcal{R}^*_{\mathbb{N}^*, \widetilde{\mathbb{H}}}$ imply that:

$$\mathbb{N}^*_\beta \tilde{\eta}_w \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big) = \tilde{\eta}_w \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big).$$

We can also write $\tilde{w}_\beta \tilde{\eta}_w = \tilde{h}' \tilde{\eta}_{s_\beta w}$ for some $\tilde{h}' \in \widetilde{\mathbb{H}}$, hence $\widetilde{\mathbb{H}} \tilde{w}_\beta \tilde{\eta}_w = \widetilde{\mathbb{H}} \tilde{\eta}_{s_\beta w}$. Thus, if $w^{-1}\beta$ is positive, it follows that:

$$\tilde{w}_\beta \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde{\eta}_w \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \subset \Big(\prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde{\eta}_{s_\beta w} \prod_{\alpha \in \Phi^+}{}' \mathbb{N}^*_\alpha.$$

Now suppose that $w^{-1}\beta$ is negative. Write $w' = s_\beta w$. Then $w'^{-1}\beta$ is positive, and $\tilde\eta_w = \tilde h \tilde w_\beta \tilde\eta_{w'}$ for some $\tilde h \in \widetilde{\mathbb{H}}$. Using the relations $\mathcal{R}^2_{\widetilde{\mathbb{M}}_\mathbb{Z},\widetilde{\mathbb{H}}}$ and $\mathcal{R}^2_{\mathbb{N}^*,\widetilde{\mathbb{M}}_\mathbb{Z}}$, and proceeding as before, it follows that:

$$\tilde w_\beta \Big(\prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde\eta_w \prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha \subset \Big(\prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde\eta_w \prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha \cup \Big(\prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha \Big) \widetilde{\mathbb{H}} \tilde\eta_{w'} \prod_{\alpha\in\Phi^+}{}' \mathbb{N}^*_\alpha.$$

This completes the proof. □

Remark. Most of the preceding proof was taken from [5] Lemme 6.11.

Corollary 4. *If $\mathcal{G}_\mathbb{G}$ and $\mathcal{R}_\mathbb{G}$ are defined by taking $\mathcal{A} := 1$ and $c := 1$ in Theorem 3, then $\langle \mathcal{G}_\mathbb{G} \,|\, \mathcal{R}_\mathbb{G} \rangle$ is a presentation of $\mathbb{G}$.* □

In the sequel we will continue to denote families of relations in the group $\widetilde{\mathbb{G}}$ as in Lemma 2, while using the notation $\mathcal{R}^2_{\mathbb{M}_\mathbb{Z},\mathbb{H}}$, $\mathcal{R}^*_{\mathbb{N},\mathbb{H}}$, etc., for the corresponding families of relations in the group $\mathbb{G}$. This should not cause any confusion.

Starting with arbitrary $\mathcal{A}$ and c, if we define $\widetilde{\mathbb{G}}$ by the presentation above, then $\widetilde{\mathbb{G}}$ is a central extension of $\mathbb{G}$. However, the subgroup of $\widetilde{\mathbb{G}}$ generated by $\mathcal{G}_\mathcal{A}$ might be a strict quotient of $\mathcal{A}$, hence $\widetilde{\mathbb{G}}$ need not be a central extension of $\mathbb{G}$ *by* $\mathcal{A}$. Matsumoto completed the above construction as follows.

Theorem 5. ([5] Théorème 5.10) *There exists a central extension of $\mathbb{G}$ by $\mathcal{A}$ with Steinberg symbol c if and only if either c is bilinear or $\mathbb{G}$ is symplectic.* □

Necessity follows from $\mathcal{S}4$. To prove sufficiency, Matsumoto first reduced to the case where $\mathbb{G}$ is either simply-laced or symplectic. Next, he showed (cf. [5] Lemme 6.6) that the abstract group $\widetilde{\mathbb{M}}$ defined by the generators $\mathcal{G}_\mathcal{A} \cup \mathcal{G}_{\widetilde{\mathbb{H}}} \cup \mathcal{G}_{\widetilde{\mathbb{M}}_\mathbb{Z}}$ subject only to the first eleven families of relations in Lemma 2 is a central extension of $\mathbb{M} := N_\mathbb{G}(\mathbb{H})$ by $\mathcal{A}$. Finally, he built the desired extension of $\mathbb{G}$ as a group of permutations on the set:

$$\mathfrak{X} := \big\{(g,\tilde m)\,\big|\, g\in\mathbb{G},\ \tilde m\in\widetilde{\mathbb{M}},\ \mathbf{m}(g)=\mathfrak{p}(\tilde m)\big\},$$

where $\mathfrak{p}$ denotes the natural projection from $\widetilde{\mathbb{M}}$ to $\mathbb{M}$, and $\mathbf{m}$ is the unique map from $\mathbb{G}$ to $\mathbb{M}$ such that $\mathbf{m}(m)=m$ for all $m\in\mathbb{M}$, and $\mathbf{m}(ngn')=\mathbf{m}(g)$ for all $n,n'\in\mathbb{N}$, $g\in\mathbb{G}$. The existence of $\mathbf{m}$ follows from the Bruhat decomposition for $\mathbb{G}$.

§2. Construction of a 2-Cocycle

We continue to use the notation of §1. Let $\mathbb{F}$ be an infinite field, $\mathbb{G}$ the $\mathbb{F}$-rational points of a simple simply-connected algebraic group that is defined and split over $\mathbb{F}$, $\mathcal{A}$ an abelian group, and $c:\mathbb{F}^\times\times\mathbb{F}^\times\to\mathcal{A}$ a Steinberg symbol that is bilinear if $\mathbb{G}$ is not symplectic (cf. §1 Theorem 5). In §1 we constructed a central extension $\widetilde{\mathbb{G}}$ of $\mathbb{G}$ by $\mathcal{A}$, together with a natural projection $\mathfrak{p}:\widetilde{\mathbb{G}}\to\mathbb{G}$. In other words, we have an exact sequence:

$$1\to\mathcal{A}\hookrightarrow\widetilde{\mathbb{G}}\xrightarrow{\mathfrak{p}}\mathbb{G}\to 1.$$

The goal of this section is to construct an explicit 2-cocycle in $Z^2(\mathbb{G};\mathcal{A})$ that represents the cohomology class in $H^2(\mathbb{G};\mathcal{A})$ of the extension $\widetilde{\mathbb{G}}$.

Recall that for any subgroup G of $\mathbb{G}$, a *section of* G (with respect to $\mathfrak{p}$) is a map $\mathfrak{s}:G\to\widetilde{\mathbb{G}}$ such that $\mathfrak{p}\big(\mathfrak{s}(g)\big)=g$ for all $g\in G$, and $\mathfrak{s}(1_{\mathbb{G}})=1_{\mathcal{A}}$. If $\mathfrak{s}$ is a homomorphism, then $\mathfrak{s}$ is said to be a *splitting of* G. In this case, $\mathfrak{s}(G)\cong G$, and $\mathfrak{p}^{-1}(G)=\mathfrak{s}(G)\cdot\mathcal{A}\cong G\times\mathcal{A}$ since $\mathcal{A}$ is central. Note that if $\mathfrak{s}$ and $\mathfrak{s}'$ are two splittings of G, the map $\mathfrak{s}'\mathfrak{s}^{-1}:G\to\mathcal{A}$, $g\mapsto\mathfrak{s}'(g)\mathfrak{s}(g)^{-1}$, is a homomorphism.

Lemma 1. *There exists a unique splitting $\mathfrak{s}_{\mathbb{N}}$ of $\mathbb{N}$ with the property:*

$$\mathfrak{s}_{\mathbb{N}}\big(n_\alpha(x)\big)=n_\alpha^*(x),\qquad \alpha\in\Phi^+,\ x\in\mathbb{F}.$$

If the exponent of $\mathcal{A}$ is finite and nonzero in $\mathbb{F}$, then $\mathfrak{s}_{\mathbb{N}}$ is the only splitting of $\mathbb{N}$.

Proof : As we mentioned earlier, Steinberg proved $\big($cf. [5] Lemme 5.1(a)$\big)$ that the map $\mathfrak{p} : \mathbb{N}^* \to \mathbb{N}$ is an isomorphism, and this immediately implies the first statement of the lemma.

Now suppose that $\mathcal{A}$ has finite exponent $z \in \mathbb{Z} \cap \mathbb{F}^\times$. If $\mathfrak{s}'$ is any splitting of $\mathbb{N}$, then $\mathfrak{s}'\mathfrak{s}_{\mathbb{N}}^{-1} : \mathbb{N} \to \mathcal{A}$, $n \mapsto \mathfrak{s}'(n)\mathfrak{s}_{\mathbb{N}}(n)^{-1}$, is a homomorphism. For all $\alpha \in \Phi^+$, $x \in \mathbb{F}$:

$$\mathfrak{s}'\mathfrak{s}_{\mathbb{N}}^{-1}\big(n_\alpha(x)\big) = \mathfrak{s}'\mathfrak{s}_{\mathbb{N}}^{-1}\big(n_\alpha(xz^{-1})^z\big) = \mathfrak{s}'\mathfrak{s}_{\mathbb{N}}^{-1}\big(n_\alpha(xz^{-1})\big)^z = 1_{\mathcal{A}}$$

since $a^z = 1_{\mathcal{A}}$ for all $a \in \mathcal{A}$. This shows that $\mathfrak{s}'\big(n_\alpha(x)\big) = \mathfrak{s}_{\mathbb{N}}\big(n_\alpha(x)\big) = n^*_\alpha(x)$, hence $\mathfrak{s}' = \mathfrak{s}_{\mathbb{N}}$ by the first statement of the lemma. $\square$

Let $d := \mathrm{rank}(\mathbb{G})$. For the remainder of the section, we fix an arbitrary ordering of the simple roots: $\Delta = \{\breve{\alpha}_1, \dots, \breve{\alpha}_d\}$. We define a section $\mathfrak{s}_{\mathbb{H}}$ of $\mathbb{H}$ as follows. First, let:

$$\mathfrak{s}_{\mathbb{H}}\big(h_\alpha(x)\big) := \tilde{h}_\alpha(x)\, c_\alpha(x,x), \qquad \alpha \in \Delta,\ x \in \mathbb{F}^\times.$$

It follows from the relations $\mathcal{R}^2_{\mathbb{H}}$ and $\mathcal{R}^*_{\mathbb{H}}$ that every $h \in \mathbb{H}$ can be uniquely expressed in the form $h = \prod_{i=1}^{d} h_{\breve{\alpha}_i}(x_i)$ with each $x_i \in \mathbb{F}^\times$, and we define:

$$\mathfrak{s}_{\mathbb{H}}(h) := \mathfrak{s}_{\mathbb{H}}\big(h_{\breve{\alpha}_d}(x_d)\big) \dots \mathfrak{s}_{\mathbb{H}}\big(h_{\breve{\alpha}_1}(x_1)\big) = \prod_{i=d}^{1} \tilde{h}_{\breve{\alpha}_i}(x_i)\ \prod_{i=1}^{d} c_{\breve{\alpha}_i}(x_i, x_i).$$

In order to extend $\mathfrak{s}_{\mathbb{H}}$ to a section $\mathfrak{s}_{\mathbb{M}}$ of the subgroup $\mathbb{M} := N_{\mathbb{G}}(\mathbb{H}) = \mathbb{H}{\cdot}\mathbb{M}_{\mathbb{Z}}$, we introduce a certain finite subset $\mathfrak{M} \subset \mathbb{M}_{\mathbb{Z}}$ as follows. Every root $\alpha \in \Phi$ defines a homomorphism $\alpha : \mathbb{H} \to \mathbb{F}^\times$, $h \mapsto h^\alpha := \alpha(h)$. The group $\mathbb{M}$ acts on Φ: for all $m \in \mathbb{M}$, $\alpha \in \Phi$, let $m\alpha$ be the unique element of Φ such that $h^{(m\alpha)} = (h^m)^\alpha$ for all $h \in \mathbb{H}$, where $h^m := m^{-1}hm$. As $\mathbb{H}$ acts trivially on Φ, we obtain a well-defined (faithful) action of the Weyl group $\mathbf{W} \cong \mathbb{M}/\mathbb{H}$ on Φ. Regarding $\mathbf{W}$ as a group of permutations on Φ, it is known that $\mathbf{W}$ is generated by the simple reflections $\big\{s_\alpha \,\big|\, \alpha \in \Delta\big\}$, where each s_α is the element of $\mathbf{W}$ associated to $w_\alpha \in \mathbb{M}$. For any $w \in \mathbf{W}$, the *length* $\ell(w)$ of w is the smallest integer ℓ such that w has an expression of the form $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_\ell}$ with each $\alpha_i \in \Delta$. For later purposes, we also define the *length* $\ell(g)$ of an arbitrary element $g \in \mathbb{G}$ by pulling back the length function on $\mathbf{W}$ via the composition:

$$\mathbb{G} \xrightarrow{\mathbf{m}} \mathbb{M} \to \mathbb{M}/\mathbb{H} \cong \mathbf{W}.$$

For all $w \in \mathbf{W}$, write $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_\ell}$ with $\ell = \ell(w)$, and let $\eta_w := w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_\ell}$. Then $\eta_w \in \mathbb{M}_{\mathbb{Z}}$, and the assignment $w \mapsto \eta_w$ is independent of the expression of w as a minimal product of simple reflections $\big($cf. [5] Lemme 6.2$\big)$. Note that $\ell(\eta_w) = \ell(w)$. We now define:

$$\mathfrak{M} := \big\{\eta_w \,\big|\, w \in \mathbf{W}\big\}.$$

The set $\mathfrak{M}$ is not a group in general, but the map $w \mapsto \eta_w$ gives a bijection between $\mathbf{W}$ and $\mathfrak{M}$, and $\mathfrak{M}$ is a complete set of distinct coset representatives for $\mathbb{M}/\mathbb{H}$.

Lemma 2. *There exists a unique section $\mathfrak{s}_{\mathbb{M}}$ of $\mathbb{M}$ with the properties:*

(a) $\mathfrak{s}_{\mathbb{M}}(w_\alpha) = \tilde{w}_\alpha$ *for all* $\alpha \in \Delta$*,*

(b) $\mathfrak{s}_{\mathbb{M}}(\eta\eta') = \mathfrak{s}_{\mathbb{M}}(\eta)\,\mathfrak{s}_{\mathbb{M}}(\eta')$ *for all* $\eta, \eta' \in \mathfrak{M}$ *such that* $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$*,*

(c) $\mathfrak{s}_{\mathbb{M}}(h\eta) = \mathfrak{s}_{\mathbb{H}}(h)\,\mathfrak{s}_{\mathbb{M}}(\eta)$ *for all* $h \in \mathbb{H}$*,* $\eta \in \mathfrak{M}$*.*

Proof : Let $\eta \in \mathfrak{M}$, and write $\eta = w_{\alpha_1} \dots w_{\alpha_\ell}$ with $\ell = \ell(\eta)$ and each $\alpha_i \in \Delta$. Then by [5] Lemme 6.2(d):

$$\mathfrak{s}_{\mathbb{M}}(\eta) := \tilde{w}_{\alpha_1} \tilde{w}_{\alpha_2} \dots \tilde{w}_{\alpha_\ell}$$

is well-defined, and $\mathfrak{s}_{\mathbb{M}}$ clearly satisfies (a) and (b). Since $\mathbb{H} \cap \mathfrak{M} = \{1_{\mathbb{G}}\}$, $\mathfrak{s}_{\mathbb{M}}$ can be extended to a section of $\mathbb{M}$ satisfying (c) as well. The uniqueness assertion is clear since $\mathbb{M} = \coprod_{\eta \in \mathfrak{M}} \mathbb{H}\eta$. $\square$

Lemma 3. *There exists a unique section $\mathfrak{s}_{\mathbb{G}}$ of $\mathbb{G}$ with the property:*

$$\mathfrak{s}_{\mathbb{G}}(nmn') = \mathfrak{s}_{\mathbb{N}}(n)\,\mathfrak{s}_{\mathbb{M}}(m)\,\mathfrak{s}_{\mathbb{N}}(n'), \qquad n, n' \in \mathbb{N},\ m \in \mathbb{M}.$$

Proof : Uniqueness follows from the Bruhat decomposition for $\mathbb{G}$. In order to prove existence, we will first verify the following assertion:

(1) if $nm = mn'$ with $n, n' \in \mathbb{N}$, $m \in \mathbb{M}$, then $\mathfrak{s}_{\mathbb{N}}(n)\, \mathfrak{s}_{\mathbb{M}}(m) = \mathfrak{s}_{\mathbb{M}}(m)\, \mathfrak{s}_{\mathbb{N}}(n')$.

First, suppose that $n = n_\alpha(x)$ with $\alpha \in \Phi^+$, $x \in \mathbb{F}$. If $x = 0$, then (1) holds trivially, thus we can assume that $x \neq 0$. If $m = h_\beta(y)$ with $\beta \in \Delta$, $y \in \mathbb{F}^\times$, then $n' = n_\alpha(xy^{-\langle\alpha,\beta\rangle})$ by $\mathcal{R}^*_{\mathbb{N},\mathbb{H}}$. But $n^*_\alpha(x)\, \tilde{h}_\beta(y) = \tilde{h}_\beta(y)\, n^*_\alpha(xy^{-\langle\alpha,\beta\rangle})$ by the relations in $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$, hence (1) follows in this case. Similarly, if $m = w_\beta$ with $\beta \in \Delta$, then $\alpha \neq \beta$ since $x \neq 0$ and $n' \in \mathbb{N}$, and $n_\alpha(x)\, w_\beta = w_\beta\, n_{s_\beta\alpha}(\mathfrak{n}_{\beta,\alpha}\, x)$ by $\mathcal{R}^*_{\mathbb{N},\mathbb{M}_\mathbb{Z}}$. By the relations in $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{M}}_\mathbb{Z}}$, $n^*_\alpha(x)\, \tilde{w}_\beta = \tilde{w}_\beta\, n^*_{s_\beta\alpha}\big(\mathfrak{n}_{\beta,\alpha}\, x\big)$, and (1) follows in this case as well. For arbitrary $m \in \mathbb{M}$, write $m = h\eta$ with $h \in \mathbb{H}$, $\eta \in \mathfrak{M}$. Factoring $h = \prod_{i=1}^{d} h_{\breve{\alpha}_i}(x_i)$ with each $x_i \in \mathbb{F}^\times$, and $\eta = w_{\alpha_1} \dots w_{\alpha_\ell}$ with $\ell = \ell(\eta)$ and each $\alpha_i \in \Delta$, we must show that:

$$n^*_\alpha(x)\, \tilde{h}_{\breve{\alpha}_d}(x_d) \dots \tilde{h}_{\breve{\alpha}_2}(x_2)\, \tilde{h}_{\breve{\alpha}_1}(x_1)\, \tilde{w}_{\alpha_1} \dots \tilde{w}_{\alpha_\ell} \prod_{i=1}^{d} c_{\breve{\alpha}_i}(x_i, x_i)$$

is equal to:

$$\tilde{h}_{\breve{\alpha}_d}(x_d) \dots \tilde{h}_{\breve{\alpha}_2}(x_2)\, \tilde{h}_{\breve{\alpha}_1}(x_1)\, \tilde{w}_{\alpha_1} \dots \tilde{w}_{\alpha_\ell} \prod_{i=1}^{d} c_{\breve{\alpha}_i}(x_i, x_i)\, \mathfrak{s}_{\mathbb{N}}(n').$$

This follows by an inductive argument using the preceding results. Thus, (1) holds if $n = n_\alpha(x)$. For arbitrary $n \in \mathbb{N}$, we factor n into a product of generators of the form $n_\alpha(x)$, and the assertion (1) follows by an inductive argument using the fact that $\mathfrak{s}_{\mathbb{N}}$ is a splitting.

Now for any $g \in \mathbb{G}$, if $g = nmn'$ with $n, n' \in \mathbb{N}$, $m \in \mathbb{M}$, then $m = \mathbf{m}(g)$ is uniquely determined. To establish the lemma, it suffices to show that if $nmn' = n_1mn'_1$, then $\mathfrak{s}_{\mathbb{N}}(n)\, \mathfrak{s}_{\mathbb{M}}(m)\, \mathfrak{s}_{\mathbb{N}}(n') = \mathfrak{s}_{\mathbb{N}}(n_1)\, \mathfrak{s}_{\mathbb{M}}(m)\, \mathfrak{s}_{\mathbb{N}}(n'_1)$. But $n_1^{-1}nm = mn'_1{n'}^{-1}$, hence (1) implies that $\mathfrak{s}_{\mathbb{N}}(n_1^{-1}n)\, \mathfrak{s}_{\mathbb{M}}(m) = \mathfrak{s}_{\mathbb{M}}(m)\, \mathfrak{s}_{\mathbb{N}}(n'_1{n'}^{-1})$. As $\mathfrak{s}_{\mathbb{N}}$ is a splitting, the result follows. □

Recall that for any group G, a *2-cocycle on G with coefficients in $\mathcal{A}$* is a map $\sigma : G \times G \to \mathcal{A}$ such that:

$$(2) \qquad \sigma(g, g')\, \sigma(gg', g'') = \sigma(g, g'g'')\, \sigma(g', g''), \qquad g, g', g'' \in G,$$

and $\sigma(1_G, 1_G) = 1_{\mathcal{A}}$, where 1_G is the identity element in G. This definition implies that $\sigma(1_G, g) = \sigma(g, 1_G) = 1_{\mathcal{A}}$ for all $g \in G$. Let $Z^2(G; \mathcal{A})$ denote the set of all such 2-cocycles.

Proposition 4. *Let $\sigma_{\mathbb{G}} \in Z^2(\mathbb{G}; \mathcal{A})$ be defined by:*

$$\sigma_{\mathbb{G}}(g, g') := \mathfrak{s}_{\mathbb{G}}(g)\, \mathfrak{s}_{\mathbb{G}}(g')\, \mathfrak{s}_{\mathbb{G}}(gg')^{-1}, \qquad g, g' \in \mathbb{G}.$$

Then $\sigma_{\mathbb{G}}$ satisfies the following properties:

(*a*) $\sigma_{\mathbb{G}}(g, n) = \sigma_{\mathbb{G}}(n, g) = 1_{\mathcal{A}}$ *for all* $n \in \mathbb{N}$, $g \in \mathbb{G}$,

(*b*) $\sigma_{\mathbb{G}}(ng, g'n') = \sigma_{\mathbb{G}}(g, g')$ *for all* $n, n' \in \mathbb{N}$, $g, g' \in \mathbb{G}$,

(*c*) $\sigma_{\mathbb{G}}(gn, g') = \sigma_{\mathbb{G}}(g, ng')$ *for all* $n \in \mathbb{N}$, $g, g' \in \mathbb{G}$,

(*d*) $\sigma_{\mathbb{G}}(h, \eta) = 1_{\mathcal{A}}$ *for all* $h \in \mathbb{H}$, $\eta \in \mathfrak{M}$,

(*e*) $\sigma_{\mathbb{G}}(\eta, \eta') = 1_{\mathcal{A}}$ *for all* $\eta, \eta' \in \mathfrak{M}$ *such that* $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$.

Proof : Property (a) follows from Lemma 3, while (b) and (c) follow from (a) and the cocycle relation (2). Properties (d) and (e) follow from Lemmas 2 and 3. □

We will next consider the restriction of the 2-cocycle $\sigma_{\mathbb{G}}$ to certain "standard" subgroups of $\mathbb{G}$. Let $\Delta^\sharp \subset \Delta$, let $\Phi^\sharp$ be the set of roots in Φ spanned by the elements of $\Delta^\sharp$, and let $\Phi^{\sharp+} := \Phi^\sharp \cap \Phi^+$. Let $\mathbb{G}^\sharp$ be the group that is generated in $\mathbb{G}$ by:

$$\left\{ h_\alpha(x) \,\middle|\, \alpha \in \Delta^\sharp,\ x \in \mathbb{F}^\times \right\} \cup \left\{ w_\alpha \,\middle|\, \alpha \in \Delta^\sharp \right\} \cup \left\{ n_\alpha(x) \,\middle|\, \alpha \in \Phi^{\sharp+},\ x \in \mathbb{F} \right\}.$$

We call $\mathbb{G}^\sharp$ a *standard subgroup* of $\mathbb{G}$ if $\mathbb{G}^\sharp$ is also the $\mathbb{F}$-rational points of a simple simply-connected algebraic group that is defined and split over $\mathbb{F}$.

Let $\mathbb{G}^\sharp$ be a standard subgroup of $\mathbb{G}$. Realize $\mathbb{G}^\sharp$ as a Chevalley group with generators $\{n_\alpha(x) \,|\, \alpha \in \Phi^\sharp,\ x \in \mathbb{F}\}$, and give $\Delta^\sharp$ the order inherited from Δ. We construct (just as we did with $\mathbb{G}$ in §1) a central extension $\widetilde{\mathbb{G}}^\sharp$ of $\mathbb{G}^\sharp$ by $\mathcal{A}$ corresponding to the Steinberg symbol c (with respect to our choices of $\{h_\alpha(x) \,|\, \alpha \in \Delta^\sharp,\ x \in \mathbb{F}^\times\}$, $\Delta^\sharp$, and $\{n_\alpha \,|\, \alpha \in \Pi^\sharp\}$). Since $\mathbb{H}^\sharp := \mathbb{H} \cap \mathbb{G}^\sharp$, $\mathbb{N}^\sharp := \mathbb{N} \cap \mathbb{G}^\sharp$, $\mathbb{M}^\sharp := \mathbb{M} \cap \mathbb{G}^\sharp$ and $\mathfrak{M}^\sharp := \mathfrak{M} \cap \mathbb{G}^\sharp$ are the analogues of $\mathbb{H}$, $\mathbb{N}$, $\mathbb{M}$ and $\mathfrak{M}$, respectively, for the group $\mathbb{G}^\sharp$, we can proceed as above to construct the section $\mathfrak{s}_{\mathbb{G}^\sharp} : \mathbb{G}^\sharp \to \widetilde{\mathbb{G}}^\sharp$ and the corresponding 2-cocycle $\sigma_{\mathbb{G}^\sharp} \in Z^2(\mathbb{G}^\sharp; \mathcal{A})$.

Lemma 5. *For every standard subgroup $\mathbb{G}^\sharp$ of $\mathbb{G}$, $\sigma_{\mathbb{G}}\big|_{\mathbb{G}^\sharp \times \mathbb{G}^\sharp} = \sigma_{\mathbb{G}^\sharp}$.*

Proof : By the results of §1, $\widetilde{\mathbb{G}}^\sharp$ is isomorphic to the group generated in $\widetilde{\mathbb{G}}$ by:

$$\mathcal{A} \cup \{\tilde{h}_\alpha(x) \,|\, \alpha \in \Delta^\sharp,\ x \in \mathbb{F}^\times\} \cup \{\tilde{w}_\alpha \,|\, \alpha \in \Delta^\sharp\} \cup \{n^*_\alpha(x) \,|\, \alpha \in \Phi^{\sharp +},\ x \in \mathbb{F}\}.$$

The natural projection $\widetilde{\mathbb{G}}^\sharp \to \mathbb{G}^\sharp$ is simply the restriction of $\mathfrak{p} : \widetilde{\mathbb{G}} \to \mathbb{G}$ to $\widetilde{\mathbb{G}}^\sharp$. The relation $\mathfrak{s}_{\mathbb{G}^\sharp} = \mathfrak{s}_{\mathbb{G}}\big|_{\mathbb{G}^\sharp}$ follows immediately from the definitions, and this implies the lemma. □

Lemma 6. *Let $\{\mathbb{G}^i \,|\, 1 \le i \le p\}$ be a collection of mutually commuting standard subgroups of $\mathbb{G}$. Then $\{\widetilde{\mathbb{G}}^i \,|\, 1 \le i \le p\}$ are mutually commuting subgroups of $\widetilde{\mathbb{G}}$. Moreover:*

$$\mathfrak{s}_{\mathbb{G}}(g_1 \ldots g_p) = \mathfrak{s}_{\mathbb{G}}(g_1) \ldots \mathfrak{s}_{\mathbb{G}}(g_p) \tag{3}$$

for all $g_i \in \mathbb{G}^i$, $1 \le i \le p$.

Proof : For each i, let Δ^i, Φ^i, Φ^{i+}, $\mathbb{H}^i$, $\mathbb{N}^i$, $\mathbb{M}^i$ and $\mathfrak{M}^i$ be defined as above for the group $\mathbb{G}^i$. Observe that if $\alpha \in \Delta^i$, $\beta \in \Delta^j$, $i \neq j$, then $\langle \alpha, \beta \rangle = 0$. Indeed, this follows from the relations $\mathcal{R}^*_{\mathbb{N},\mathbb{H}}$. Using the relations in $\mathcal{R}_{\mathcal{A}}$, $\mathcal{R}^*_{\widetilde{\mathbb{H}}}$, $\mathcal{R}_{\widetilde{\mathbb{H}},\mathcal{A}}$, $\mathcal{R}^0_{\widetilde{\mathbb{M}}_{\mathbb{Z}}}$, $\mathcal{R}_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\mathcal{A}}$, $\mathcal{R}^*_{\widetilde{\mathbb{M}}_{\mathbb{Z}},\widetilde{\mathbb{H}}}$, $\mathcal{R}^*_{\mathbb{N}^*}$, $\mathcal{R}_{\mathbb{N}^*,\mathcal{A}}$, $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{H}}}$, and $\mathcal{R}^*_{\mathbb{N}^*,\widetilde{\mathbb{M}}_{\mathbb{Z}}}$, it follows that if $\tilde{g}_i$ [resp. $\tilde{g}_j$] is a *generator* of $\widetilde{\mathbb{G}}^i$ [resp. $\widetilde{\mathbb{G}}^j$], $i \neq j$, then $\tilde{g}_i\tilde{g}_j = \tilde{g}_j\tilde{g}_i$. This implies the first statement of the lemma.

We first establish (3) in the case where $g_i = h_i \in \mathbb{H}^i$, $1 \leq i \leq p$. The product $h_1 \dots h_p$ can be uniquely expressed in the form $\prod_{i=1}^{d} h_{\check{\alpha}_i}(x_i)$ with each $x_i \in \mathbb{F}^\times$. Let $l := \operatorname{card}\{i \,|\, x_i \neq 1\}$. We induct on l, the case $l = 0$ being trivial. Thus, suppose that $l \geq 1$, and let k be the largest integer such that $x_k \neq 1$. Let i be such that $\check{\alpha}_k \in \Delta^i$. From the definition of $\mathfrak{s}_{\mathbb{H}}$:

$$\mathfrak{s}_{\mathbb{H}}(h_1 \dots h_p) = \mathfrak{s}_{\mathbb{H}}\big(h_{\check{\alpha}_k}(x_k)\, h_1 \dots \widehat{h}_i \dots h_p\big) = \mathfrak{s}_{\mathbb{H}}\big(h_{\check{\alpha}_k}(x_k)\big)\, \mathfrak{s}_{\mathbb{H}}(h_1 \dots \widehat{h}_i \dots h_p),$$

where $\widehat{h}_i := h_{\check{\alpha}_k}(x_k)^{-1} h_i$. By induction on l, this expression equals:

$$\mathfrak{s}_{\mathbb{H}}\big(h_{\check{\alpha}_k}(x_k)\big)\, \mathfrak{s}_{\mathbb{H}}\big(h_1\big) \dots \mathfrak{s}_{\mathbb{H}}\big(\widehat{h}_i\big) \dots \mathfrak{s}_{\mathbb{H}}\big(h_p\big) = \mathfrak{s}_{\mathbb{H}}\big(h_1\big) \dots \mathfrak{s}_{\mathbb{H}}\big(h_{\check{\alpha}_k}(x_k)\big)\, \mathfrak{s}_{\mathbb{H}}\big(\widehat{h}_i\big) \dots \mathfrak{s}_{\mathbb{H}}\big(h_p\big),$$

since $\widetilde{\mathbb{H}}^i$ commutes with $\widetilde{\mathbb{H}}^j$ if $i \neq j$. But $\mathfrak{s}_{\mathbb{H}}\big(h_{\check{\alpha}_k}(x_k)\big)\, \mathfrak{s}_{\mathbb{H}}\big(\widehat{h}_i\big) = \mathfrak{s}_{\mathbb{H}}(h_i)$, hence (3) holds in this case.

Next, suppose that $g_i = \eta_i \in \mathfrak{M}^i$, $1 \leq i \leq p$. Since the elements of $\mathfrak{M}^i$ commute with the elements of $\mathfrak{M}^j$ if $i \neq j$, it follows that $\ell(\eta_1 \dots \eta_p) = \ell(\eta_1) + \dots + \ell(\eta_p)$. By an inductive argument using Lemma 2(b):

$$\mathfrak{s}_{\mathbb{M}}(\eta_1 \dots \eta_p) = \mathfrak{s}_{\mathbb{M}}(\eta_1) \dots \mathfrak{s}_{\mathbb{M}}(\eta_p),$$

and (3) holds in this case as well.

To establish (3) in general, let $g_i \in \mathbb{G}^i$, $1 \leq i \leq p$, and factor each $g_i = n_i h_i \eta_i n_i'$ with $n_i, n_i' \in \mathbb{N}^i$, $h_i \in \mathbb{H}^i$, and $\eta_i \in \mathfrak{M}^i$. Then:

$$\begin{aligned}
\mathfrak{s}_{\mathbb{G}}(g_1 \dots g_p) &= \mathfrak{s}_{\mathbb{G}}(n_1 h_1 \eta_1 n_1' \dots n_p h_p \eta_p n_p') \\
&= \mathfrak{s}_{\mathbb{G}}(n_1 \dots n_p h_1 \dots h_p \eta_1 \dots \eta_p n_1' \dots n_p') \\
&= \mathfrak{s}_{\mathbb{N}}(n_1 \dots n_p)\, \mathfrak{s}_{\mathbb{H}}(h_1 \dots h_p)\, \mathfrak{s}_{\mathbb{M}}(\eta_1 \dots \eta_p)\, \mathfrak{s}_{\mathbb{N}}(n_1' \dots n_p').
\end{aligned}$$

Using the results above and the fact that $\mathfrak{s}_{\mathbb{N}}$ is a splitting, the last expression equals:

$$\mathfrak{s}_{\mathbb{N}}(n_1) \dots \mathfrak{s}_{\mathbb{N}}(n_p)\, \mathfrak{s}_{\mathbb{H}}(h_1) \dots \mathfrak{s}_{\mathbb{H}}(h_p)\, \mathfrak{s}_{\mathbb{M}}(\eta_1) \dots \mathfrak{s}_{\mathbb{M}}(\eta_p)\, \mathfrak{s}_{\mathbb{N}}(n_1') \dots \mathfrak{s}_{\mathbb{N}}(n_p').$$

Since $\widetilde{\mathbb{G}}^i$ commutes with $\widetilde{\mathbb{G}}^j$ if $i \neq j$, this expression equals:

$$\mathfrak{s}_{\mathbb{N}}(n_1)\,\mathfrak{s}_{\mathbb{H}}(h_1)\,\mathfrak{s}_{\mathbb{M}}(\eta_1)\,\mathfrak{s}_{\mathbb{N}}(n_1')\ldots\mathfrak{s}_{\mathbb{N}}(n_p)\,\mathfrak{s}_{\mathbb{H}}(h_p)\,\mathfrak{s}_{\mathbb{M}}(\eta_p)\,\mathfrak{s}_{\mathbb{N}}(n_p').$$

As $\mathfrak{s}_{\mathbb{N}}(n_i)\,\mathfrak{s}_{\mathbb{H}}(h_i)\,\mathfrak{s}_{\mathbb{M}}(\eta_i)\,\mathfrak{s}_{\mathbb{N}}(n_i') = \mathfrak{s}_{\mathbb{G}}(g_i)$ for each i, the result follows. □

Theorem 7. *Let $\{\mathbb{G}^i \,|\, 1 \leq i \leq p\}$ be a collection of mutually commuting standard subgroups of $\mathbb{G}$. Then:*

$$\sigma_{\mathbb{G}}(g_1 \ldots g_p, g_1' \ldots g_p') = \prod_{i=1}^{p} \sigma_{\mathbb{G}^i}(g_i, g_i')$$

for all $g_i, g_i' \in \mathbb{G}^i$, $1 \leq i \leq p$.

Proof : By Lemma 6:

$$\mathfrak{s}_{\mathbb{G}}(g_1 \ldots g_p) = \mathfrak{s}_{\mathbb{G}}(g_1)\ldots\mathfrak{s}_{\mathbb{G}}(g_p), \qquad \mathfrak{s}_{\mathbb{G}}(g_1' \ldots g_p') = \mathfrak{s}_{\mathbb{G}}(g_1')\ldots\mathfrak{s}_{\mathbb{G}}(g_p').$$

Since $\widetilde{\mathbb{G}}^i$ commutes with $\widetilde{\mathbb{G}}^j$ if $i \neq j$:

$$\begin{aligned}\mathfrak{s}_{\mathbb{G}}(g_1 \ldots g_p)\,\mathfrak{s}_{\mathbb{G}}(g_1' \ldots g_p') &= \mathfrak{s}_{\mathbb{G}}(g_1)\,\mathfrak{s}_{\mathbb{G}}(g_1')\ldots\mathfrak{s}_{\mathbb{G}}(g_p)\,\mathfrak{s}_{\mathbb{G}}(g_p') \\ &= \mathfrak{s}_{\mathbb{G}}(g_1 g_1')\ldots\mathfrak{s}_{\mathbb{G}}(g_p g_p')\prod_{i=1}^{p}\sigma_{\mathbb{G}^i}(g_i, g_i')\end{aligned}$$

by Lemma 5. On the other hand:

$$\mathfrak{s}_{\mathbb{G}}(g_1 \ldots g_p g_1' \ldots g_p') = \mathfrak{s}_{\mathbb{G}}(g_1 g_1' \ldots g_p g_p') = \mathfrak{s}_{\mathbb{G}}(g_1 g_1')\ldots\mathfrak{s}_{\mathbb{G}}(g_p g_p').$$

Comparing these expressions, the theorem follows. □

§3. The 2-Cocycle for the General Linear Group

We continue to use the notation of the previous two sections. For the remainder of the paper, let $G^\flat$ be the general linear group $GL_{n+1}(\mathbb{F})$, and let $\mathbb{G}^\flat \subset G^\flat$ be the special linear group $SL_{n+1}(\mathbb{F})$, where n is a fixed positive integer. Let $T^\flat$ be the subgroup of diagonal matrices in $G^\flat$, and for $1 \le i \le n+1$, let τ_i be the i-th coordinate homomorphism on $T^\flat$:

$$\tau_i : T^\flat \to \mathbb{F}^\times, \qquad t = \mathrm{diag}(t_1, \dots, t_{n+1}) \mapsto t_i.$$

The set $\Phi^\flat$ of roots of $G^\flat$ relative to $T^\flat$ can be identified with the set of pairs $\{(i,j) \mid 1 \le i, j \le n+1,\ i \ne j\}$, where:

$$t^\alpha := \frac{\tau_i(t)}{\tau_j(t)}, \qquad t \in T^\flat,\ \alpha = (i,j) \in \Phi^\flat.$$

We call a root (i,j) *positive* if $i < j$ and *negative* if $i > j$. Let $\Phi^{\flat+}$ be the set of positive roots in $\Phi^\flat$, and $\Delta^\flat$ its ordered base of simple roots $\{\check{\alpha}_i := (i, i+1) \mid 1 \le i \le n\}$. Let $\mathbb{H}^\flat := T^\flat \cap \mathbb{G}^\flat$, and let $\mathbb{N}^\flat$, $\mathbb{M}^\flat_{\mathbb{Z}}$, $\mathbb{M}^\flat$ and $\mathfrak{M}^\flat$ be defined for the simple Chevalley group $\mathbb{G}^\flat$ as in §§1-2. Note that $N^\flat := \mathbb{N}^\flat$ is the standard unipotent subgroup of $G^\flat$ associated to our choice of positive roots, and $M^\flat := T^\flat{\cdot}\mathbb{M}^\flat_{\mathbb{Z}} = \coprod_{\eta \in \mathfrak{M}^\flat} T^\flat \eta$ is the subgroup of monomial matrices.

For all $g, h \in G^\flat$, let $g^h := h^{-1} g h$, and ${}^h g := h g h^{-1} = g^{h^{-1}}$. As in §2, the action of $M^\flat$ on $T^\flat$ by conjugation induces an action on $\Phi^\flat$, where for all $m \in M^\flat$, $\alpha \in \Phi^\flat$, $m\alpha$ is the unique element of $\Phi^\flat$ such that $t^{(m\alpha)} = (t^m)^\alpha$ for all $t \in T^\flat$. The group $M^\flat$ also acts on the finite set $\mathfrak{N}_n := \{1, \dots, n\}$, where for all $m \in M^\flat$, $i \in \mathfrak{N}_n$, mi is the unique element of $\mathfrak{N}_n$ such that:

$$\tau_{mi}(t) = \tau_i(t^m), \qquad t \in T^\flat.$$

The relation:

$$(t^m)^{(i,j)} = \frac{\tau_i(t^m)}{\tau_j(t^m)} = \frac{\tau_{mi}(t)}{\tau_{mj}(t)} = t^{(mi,mj)}$$

shows that:

$$m(i,j) = (mi, mj), \qquad m \in M^\flat,\ (i,j) \in \Phi^\flat.$$

A set of generators of $\mathbb{G}^\flat$ can be explicitly described as follows. For every $\alpha \in \Phi^{\flat+}$, $x \in \mathbb{F}$, the generator $n_\alpha(x)$ is the matrix $1_{\mathbb{G}^\flat} + x\,\mathbf{e}_\alpha$, where $1_{\mathbb{G}^\flat}$ is the identity matrix in $\mathbb{G}^\flat$, and $\mathbf{e}_\alpha$ is the elementary matrix with 1 in the α-th position and

0's elsewhere. For every $\alpha = \check{\alpha}_i \in \Delta^\flat$, the generator $w_\alpha := n_\alpha(-1)\, n_{-\alpha}(1)\, n_\alpha(-1)$ is the monomial matrix with -1 in the α-th position, 1 in the $-\alpha$-th position, 1 in the j-th diagonal entry for all $j \neq i, i+1$, and 0's elsewhere. For every $\alpha = \check{\alpha}_i \in \Delta^\flat$, $x \in \mathbb{F}^\times$, the generator $h_\alpha(x) := n_\alpha(x)\, n_{-\alpha}(-x^{-1})\, n_\alpha(x)\, w_\alpha$ is the diagonal matrix with x in the i-th diagonal entry, x^{-1} in the $(i+1)$-th entry, and 1's elsewhere along the diagonal. For example, our generators for $SL_2(\mathbb{F})$ have the form:

$$h_{\check{\alpha}_1}(x) := \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}, \qquad w_{\check{\alpha}_1} := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad n_{\check{\alpha}_1}(x) := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

For the remainder of the paper, let $\mathcal{A}$ be an abelian group, and $c : \mathbb{F}^\times \times \mathbb{F}^\times \to \mathcal{A}$ a *bilinear* Steinberg symbol. Note that $\mathbb{F}^\times \to \mathcal{A}$, $x \mapsto c(x, x)$, is a homomorphism in this case, and every element $c(x, x) = c(x, -1) \in \mathcal{A}$ has order at most two.

According to §1 Theorem 5, there exists a central extension $\widetilde{\mathbb{G}}^\flat$ of $\mathbb{G}^\flat$ by $\mathcal{A}$ with Steinberg symbol c. Notice that $c_\alpha = c$ for all $\alpha \in \Delta^\flat$. Using our ordering on $\Delta^\flat$, the section $\mathfrak{s}_{\mathbb{G}^\flat} : \mathbb{G}^\flat \to \widetilde{\mathbb{G}}^\flat$ and the corresponding 2-cocycle $\sigma_{\mathbb{G}^\flat} \in Z^2(\mathbb{G}^\flat; \mathcal{A})$ are defined as in §2.

Now let $G := GL_n(\mathbb{F})$, let $\mathbb{G} := SL_n(\mathbb{F}) \subset G$, and let T, Φ, Φ^+, Δ, $\mathbb{H}$, $\mathbb{N}$, $\mathbb{M}_\mathbb{Z}$, $\mathbb{M}$, $\mathfrak{M}$, N and M be defined as above with $n+1$ replaced by n. In particular, $\Delta = \{\check{\alpha}_i \,|\, 1 \le i \le n-1\}$. The goal of this section is to study the 2-cocycle σ_n in $Z^2(G; \mathcal{A})$ that is defined as follows. Consider the embedding of G into $\mathbb{G}^\flat$:

$$\iota : G \hookrightarrow \mathbb{G}^\flat, \qquad g \mapsto \begin{pmatrix} g & \\ & \det(g)^{-1} \end{pmatrix}.$$

Then σ_n is the 2-cocycle defined by:

$$\sigma_n(g, g') := \sigma_{\mathbb{G}^\flat}\big(\iota(g), \iota(g')\big)\, c\big(\det(g), \det(g')\big)^{-1}, \qquad g, g' \in G.$$

Lemma 1. *For all $t, t' \in T$, $\sigma_n(t, t') = \prod_{i<j} c\big(\tau_i(t), \tau_j(t')\big)^{-1}$.*

Proof : Let $t_i := \tau_i(t)$ and $t'_i := \tau_i(t')$ for $1 \le i \le n$. Then:

$$\iota(t) = \prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i), \qquad \iota(t') = \prod_{i=1}^{n} h_{\check{\alpha}_i}(x'_i),$$

where $x_i := \prod_{j=1}^{i} t_j$ and $x'_i := \prod_{j=1}^{i} t'_j$ for $1 \le i \le n$. Then:

$$\begin{aligned}\sigma_{\mathbb{G}^\flat}\big(\iota(t), \iota(t')\big) &= \mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)\big)\, \mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x'_i)\big)\, \mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i x'_i)\big)^{-1} \\ &= \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x'_i) \big(\prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i x'_i)\big)^{-1},\end{aligned}$$

since:

$$\prod_{i=1}^{n} c(x_i, x_i) \prod_{i=1}^{n} c(x'_i, x'_i) \big(\prod_{i=1}^{n} c(x_i x'_i, x_i x'_i)\big)^{-1} = 1_{\mathcal{A}}.$$

Using the relations in $\mathcal{R}_{\widetilde{\mathbb{H}}}$:

$$\begin{aligned}\prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x'_i) &= \prod_{i=n}^{1} \big(\tilde{h}_{\check{\alpha}_i}(x_i)\, \tilde{h}_{\check{\alpha}_i}(x'_i)\big) \prod_{i=1}^{n-1} c(x_i, x'_{i+1})^{-1} \\ &= \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i x'_i) \prod_{i=1}^{n} c(x_i, x'_i) \prod_{i=1}^{n-1} c(x_i, x'_{i+1})^{-1}.\end{aligned}$$

Consequently:

$$\sigma_{\mathbb{G}^\flat}\big(\iota(t), \iota(t')\big) = c(x_n, x'_n) \prod_{i=1}^{n-1} c(x_i, x'_{i+1} {x'_i}^{-1})^{-1} = c(x_n, x'_n) \prod_{i=1}^{n-1} c(x_i, t'_{i+1})^{-1}.$$

Since $x_n = \det(t)$, $x'_n = \det(t')$, we have:

$$\sigma_n(t, t') = \sigma_{\mathbb{G}^\flat}\big(\iota(t), \iota(t')\big)\, c(x_n, x'_n)^{-1} = \prod_{j=2}^{n} c(x_{j-1}, t'_j)^{-1} = \prod_{j=2}^{n} \prod_{i=1}^{j-1} c(t_i, t'_j)^{-1}.$$

The result follows. ∎

Lemma 2. *We have:*

(a) $\sigma_n(t, \eta) = 1_{\mathcal{A}}$ *for all* $t \in T$, $\eta \in \mathfrak{M}$,

(b) $\sigma_n(\eta, \eta') = 1_{\mathcal{A}}$ *for all* $\eta, \eta' \in \mathfrak{M}$ *such that* $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$.

Proof : Observe that $\iota(T) \subset \mathbb{H}^\flat$, and $\iota(\mathfrak{M}) \subset \mathfrak{M}^\flat$. Then for all $t \in T$, $\eta \in \mathfrak{M}$:

$$\sigma_n(t,\eta) = \sigma_{\mathbb{G}^\flat}\big(\iota(t),\iota(\eta)\big)\, c\big(\det(t),1\big)^{-1} = 1_{\mathcal{A}}$$

by §2 Proposition 4(d). Similarly, if $\eta, \eta' \in \mathfrak{M}$ with $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$, then $\ell\big(\iota(\eta)\,\iota(\eta')\big) = \ell\big(\iota(\eta)\big) + \ell\big(\iota(\eta')\big)$, hence:

$$\sigma_n(\eta,\eta') = \sigma_{\mathbb{G}^\flat}\big(\iota(\eta),\iota(\eta')\big)\, c(1,1)^{-1} = 1_{\mathcal{A}}$$

by §2 Proposition 4(e). □

Lemma 3. *For all $t \in T$, $\eta \in \mathfrak{M}$:*

(1) $$\sigma_n(\eta,t) = \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta\alpha<0}} c\big(-\tau_j(t),\tau_i(t)\big)^{-1}.$$

Proof : We identify $\mathfrak{M}$ with its image $\iota(\mathfrak{M})$ in $\mathbb{G}^\flat$. If $\ell(\eta) = 0$, (1) holds trivially. Now suppose that η is a simple generator of the form w_α with $\alpha = \check{\alpha}_k \in \Delta$. As in Lemma 1, let $t_i := \tau_i(t)$ for $1 \le i \le n$, and write $\iota(t) = \prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)$ with $x_i := \prod_{j=1}^{i} t_j$. Let $t' := {}^{w_\alpha}t$, $t'_i := \tau_i(t') = \tau_{w_\alpha^{-1}i}(t)$ for $1 \le i \le n$, and write $\iota(t') = \prod_{i=1}^{n} h_{\check{\alpha}_i}(x'_i)$ with $x'_i := \prod_{j=1}^{i} t'_j$. Then:

(2) $$x_i = \begin{cases} t^\alpha x'_i & \text{if } i = k, \\ x'_i & \text{otherwise.} \end{cases}$$

Since $\det(\eta) = 1$, we have that:

$$\sigma_n(\eta,t) = \sigma_{\mathbb{G}^\flat}\big(w_\alpha,\iota(t)\big) = \mathfrak{s}_{\mathbb{G}^\flat}(w_\alpha)\,\mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)\big)\,\mathfrak{s}_{\mathbb{G}^\flat}\big(w_\alpha \prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)\big)^{-1}.$$

Now:

$$\mathfrak{s}_{\mathbb{G}^\flat}(w_\alpha)\,\mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)\big) = \tilde{w}_\alpha \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) \prod_{i=1}^{n} c(x_i,x_i),$$

$$\mathfrak{s}_{\mathbb{G}^\flat}\big(w_\alpha \prod_{i=1}^{n} h_{\check{\alpha}_i}(x_i)\big) = \mathfrak{s}_{\mathbb{G}^\flat}\big(\prod_{i=1}^{n} h_{\check{\alpha}_i}(x'_i)\, w_\alpha\big) = \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x'_i)\,\tilde{w}_\alpha \prod_{i=1}^{n} c(x'_i,x'_i),$$

thus $\sigma_n(\eta, t)$ is the product of:

$$\tilde{w}_\alpha \prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) \, \big(\prod_{i=n}^{1} \tilde{h}_{\check{\alpha}_i}(x_i') \; \tilde{w}_\alpha \big)^{-1} \tag{3}$$

and:

$$\prod_{i=1}^{n} c(x_i, x_i) \, \big(\prod_{i=1}^{n} c(x_i', x_i') \big)^{-1} = c(t^\alpha, t^\alpha).$$

Applying the relations in $\mathcal{R}^*_{\widetilde{\mathbb{M}_\mathbb{Z}},\widetilde{\mathbb{H}}}$, $\mathcal{R}^2_{\widetilde{\mathbb{H}}}$, and $\mathcal{R}^*_{\widetilde{\mathbb{H}}}$, it follows that:

$$\begin{aligned}
\tilde{w}_\alpha \prod_{i=n}^{k+2} \tilde{h}_{\check{\alpha}_i}(x_i) &= \prod_{i=n}^{k+2} \tilde{h}_{\check{\alpha}_i}(x_i) \; \tilde{w}_\alpha, \\
\tilde{w}_\alpha \, \tilde{h}_{\check{\alpha}_{k+1}}(x_{k+1}) &= c(x_{k+1}, x_{k+1}) \, \tilde{h}_{\check{\alpha}_{k+1}}(x_{k+1}) \, \tilde{h}_{\check{\alpha}_k}(x_{k+1}) \, \tilde{w}_\alpha, \\
\tilde{w}_\alpha \, \tilde{h}_{\check{\alpha}_k}(x_k) &= \tilde{h}_{\check{\alpha}_k}(x_k^{-1}) \, \tilde{w}_\alpha \\
\tilde{w}_\alpha \, \tilde{h}_{\check{\alpha}_{k-1}}(x_{k-1}) &= \tilde{h}_{\check{\alpha}_k}(x_{k-1}) \, \tilde{h}_{\check{\alpha}_{k-1}}(x_{k-1}) \, \tilde{w}_\alpha, \\
\tilde{w}_\alpha \prod_{i=k-2}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) &= \prod_{i=k-2}^{1} \tilde{h}_{\check{\alpha}_i}(x_i) \; \tilde{w}_\alpha.
\end{aligned}$$

By a straight-forward calculation, it can be shown that:

$$c(x_{k+1}, x_{k+1}) \, \tilde{h}_{\check{\alpha}_k}(x_{k+1}) \, \tilde{h}_{\check{\alpha}_k}(x_k^{-1}) \, \tilde{h}_{\check{\alpha}_k}(x_{k-1}) = c(t^\alpha, t_{k+1}) \, \tilde{h}_{\check{\alpha}_k}(t^{-\alpha} \, x_k).$$

Using (2), it follows that (3) is equal to $c(t^\alpha, t_{k+1})$. Consequently:

$$\sigma_n(\eta, t) = c(t^\alpha, t_{k+1}) \, c(t^\alpha, t^\alpha) = c(t_k t_{k+1}^{-1}, t_k) = c(-t_{k+1}^{-1}, t_k) = c(-t_{k+1}, t_k)^{-1}.$$

Since $\big\{\alpha \in \Phi^+ \,\big|\, \eta\alpha < 0\big\} = \{\check{\alpha}_k\}$, this proves (1) when $\ell(\eta) = 1$.

Now suppose that (1) has been established for all $t \in T$, $\eta \in \mathfrak{M}$, with $\ell(\eta) < l$, $l \geq 2$. Let $t \in T$, $\eta \in \mathfrak{M}$, with $\ell(\eta) = l$. We can write $\eta = \eta_1 \eta_2$ for some $\eta_1, \eta_2 \in \mathfrak{M}$ such that $\ell(\eta) = \ell(\eta_1) + \ell(\eta_2)$, and $\ell(\eta_i) < l$ for $i = 1, 2$. Then:

$$\sigma_n(\eta, t) = \sigma_n(\eta_1\eta_2, t) = \sigma_n(\eta_1\eta_2, t) \, \sigma_n(\eta_1, \eta_2) = \sigma_n(\eta_1, \eta_2 t) \, \sigma_n(\eta_2, t)$$

by Lemma 2(b). Now:

$$\sigma_n(\eta_1, \eta_2 t) = \sigma_n(\eta_1, {}^{\eta_2}t \, \eta_2) = \sigma_n(\eta_1, {}^{\eta_2}t \, \eta_2) \, \sigma_n({}^{\eta_2}t, \eta_2) = \sigma_n(\eta_1 {}^{\eta_2}t, \eta_2) \, \sigma_n(\eta_1, {}^{\eta_2}t)$$

by Lemma 2(a). Also:

$$\sigma_n({\eta_1}^{\eta_2}t, \eta_2) = \sigma_n({}^{\eta_1\eta_2}t\,\eta_1, \eta_2) = \sigma_n({}^{\eta_1\eta_2}t\,\eta_1, \eta_2)\,\sigma_n({}^{\eta_1\eta_2}t, \eta_1)$$
$$= \sigma_n({}^{\eta_1\eta_2}t, \eta_1\eta_2)\,\sigma_n(\eta_1, \eta_2) = 1_{\mathcal{A}}.$$

Thus, by the inductive hypothesis, $\sigma_n(\eta, t)$ is equal to:

$$\sigma_n(\eta_1, {}^{\eta_2}t)\,\sigma_n(\eta_2, t) = \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta_1\alpha<0}} c\big(-\tau_j({}^{\eta_2}t), \tau_i({}^{\eta_2}t)\big)^{-1} \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta_2\alpha<0}} c\big(-\tau_j(t), \tau_i(t)\big)^{-1}.$$

The first product can be re-expressed as:

$$\prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta_1\alpha<0}} c\big(-\tau_{\eta_2^{-1}j}(t), \tau_{\eta_2^{-1}i}(t)\big)^{-1} = \prod_{\substack{\alpha=(i,j)\in\eta_2^{-1}\Phi^+ \\ \eta_1\eta_2\alpha<0}} c\big(-\tau_j(t), \tau_i(t)\big)^{-1}.$$

Since $\ell(\eta_1\eta_2) = \ell(\eta_1) + \ell(\eta_2)$, it is easily shown that:

$$\big\{\alpha \in \Phi^+ \,\big|\, \eta_1\eta_2\alpha < 0\big\} = \big\{\alpha \in \Phi^+ \,\big|\, \eta_2\alpha < 0\big\} \coprod \big\{\alpha \in \eta_2^{-1}\Phi^+ \,\big|\, \eta_1\eta_2\alpha < 0\big\}.$$

Thus, $\sigma_n(\eta, t)$ is equal to:

$$\prod_{\substack{\alpha=(i,j)\in\eta_2^{-1}\Phi^+ \\ \eta_1\eta_2\alpha<0}} c\big(-\tau_j(t), \tau_i(t)\big)^{-1} \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta_2\alpha<0}} c\big(-\tau_j(t), \tau_i(t)\big)^{-1} = \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta_1\eta_2\alpha<0}} c\big(-\tau_j(t), \tau_i(t)\big)^{-1}.$$

Since $\eta = \eta_1\eta_2$, this completes the proof. □

Lemma 4. *For all $n, n' \in N$, $g, g' \in G$:*

$$\sigma_n(n, g) = \sigma_n(g, n) = 1_{\mathcal{A}},$$
$$\sigma_n(ng, g'n') = \sigma_n(g, g'),$$
$$\sigma_n(gn, g') = \sigma_n(g, ng').$$

Proof : Observe that $\iota(N) \subset \mathbb{N}^\flat$. Then:

$$\sigma_n(n, g) = \sigma_{\mathbb{G}^\flat}\big(\iota(n), \iota(g)\big)\,c\big(1, \det(g)\big)^{-1} = 1_{\mathcal{A}},$$
$$\sigma_n(g, n) = \sigma_{\mathbb{G}^\flat}\big(\iota(g), \iota(n)\big)\,c\big(\det(g), 1\big)^{-1} = 1_{\mathcal{A}},$$

for all $n \in N$, $g \in G$, by §2 Proposition 4(a). The other statements follow from these and the cocycle relation. ☐

Lemma 5. *For all $\alpha \in \Delta$, $x \in \mathbb{F}$:*

$$\sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha\big) = \begin{cases} c(x,x) & \text{if } x \neq 0, \\ c(-1,-1) & \text{if } x = 0. \end{cases}$$

Proof : If we identify $\mathfrak{M}$ [resp. N] with its image $\iota(\mathfrak{M})$ [resp. $\iota(N)$] in $\mathbb{G}^\flat$, then $\mathfrak{M} \subset \mathfrak{M}^\flat$, and $N \subset N^\flat$. Suppose that $x \in \mathbb{F}^\times$. Since $\det(w_\alpha) = 1$, $\sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha\big)$ is equal to:

$$\begin{aligned}
\sigma_{\mathbb{G}^\flat}\big(w_\alpha, n_\alpha(x)w_\alpha\big) &= \mathfrak{s}_{\mathbb{G}^\flat}(w_\alpha)\, \mathfrak{s}_{\mathbb{G}^\flat}\big(n_\alpha(x)w_\alpha\big)\, \mathfrak{s}_{\mathbb{G}^\flat}\big(w_\alpha n_\alpha(x)w_\alpha\big)^{-1} \\
&= \tilde{w}_\alpha\, n^*_\alpha(x)\, \tilde{w}_\alpha\, \mathfrak{s}_{\mathbb{G}^\flat}\big(n_\alpha(-x^{-1})h_\alpha(x^{-1})w_\alpha n_\alpha(-x^{-1})\big)^{-1} \\
&= \tilde{w}_\alpha n^*_\alpha(x)\tilde{w}_\alpha\big(n^*_\alpha(-x^{-1})\tilde{h}_\alpha(x^{-1})\tilde{w}_\alpha n^*_\alpha(-x^{-1})c(x^{-1},x^{-1})\big)^{-1} \\
&= \tilde{w}_\alpha\, n^*_\alpha(x)\, \tilde{w}_\alpha\big(\tilde{w}_\alpha\, n^*_\alpha(x)\, \tilde{w}_\alpha\big)^{-1} c(x^{-1},x^{-1})^{-1} = c(x,x).
\end{aligned}$$

The case $x = 0$ is similar. ☐

Recall that $\mathfrak{M}$ is a complete set of distinct coset representatives for $M/T \cong \mathbb{M}/\mathbb{H}$. Then G has the Bruhat decomposition:

$$G = \coprod_{\eta \in \mathfrak{M}} NT\eta N,$$

and there exists a unique map $\mathbf{t} : G \to T$ such that:

$$\mathbf{t}(nt\eta n') = t, \qquad n, n' \in N,\ t \in T,\ \eta \in \mathfrak{M}.$$

Note that $\mathbf{t}(g)$ can be easily computed for any $g \in G$ (cf. §4).

Proposition 6. *For all $\alpha \in \Delta$, $g \in G$:*

$$\sigma_n(w_\alpha, g) = \sigma_n\big(\mathbf{t}(w_\alpha g)\, \mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big). \tag{4}$$

Proof : For the proof, we will only need the fact that σ_n lies in $Z^2(G;\mathcal{A})$ and satisfies the properties of Lemmas 1-5, which we apply repeatedly without comment. For any $g \in G$, we factor $g = nt\eta n'$ with $n, n' \in N$, $t \in T$, $\eta \in \mathfrak{M}$. Without loss of generality, we can always assume that $n^\eta \in N^-$, where $N^- := \prod_{\alpha\in\Phi^+} N_{-\alpha}$ is the standard unipotent subgroup of G *opposite* to N.

Case I. Suppose that ${}^{w_\alpha}n \notin N$. Then it is possible to factor $n = n''n_\alpha(x)$ with $x \in \mathbb{F}^\times$ and $n'' \in N$ such that ${}^{w_\alpha}n'' \in N$, $(n'')^\eta \in N^-$. Since $n_\alpha(x)^\eta$ must lie in N^-, it follows that $\eta^{-1}\alpha < 0$, hence $\eta = w_\alpha\eta'$ for some $\eta' \in \mathfrak{M}$ such that $\ell(w_\alpha\eta') = \ell(\eta') + 1$. Using the fact that $n_\alpha(-x^{-1})^{(t^{w_\alpha}\eta')} \in N$, it follows that $\sigma_n(w_\alpha, g)$ is equal to:

$$\begin{aligned}
&\sigma_n\big(w_\alpha, n''n_\alpha(x)tw_\alpha\eta'n'\big) = \sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha t^{w_\alpha}\eta'\big)\\
&\quad= \sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha t^{w_\alpha}\eta'\big)\,\sigma_n\big(n_\alpha(x)w_\alpha, t^{w_\alpha}\eta'\big)\,\sigma_n(w_\alpha, t^{w_\alpha}\eta')^{-1}\\
&\quad= \sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha\big)\sigma_n\big(n_\alpha(-x^{-1})h_\alpha(x^{-1})w_\alpha n_\alpha(-x^{-1}), t^{w_\alpha}\eta'\big)\sigma_n(w_\alpha, t^{w_\alpha}\eta')^{-1}\\
&\quad= c(x,x)\,\sigma_n\big(h_\alpha(x^{-1})w_\alpha, t^{w_\alpha}\eta'\big)\,\sigma_n\big(h_\alpha(x^{-1}), w_\alpha\big)\,\sigma_n(w_\alpha, t^{w_\alpha}\eta')^{-1}\\
&\quad= c(x,x)\,\sigma_n\big(h_\alpha(x^{-1}), t\eta\big)\,\sigma_n(t,\eta)\\
&\quad= c(x,x)\,\sigma_n\big(h_\alpha(x^{-1}), t\big)\,\sigma_n\big(h_\alpha(x^{-1})t, \eta\big)\\
&\quad= c(x^{-1},-1)^{-1}\,c\big(x^{-1}, \tau_{k+1}(t)\big)^{-1} = c\big(x^{-1}, -\tau_{k+1}(t)\big)^{-1}
\end{aligned}$$

if $\alpha = \check{\alpha}_k$. On the other hand:

$$\begin{aligned}
\mathbf{t}(w_\alpha g) &= \mathbf{t}\big(w_\alpha n''n_\alpha(x)tw_\alpha\eta'n''\big) = \mathbf{t}\big(w_\alpha n_\alpha(x)w_\alpha t^{w_\alpha}\eta'\big)\\
&= \mathbf{t}\big(n_\alpha(-x^{-1})h_\alpha(x^{-1})w_\alpha n_\alpha(-x^{-1})t^{w_\alpha}\eta'\big) = h_\alpha(x^{-1})\,t,
\end{aligned}$$

hence:

$$\sigma_n\big(\mathbf{t}(w_\alpha g)\mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big) = \sigma_n\big(h_\alpha(x^{-1}), -t\big) = c\big(x^{-1}, -\tau_{k+1}(t)\big)^{-1}.$$

This proves (4) in this case.

Case II. Suppose that ${}^{w_\alpha}n \in N$, and $\ell(w_\alpha\eta) = \ell(\eta) + 1$. Then:

$$\begin{aligned}
\sigma_n(w_\alpha, g) &= \sigma_n(w_\alpha, nt\eta n') = \sigma_n(w_\alpha, t\eta)\,\sigma_n(t,\eta)\\
&= \sigma_n(w_\alpha, t)\,\sigma_n({}^{w_\alpha}tw_\alpha, \eta)\,\sigma_n({}^{w_\alpha}t, w_\alpha)\\
&= \sigma_n(w_\alpha, t)\,\sigma_n({}^{w_\alpha}t, w_\alpha\eta)\,\sigma_n(w_\alpha, \eta) = c\big(-\tau_{k+1}(t), \tau_k(t)\big)^{-1}
\end{aligned}$$

if $\alpha = \check{\alpha}_k$. On the other hand:

$$\mathbf{t}(w_\alpha g) = \mathbf{t}(w_\alpha n t \eta n') = \mathbf{t}(w_\alpha t \eta) = {}^{w_\alpha} t = h_\alpha(t^{-\alpha})\, t.$$

Thus:

$$\begin{aligned} \sigma_n\big(\mathbf{t}(w_\alpha g)\mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big) &= \sigma_n\big(h_\alpha(t^{-\alpha}), -t\big) \\ &= c\big(\tau_{k+1}(t)\tau_k(t)^{-1}, -\tau_{k+1}(t)\big)^{-1} = c\big(-\tau_{k+1}(t), \tau_k(t)\big)^{-1}. \end{aligned}$$

Thus, (4) holds in this case as well.

Case III. Suppose that ${}^{w_\alpha} n \in N$, and $\ell(w_\alpha \eta) = \ell(\eta) - 1$. We can write $\eta = w_\alpha \eta'$ with $\eta' \in \mathfrak{M}$ such that $\ell(w_\alpha \eta') = \ell(\eta') + 1$. Then:

$$\sigma_n(w_\alpha, g) = \sigma_n(w_\alpha, n t w_\alpha \eta' n') = \sigma_n(w_\alpha, t w_\alpha \eta')\, \sigma_n(t w_\alpha, \eta'),$$

since:

$$\sigma_n(t w_\alpha, \eta') = \sigma_n(t w_\alpha, \eta')\, \sigma_n(t, w_\alpha) = \sigma_n(t, w_\alpha \eta')\, \sigma_n(w_\alpha, \eta') = 1_{\mathcal{A}}.$$

Then $\sigma_n(w_\alpha, g)$ is equal to:

$$\begin{aligned} \sigma_n(w_\alpha, t w_\alpha)\, \sigma_n(w_\alpha t w_\alpha, \eta') &= \sigma_n(w_\alpha, t w_\alpha)\, \sigma_n(t, w_\alpha) \\ &= \sigma_n(w_\alpha, t)\, \sigma_n({}^{w_\alpha} t w_\alpha, w_\alpha)\, \sigma_n({}^{w_\alpha} t, w_\alpha) \\ &= \sigma_n(w_\alpha, t)\, \sigma_n\big({}^{w_\alpha} t, h_\alpha(-1)\big)\, \sigma_n(w_\alpha, w_\alpha) \\ &= c\big(-\tau_{k+1}(t), \tau_k(t)\big)^{-1} c\big(\tau_{k+1}(t), -1\big)^{-1} c(-1,-1)^{-1} \\ &= c\big(-\tau_{k+1}(t), -\tau_k(t)\big)^{-1} \end{aligned}$$

if $\alpha = \check{\alpha}_k$. On the other hand:

$$\mathbf{t}(w_\alpha g) = \mathbf{t}(w_\alpha n t w_\alpha \eta' n') = \mathbf{t}(w_\alpha t w_\alpha \eta') = h_\alpha(-1)\, t^{w_\alpha} = h_\alpha(-1) h_\alpha(t^{-\alpha})\, t.$$

Hence:

$$\begin{aligned} \sigma_n\big(\mathbf{t}(w_\alpha g)\mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big) &= \sigma_n\big(h_\alpha(-1) h_\alpha(t^{-\alpha}), -t\big) \\ &= c\big(-\tau_{k+1}(t)\tau_k(t)^{-1}, -\tau_{k+1}(t)\big)^{-1} = c\big(-\tau_{k+1}(t), -\tau_k(t)\big)^{-1}, \end{aligned}$$

Thus, (4) holds in this case, and the proof is complete. ∎

Theorem 7. *The 2-cocycle σ_n is the unique element of $Z^2(G;\mathcal{A})$ satisfying:*

(a) $\sigma_n(t,t') = \prod_{i<j} c\big(\tau_i(t),\tau_j(t')\big)^{-1}$ *for all* $t,t' \in T$,

(b) $\sigma_n(t,\eta) = 1_{\mathcal{A}}$ *for all* $t \in T$, $\eta \in \mathfrak{M}$,

(c) $\sigma_n(\eta,\eta') = 1_{\mathcal{A}}$ *for all* $\eta,\eta' \in \mathfrak{M}$ *such that* $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$,

(d) $\sigma_n(\eta,t) = \prod_{\substack{\alpha=(i,j)\in\Phi^+ \\ \eta\alpha<0}} c\big(-\tau_j(t),\tau_i(t)\big)^{-1}$ *for all* $t \in T$, $\eta \in \mathfrak{M}$,

(e) $\sigma_n(n,g) = \sigma_n(g,n) = 1_{\mathcal{A}}$ *for all* $n \in N$, $g \in G$,

(f) $\sigma_n(ng,g'n') = \sigma_n(g,g')$ *for all* $n,n' \in N$, $g,g' \in G$,

(g) $\sigma_n(gn,g') = \sigma_n(g,ng')$ *for all* $n \in N$, $g,g' \in G$,

(h) $\sigma_n\big(w_\alpha, n_\alpha(x)w_\alpha\big) = c(x,x)$ *for all* $\alpha \in \Delta$, $x \in \mathbb{F}^\times$.

(i) $\sigma_n(w_\alpha,w_\alpha) = c(-1,-1)$.

Proof : By Lemmas 1-5, σ_n satisfies the above properties. Conversely, suppose that $\sigma \in Z^2(G;\mathcal{A})$ satisfies (a)-(i). Clearly:

$$\sigma(t,g) = \sigma\big(t,\mathbf{t}(g)\big), \qquad t \in T,\ g \in G. \tag{5}$$

Also, the proof of Proposition 6 shows that:

$$\sigma(w_\alpha,g) = \sigma\big(\mathbf{t}(w_\alpha g)\mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big), \qquad \alpha \in \Delta,\ g \in G. \tag{6}$$

Now let $g,g' \in G$, and factor $g = nt\eta n'$ with $n,n' \in N$, $t \in T$, $\eta \in \mathfrak{M}$. If $\ell(\eta) = \ell$, we can express η in the form $\eta = w_{\alpha_1}\dots w_{\alpha_\ell}$ with each $\alpha_i \in \Delta$. By an easy argument:

$$\sigma(g,g') = \sigma(t,\eta n'g')\,\sigma(w_{\alpha_1}, w_{\alpha_2}\dots w_{\alpha_\ell}n'g')\dots\sigma(w_{\alpha_{\ell-1}}, w_{\alpha_\ell}n'g')\,\sigma(w_{\alpha_\ell}, n'g'),$$

and all of the terms on the right can be computed using (5) and (6). Since $\sigma_n(g,g')$ has a similar expansion, and σ and σ_n must agree on $T \times G$ and on $\{w_\alpha\} \times G$ for every $\alpha \in \Delta$, it follows that $\sigma = \sigma_n$. ∎

Corollary 8. *The 2-cocycle σ_1 on $GL_1(\mathbb{F}) \cong \mathbb{F}^\times$ is trivial. The 2-cocycle σ_2 on $GL_2(\mathbb{F})$ agrees with the Kubota 2-cocycle on $GL_2(\mathbb{F})$.*

Proof : If $n = 1$, then $T = G$, hence σ_1 is trivial by Theorem 7(a). For $n = 2$, the Kubota 2-cocycle $\sigma_K \in Z^2(G; \mathcal{A})$ is defined by:

$$\sigma_K(g, g') := c\left(\frac{\mathbf{x}(gg')}{\mathbf{x}(g)}, \frac{\mathbf{x}(gg')}{\mathbf{x}(g')\det(g)}\right)^{-1}, \qquad g, g' \in G,$$

where for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$:

$$\mathbf{x}(g) := \begin{cases} c & \text{if } c \neq 0, \\ d & \text{otherwise.} \end{cases}$$

As σ_K satisfies properties (a)-(i) of Theorem 7, $\sigma_K = \sigma_2$. □

Corollary 9. *Let $G^\sharp := GL_m(\mathbb{F})$, $m \leq n$, and let $\jmath : G^\sharp \hookrightarrow G$ be the embedding defined by:*

$$g \mapsto \begin{pmatrix} 1_a & & \\ & g & \\ & & 1_b \end{pmatrix},$$

where $a, b \geq 0$, $n = a + m + b$, and 1_a [resp. 1_b] is the $a \times a$ [resp. $b \times b$] matrix with 1's along the diagonal and 0's elsewhere. Then:

$$\sigma_n\big(\jmath(g), \jmath(g')\big) = \sigma_m(g, g'), \qquad g, g' \in G^\sharp.$$

Proof : Let $\sigma^\jmath(g, g') := \sigma_n\big(\jmath(g), \jmath(g')\big)$ for all $g, g' \in G^\sharp$. Let $T^\sharp$, $\Phi^\sharp$, $\Phi^{\sharp+}$, $\Delta^\sharp$, $\mathbb{H}^\sharp$, $\mathbb{N}^\sharp$, $\mathbb{M}^\sharp_\mathbb{Z}$, $\mathbb{M}^\sharp$, $\mathfrak{M}^\sharp$, $N^\sharp$ and $M^\sharp$ be defined as above with n replaced by m. In particular, $\Delta^\sharp = \big\{\breve{\alpha}_i \,\big|\, a + 1 \leq i \leq a + m - 1\big\} \subset \Delta$. Also, let $\tau_i^\sharp : T^\sharp \to \mathbb{F}^\times$ be the i-th coordinate homomorphism on $T^\sharp$: $\tau_i^\sharp(t) := t_i$ for all $t = \operatorname{diag}(t_1, \ldots, t_m) \in T^\sharp$. Note that:

$$\tau_i\big(\jmath(t)\big) = \begin{cases} \tau^\sharp_{i-a}(t) & \text{if } a + 1 \leq i \leq a + m, \\ 1 & \text{otherwise,} \end{cases}$$

for all $t \in T^\sharp$. Then for all $t, t' \in T^\sharp$:

$$\sigma^\jmath(t,t') = \prod_{i<j} c\big(\tau_i\big(\jmath(t)\big), \tau_j\big(\jmath(t)\big)\big)^{-1} = \prod_{a+1\le i<j\le a+m} c\big(\tau^\sharp_{i-a}(t), \tau^\sharp_{j-a}(t)\big)^{-1}$$
$$= \prod_{1\le i<j\le m} c\big(\tau^\sharp_i(t), \tau^\sharp_j(t)\big)^{-1} = \sigma_m(t,t').$$

Since $\jmath(\mathfrak{M}^\sharp) \subset \mathfrak{M}$, the relation $\sigma^\jmath(t,\eta) = 1_{\mathcal{A}}$ for all $t \in T^\sharp$, $\eta \in \mathfrak{M}^\sharp$, is clear. Moreover, $\jmath(w_{\breve{\alpha}_i}) = w_{\breve{\alpha}_{i+a}}$ for $1 \le i \le m-1$, hence $\jmath$ is length preserving on $\mathfrak{M}^\sharp$. Consequently, $\sigma^\jmath(\eta,\eta') = 1_{\mathcal{A}}$ for all $\eta,\eta' \in \mathfrak{M}^\sharp$ such that $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$. For all $t \in T^\sharp$, $\eta \in \mathfrak{M}^\sharp$, $\sigma^\jmath(\eta,t)$ is equal to:

$$\prod_{\substack{\alpha=(i,j)\in\Phi^+\\ \jmath(\eta)\alpha<0}} c\big(-\tau_j\big(\jmath(t)\big), \tau_i\big(\jmath(t)\big)\big)^{-1} = \prod_{\substack{\alpha=(i,j)\in\Phi^+\\ \jmath(\eta)\alpha<0\\ a+1\le i,j\le a+m}} c\big(-\tau^\sharp_{j-a}(t), \tau^\sharp_{i-a}(t)\big)^{-1}$$
$$= \prod_{\substack{\alpha=(i+a,j+a)\in\Phi^+\\ \jmath(\eta)\alpha<0\\ 1\le i,j\le m}} c\big(-\tau^\sharp_j(t), \tau^\sharp_i(t)\big)^{-1}.$$

For all $i \in \mathfrak{N}_n = \{1,\dots,n\}$, $\eta \in \mathfrak{M}^\sharp$:

$$\jmath(\eta)i = \begin{cases} a+\eta(i-a) & \text{if } a+1 \le i \le a+m,\\ i & \text{otherwise.}\end{cases}$$

This relation implies that:

$$\big\{(i,j) \,\big|\, 1 \le i < j \le m,\ \jmath(\eta)(i+a,j+a) < 0\big\} = \big\{(i,j) \in \Phi^{\sharp+} \,\big|\, \eta(i,j) < 0\big\}.$$

Thus:

$$\sigma^\jmath(\eta,t) = \prod_{\substack{\alpha=(i,j)\in\Phi^{\sharp+}\\ \eta\alpha<0}} c\big(-\tau^\sharp_j(t), \tau^\sharp_i(t)\big)^{-1} = \sigma_m(\eta,t).$$

Since $\jmath(N^\sharp) \subset N$, $\sigma^\jmath(n,g) = \sigma^\jmath(g,n) = 1_{\mathcal{A}}$ for all $n \in N^\sharp$, $g \in G^\sharp$, and it follows that $\sigma^\jmath(ng,g'n') = \sigma^\jmath(g,g')$ and $\sigma^\jmath(gn,g') = \sigma^\jmath(g,ng')$ for all $n,n' \in N^\sharp$, $g,g' \in G^\sharp$. Finally, $\jmath\big(n_{\breve{\alpha}_i}(x)\big) = n_{\breve{\alpha}_{i+a}}(x)$ for all $x \in \mathbb{F}$, $1 \le i \le m-1$, hence:

$$\sigma^\jmath\big(w_{\breve{\alpha}_i}, n_{\breve{\alpha}_i}(x)w_{\breve{\alpha}_i}\big) = \sigma_n\big(w_{\breve{\alpha}_{i+a}}, n_{\breve{\alpha}_{i+a}}(x)w_{\breve{\alpha}_{i+a}}\big) = \begin{cases} c(x,x) & \text{if } x \ne 0,\\ c(-1,-1) & \text{if } x = 0.\end{cases}$$

We have shown that $\sigma^{\jmath}$ satisfies properties (a)-(i) of Theorem 7 with n replaced by m. By the uniqueness assertion, it follows that $\sigma^{\jmath} = \sigma_m$. $\square$

Lemma 10. *For all $t, t' \in T$, $\eta, \eta' \in \mathfrak{M}$:*

$$\sigma_n(t\eta, t'\eta') = \sigma_n(t, {}^{\eta}t')\, \sigma_n(\eta, t')\, \sigma_n\big({}^{\eta}t't, \mathbf{t}(\eta\eta')\big)\, \sigma_n(\eta, \eta').$$

In particular, if $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$, $\sigma_n(t\eta, t'\eta') = \sigma_n(t, {}^{\eta}t')\, \sigma_n(\eta, t')$.

Proof : We have:

$$\begin{aligned}
\sigma_n(t\eta, t'\eta') &= \sigma_n(t\eta, t'\eta')\, \sigma_n(t', \eta') = \sigma_n(t\eta, t')\, \sigma_n(t, \eta)\, \sigma_n({}^{\eta}t't\eta, \eta')\, \sigma_n({}^{\eta}t't, \eta) \\
&= \sigma_n(t, {}^{\eta}t'\eta)\, \sigma_n({}^{\eta}t', \eta)\, \sigma_n(\eta, t')\, \sigma_n({}^{\eta}t't, \eta\eta')\, \sigma_n(\eta, \eta') \\
&= \sigma_n(t, {}^{\eta}t')\, \sigma_n(\eta, t')\, \sigma_n\big({}^{\eta}t't, \mathbf{t}(\eta\eta')\big)\, \sigma_n(\eta, \eta').
\end{aligned}$$

If $\ell(\eta\eta') = \ell(\eta) + \ell(\eta')$, then $\eta\eta' \in \mathfrak{M}$ $\big($hence $\mathbf{t}(\eta\eta') = 1_G\big)$, and the second statement follows easily. $\square$

Theorem 11. *For every standard Levi subgroup of G, the following block formula holds:*

$$\sigma_n\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_p \end{pmatrix}, \begin{pmatrix} g'_1 & & \\ & \ddots & \\ & & g'_p \end{pmatrix}\right) = \prod_{i=1}^{p} \sigma_{n_i}(g_i, g'_i) \prod_{i<j} c\big(\det(g_i), \det(g'_j)\big)^{-1},$$

where $n = n_1 + \ldots + n_p$, and $g_i, g'_i \in GL_{n_i}(\mathbb{F})$ for $1 \leq i \leq p$.

Proof : We induct on the number of blocks. When $p = 1$, there is nothing to prove. Suppose that $p = 2$, $n = n_1 + n_2$, and let $G^1 := GL_{n_1}(\mathbb{F})$, $G^2 := GL_{n_2}(\mathbb{F})$. For $i = 1, 2$, we define T^i, Φ^i, Φ^{i+}, Δ^i, $\mathbb{H}^i$, $\mathbb{N}^i$, $\mathbb{M}^i_{\mathbb{Z}}$, $\mathbb{M}^i$, $\mathfrak{M}^i$, N^i and M^i as before with n replaced by n_i. Let:

$$\jmath^1 : G^1 \hookrightarrow G, \qquad g_1 \mapsto \begin{pmatrix} g_1 & \\ & 1_{n_2} \end{pmatrix},$$

$$\jmath^2 : G^2 \hookrightarrow G, \qquad g_2 \mapsto \begin{pmatrix} 1_{n_1} & \\ & g_2 \end{pmatrix},$$

where 1_{n_i} denotes the identity matrix in G^i. According to Corollary 9, if we identify each G^i with its image $\jmath^i(G^i)$ in G, then:

$$\sigma_n(g_i, g_i') = \sigma_{n_i}(g_i, g_i'), \qquad g_i, g_i' \in G^i. \tag{7}$$

We claim that:

$$\begin{aligned} &\sigma_n(g_1, g_2) = c\big(\det(g_1), \det(g_2)\big)^{-1}, \\ &\sigma_n(g_2, g_1) = 1_{\mathcal{A}}, \end{aligned} \tag{8}$$

for all $g_1 \in G^1$, $g_2 \in G^2$. Indeed, if we factor each $g_i = n_i m_i n_i'$ with $n_i, n_i' \in N^i$, $m_i \in M^i$, then:

$$\sigma_n(g_1, g_2) = \sigma_n(g_1, n_2 m_2 n_2') = \sigma_n(g_1 n_2, m_2) = \sigma_n(n_2 g_1, m_2) = \sigma_n(g_1, m_2).$$

Similarly, $\sigma_n(g_1, m_2) = \sigma_n(m_1, m_2)$. Now factor each $m_i = t_i \eta_i$ with $t_i \in T^i$, $\eta_i \in \mathfrak{M}^i$. Since $\ell(\eta_1 \eta_2) = \ell(\eta_1) + \ell(\eta_2)$, and G^1 commutes with G^2, Lemma 10 implies:

$$\sigma_n(m_1, m_2) = \sigma_n(t_1, {}^{\eta_1}t_2)\, \sigma_n(\eta_1, t_2) = \sigma_n(t_1, t_2)\, \sigma_n(\eta_1, t_2).$$

Since $\tau_i(t_2) = \tau_j(t_2) = 1$ for all $(i, j) \in \Phi^+$ such that $\eta_1(i, j) < 0$, $\sigma_n(\eta_1, t_2) = 1_{\mathcal{A}}$. Thus:

$$\sigma_n(g_1, g_2) = \sigma_n(t_1, t_2) = c\big(\det(t_1), \det(t_2)\big)^{-1} = c\big(\det(g_1), \det(g_2)\big)^{-1}.$$

The proof that $\sigma_n(g_2, g_1) = 1_{\mathcal{A}}$ is similar. This establishes the claim.

For all $g_1, g_1' \in G^1$, $g_2, g_2' \in G^2$, we compute using (7) and (8):

$$\begin{aligned} \sigma_n(g_1 g_2, g_1' g_2') &= \sigma_n(g_2 g_1, g_2' g_1')\, \sigma_n(g_2', g_1') \\ &= \sigma_n(g_2 g_1, g_2')\, \sigma_n(g_2, g_1)\, \sigma_n(g_2 g_2' g_1, g_1')\, \sigma_n(g_2 g_2', g_1) \\ &= \sigma_n(g_2, g_2' g_1)\, \sigma_n(g_2', g_1)\, \sigma_n(g_1, g_2')\, \sigma_n(g_2 g_2', g_1 g_1')\, \sigma_n(g_1, g_1') \\ &= \sigma_n(g_2, g_2')\, \sigma_n(g_2 g_2', g_1)\, \sigma_n(g_1, g_2')\, \sigma_n(g_1, g_1') \\ &= \sigma_{n_1}(g_1, g_1')\, \sigma_{n_2}(g_2, g_2')\, c\big(\det(g_1), \det(g_2')\big)^{-1}. \end{aligned}$$

This proves the theorem in the case $p = 2$. For arbitrary $p \geq 2$, we proceed by induction:

$$\sigma_n\left(\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_p \end{pmatrix}, \begin{pmatrix} g_1' & & \\ & \ddots & \\ & & g_p' \end{pmatrix}\right)$$

$$= \sigma_{n_1}(g_1, g_1')\, \sigma_{n-n_1}\left(\begin{pmatrix} g_2 & & \\ & \ddots & \\ & & g_p \end{pmatrix}, \begin{pmatrix} g_2' & & \\ & \ddots & \\ & & g_p' \end{pmatrix}\right) c\big(\det(g_1), \det(g_2' \dots g_p')\big)^{-1}$$

$$= \sigma_{n_1}(g_1, g_1') \prod_{i=2}^{p} \sigma_{n_i}(g_i, g_i') \prod_{2 \leq i < j \leq p} c\big(\det(g_i), \det(g_j')\big)^{-1} \prod_{j=2}^{p} c\big(\det(g_1), \det(g_j')\big)^{-1}$$

$$= \prod_{i=1}^{p} \sigma_{n_i}(g_i, g_i') \prod_{i<j} c\big(\det(g_i), \det(g_j')\big)^{-1}.$$

This completes the proof. ◻

§4. On Calculating the 2-Cocycle

We continue to use the notation of §3. In order to compute $\sigma_n(g, g')$ for arbitrary $g, g' \in G = GL_n(\mathbb{F})$, we will need to have a method of computing $\mathbf{t}(g)$ for any $g \in G$. Here $\mathbf{t}$ is the unique map from G to T such that:

$$\mathbf{t}(nt\eta n') = t, \qquad n, n' \in N,\ t \in T,\ \eta \in \mathfrak{M}.$$

To this end, consider the set of functions $\{\mathbf{x}_i : G \to \mathbb{F}^\times \,\big|\, 1 \leq i \leq n+1\}$ defined as follows. First, let $\mathbf{x}_{n+1}(g) := 1$ for all $g \in G$. Next, if $1 \leq i \leq n$ and $g \in G$, let $\mathbf{x}_i(g)$ be the first nonzero $(n+1-i) \times (n+1-i)$-minor formed from the last $n+1-i$ rows of g, where the minors are ordered lexicographically according to the columns they involve. In particular, $\mathbf{x}_1(g) = \det(g)$ for all $g \in G$. For each i, it follows from the definition of $\mathbf{x}_i$ that:

$$\begin{aligned} \mathbf{x}_i(tg) &= \mathbf{x}_i(t)\,\mathbf{x}_i(g), \qquad t \in T,\ g \in G, \\ \mathbf{x}_i(ngn') &= \mathbf{x}_i(g), \qquad n, n' \in N,\ g \in G. \end{aligned} \tag{1}$$

The computation of $\mathbf{t}(g)$ is now described by the following lemma.

Lemma 1. *The set* $\mathfrak{M}$ *is characterized by:*

$$\mathfrak{M} = \big\{ m \in M \,\big|\, \mathbf{x}_i(m) = 1 \text{ for } 1 \le i \le n \big\}.$$

Consequently:

$$\tau_i\big(\mathbf{t}(g)\big) = \frac{\mathbf{x}_i(g)}{\mathbf{x}_{i+1}(g)}, \qquad g \in G,\ 1 \le i \le n.$$

Proof : Let $M^1 := \big\{ m \in M \,\big|\, \mathbf{x}_i(m) = 1 \text{ for } 1 \le i \le n \big\}$. For any monomial matrix $m \in M$, let m_i denote the nonzero entry in the i-th row for each $1 \le i \le n$. As m_i lies in the $(m^{-1}i)$-th column, it follows that:

$$\mathbf{x}_i(m) = \mathrm{sign}_i(m)\, m_i\, \mathbf{x}_{i+1}(m),$$

where:

$$\mathrm{sign}_i(m) := \prod_{\substack{j>i \\ m^{-1}j<m^{-1}i}} (-1).$$

Consequently, $m \in M^1$ if and only if $m_i = \mathrm{sign}_i(m)$ for $1 \le i \le n$.

We will show that $\mathfrak{M} \subset M^1$. If $\eta \in \mathfrak{M}$ with $\ell(\eta) = 0$, then $\eta = 1_G$, hence $\eta \in M^1$. We proceed by induction. Suppose that $\eta \in M^1$ for all $\eta \in \mathfrak{M}$ with $\ell(\eta) < l$, $l \ge 1$. Let $\eta \in \mathfrak{M}$ with $\ell(\eta) = l$. Factor $\eta = w_\alpha \eta'$ with $\alpha = \check{\alpha}_k \in \Delta$, $\eta' \in \mathfrak{M}$, and $\ell(\eta') = l - 1$. Since $\det(w_\alpha) = 1$, it follows from the definitions that $\mathbf{x}_i(w_\alpha \eta') = \mathbf{x}_i(\eta')$ if $i \ne k+1$. Since $\eta' \in M^1$ by the inductive hypothesis, $\mathbf{x}_i(\eta') = 1$ for all i, hence $\mathbf{x}_i(\eta) = 1$ if $i \ne k+1$. Also:

$$\mathbf{x}_{k+1}(\eta) = \mathrm{sign}_{k+1}(\eta)\, \eta_{k+1}\, \mathbf{x}_{k+2}(\eta) = \mathrm{sign}_{k+1}(\eta)\, \eta_{k+1},$$

and we have that:

$$\mathrm{sign}_{k+1}(\eta) = \mathrm{sign}_{k+1}(w_\alpha \eta') = \prod_{\substack{j>k+1 \\ (w_\alpha\eta')^{-1}j<(w_\alpha\eta')^{-1}(k+1)}} (-1) \quad = \prod_{\substack{j>k+1 \\ \eta'^{-1}j<\eta'^{-1}k}} (-1),$$

and:

$$\eta_{k+1} = (w_\alpha \eta')_{k+1} = (\eta')_k = \mathrm{sign}_k(\eta') = \prod_{\substack{j>k \\ \eta'^{-1}j<\eta'^{-1}k}} (-1) = \prod_{\substack{j>k+1 \\ \eta'^{-1}j<\eta'^{-1}k}} (-1),$$

since the relation $\ell(w_\alpha\eta') = \ell(\eta') + 1$ implies that $\eta'^{-1}(k+1) > \eta'^{-1}k$. Thus $\mathbf{x}_{k+1}(\eta) = 1$ as well, and therefore $\eta \in M^1$. This shows that $\mathfrak{M} \subset M^1$. On the other hand, both $\mathfrak{M}$ and M^1 have cardinality $n!$, hence $\mathfrak{M} = M^1$.

To prove the second statement, first observe that if $t \in T$, then $\mathbf{x}_i(t) = \prod_{j\geq i} \tau_j(t)$. Consequently:

$$\tau_i(t) = \frac{\mathbf{x}_i(t)}{\mathbf{x}_{i+1}(t)}, \qquad t \in T.$$

By (1), it follows that:

$$\mathbf{x}_i(nt\eta n') = \mathbf{x}_i(t), \qquad n, n' \in N,\ t \in T,\ \eta \in \mathfrak{M},$$

since $\mathbf{x}_i(\eta) = 1$ for all $\eta \in \mathfrak{M} = M^1$. In other words, $\mathbf{x}_i(g) = \mathbf{x}_i\big(\mathbf{t}(g)\big)$ for all $g \in G$. Thus:

$$\tau_i\big(\mathbf{t}(g)\big) = \frac{\mathbf{x}_i\big(\mathbf{t}(g)\big)}{\mathbf{x}_{i+1}\big(\mathbf{t}(g)\big)} = \frac{\mathbf{x}_i(g)}{\mathbf{x}_{i+1}(g)}$$

for all $g \in G$. □

In order to compute $\sigma_n(g, g')$ for arbitrary $g, g' \in G$, we use the method of §3 Theorem 7. First, factor $g = ntw_{\alpha_1} \dots w_{\alpha_\ell} n'$ with $n, n' \in N$, $t \in T$, and each $\alpha_i \in \Delta$. Here $\ell = \ell(g)$. Then $\sigma_n(g, g')$ is equal to:

$$\sigma_n(t, w_{\alpha_1} \dots w_{\alpha_\ell} n'g')\, \sigma_n(w_{\alpha_1}, w_{\alpha_2} \dots w_{\alpha_\ell} n'g') \dots \sigma_n(w_{\alpha_{\ell-1}}, w_{\alpha_\ell} n'g')\, \sigma_n(w_{\alpha_\ell}, n'g').$$

Each of these terms can be computed using §3 Theorem 7(a) and the relations:

$$\sigma_n(t, g) = \sigma_n\big(t, \mathbf{t}(g)\big), \qquad t \in T,\ g \in G,$$
$$\sigma_n(w_\alpha, g) = \sigma_n\big(\mathbf{t}(w_\alpha g)\mathbf{t}(g)^{-1}, -\mathbf{t}(g)\big), \qquad \alpha \in \Delta,\ g \in G.$$

§5. Calculations on the Weyl Group

We continue to use the notation of §§3-4. Let W denote the Weyl group of permutation matrices in G, that is, matrices with a 1 in each row and column, and 0's elsewhere. For every $\alpha = \check{\alpha}_i \in \Delta$, let s_α be the monomial matrix with 1 in the α-th and $-\alpha$-th positions, 1 in the j-th diagonal entry for all $j \neq i, i+1$, and 0's elsewhere. The elements $\{s_\alpha \,|\, \alpha \in \Delta\}$ generate W. If $\alpha = \check{\alpha}_k \in \Delta$, and $\beta = \check{\alpha}_l \in \Delta$, we will write $\alpha \prec \beta$ if $k < l$, and $\alpha \succ \beta$ if $k > l$.

Lemma 1. *For all $\alpha, \beta \in \Delta$:*

$$\sigma_n(s_\alpha, s_\beta) = \begin{cases} c(-1,-1) & \textit{if } \langle \alpha, \beta \rangle = 0 \textit{ and } \alpha \prec \beta, \\ 1_{\mathcal{A}} & \textit{otherwise.} \end{cases}$$

Proof : First, suppose that $\alpha = \check{\alpha}_k$, $\beta = \check{\alpha}_l$, and $k \neq l$. By §4 Lemma 1:

$$\tau_i\big(\mathbf{t}(s_\alpha)\big) = \begin{cases} -1 & \text{if } i = k, \\ 1 & \text{otherwise,} \end{cases}$$
$$\tau_i\big(\mathbf{t}(s_\beta)\big) = \begin{cases} -1 & \text{if } i = l, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, $\ell(w_\alpha w_\beta) = \ell(w_\alpha) + \ell(w_\beta)$, hence:

$$\sigma_n(s_\alpha, s_\beta) = \sigma_n\big(\mathbf{t}(s_\alpha)w_\alpha, \mathbf{t}(s_\beta)w_\beta\big) = \sigma_n\big(\mathbf{t}(s_\alpha), {}^{w_\alpha}\mathbf{t}(s_\beta)\big)\,\sigma_n\big(w_\alpha, \mathbf{t}(s_\beta)\big)$$

by §3 Lemma 10. Now:

$$\begin{aligned} \sigma_n\big(\mathbf{t}(s_\alpha), {}^{w_\alpha}\mathbf{t}(s_\beta)\big) &= \prod_{i<j} c\Big(\tau_i\big(\mathbf{t}(s_\alpha)\big), \tau_{w_\alpha^{-1}j}\big(\mathbf{t}(s_\beta)\big)\Big)^{-1} \\ &= \prod_{k<j} c\Big(-1, \tau_{w_\alpha^{-1}j}\big(\mathbf{t}(s_\beta)\big)\Big) \\ &= \begin{cases} c(-1,-1) & \text{if } w_\alpha\, l > k, \\ 1_{\mathcal{A}} & \text{otherwise,} \end{cases} \end{aligned}$$

and:

$$\sigma_n\big(w_\alpha, \mathbf{t}(s_\beta)\big) = c\Big(-\tau_{k+1}\big(\mathbf{t}(s_\beta)\big), \tau_k\big(\mathbf{t}(s_\beta)\big)\Big)^{-1} = 1_{\mathcal{A}}$$

since $k \neq l$. This proves the lemma when $\alpha \neq \beta$. If $\alpha = \beta$, then:

$$\sigma_n(s_\alpha, s_\alpha) = \sigma_K\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) = 1_{\mathcal{A}}$$

by Corollaries 8 and 9 of §3. $\square$

Proposition 2. *Let $w \in W$, $\ell(w) > 0$, and let α be the largest element of Δ such that $w^{-1}\alpha < 0$. Then $\sigma_n(s_\alpha, w) = \sigma_n(s_\alpha, s_\alpha w) = 1_{\mathcal{A}}$.*

Proof : As in the proof of §4 Lemma 1:

$$\mathbf{x}_i(w) = \mathrm{sign}_i(w)\, \mathbf{x}_{i+1}(w), \qquad w \in W,\ 1 \leq i \leq n,$$

since each $w_i = 1$. Then:

$$\tau_i\big(\mathbf{t}(w)\big) = \frac{\mathbf{x}_i(w)}{\mathbf{x}_{i+1}(w)} = \mathrm{sign}_i(w) = \prod_{\substack{j>i \\ w^{-1}j<w^{-1}i}} (-1).$$

Suppose that $\alpha = \check{\alpha}_k \in \Delta$ and $w \in W$ satisfy the conditions of the proposition. Since $\mathbf{x}_i(s_\alpha w) = \mathbf{x}_i(w)$ for $i \geq k+2$, we have $\tau_i\big(\mathbf{t}(s_\alpha w)\big) = \tau_i\big(\mathbf{t}(w)\big)$ if $i \geq k+2$. Also:

$$\tau_k\big(\mathbf{t}(s_\alpha w)\big) = \prod_{\substack{j>k \\ (s_\alpha w)^{-1}j<(s_\alpha w)^{-1}k}} (-1).$$

Since $w^{-1}\alpha < 0$, it follows that $(s_\alpha w)^{-1}(k+1) > (s_\alpha w)^{-1}k$, hence:

$$\tau_k\big(\mathbf{t}(s_\alpha w)\big) = \prod_{\substack{j>k+1 \\ (s_\alpha w)^{-1}j<(s_\alpha w)^{-1}k}} (-1) \quad = \prod_{\substack{j>k+1 \\ w^{-1}j<w^{-1}(k+1)}} (-1) = \tau_{k+1}\big(\mathbf{t}(w)\big).$$

For every $\beta := \check{\alpha}_l \in \Delta$, $\beta \succ \alpha$, we have $w^{-1}\beta > 0$, or $w^{-1}l < w^{-1}(l+1)$. Thus:

$$w^{-1}(k+1) < w^{-1}(k+2) < \ldots < w^{-1}(n-1),$$

and this implies $\tau_i\big(\mathbf{t}(w)\big) = \mathrm{sign}_i(w) = 1$ for all $i > k$. Consequently:

$$(1) \qquad \tau_i\big(\mathbf{t}(s_\alpha w)\big) = 1 \qquad \text{if } i = k \text{ or } i \geq k+2.$$

Now factor $s_\alpha w = \mathbf{t}(s_\alpha w)\eta$ with $\eta \in \mathfrak{M}$. Since $\ell(w_\alpha \eta) = \ell(w_\alpha) + \ell(\eta)$:

$$\sigma_n(s_\alpha, s_\alpha w) = \sigma_n\big(\mathbf{t}(s_\alpha)w_\alpha, \mathbf{t}(s_\alpha w)\eta\big) = \sigma_n\big(\mathbf{t}(s_\alpha), {}^{w_\alpha}\mathbf{t}(s_\alpha w)\big)\, \sigma_n\big(w_\alpha, \mathbf{t}(s_\alpha w)\big)$$

by §3 Lemma 10. Using (1), we have:

$$\begin{aligned}\sigma_n\big(\mathbf{t}(s_\alpha), {}^{w_\alpha}\mathbf{t}(s_\alpha w)\big) &= \prod_{i<j} c\big(\tau_i\big(\mathbf{t}(s_\alpha)\big), \tau_{w_\alpha^{-1}j}\big(\mathbf{t}(s_\alpha w)\big)\big)^{-1} \\ &= \prod_{j>k} c\big(-1, \tau_{w_\alpha^{-1}j}\big(\mathbf{t}(s_\alpha w)\big)\big)^{-1}, \\ &= c\big(-1, \tau_k\big(\mathbf{t}(s_\alpha w)\big)\big)^{-1} \prod_{j>k+1} c\big(-1, \tau_j\big(\mathbf{t}(s_\alpha w)\big)\big)^{-1} = 1_{\mathcal{A}},\end{aligned}$$

and:

$$\sigma_n\big(w_\alpha, \mathbf{t}(s_\alpha w)\big) = c\big(-\tau_{k+1}\big(\mathbf{t}(s_\alpha w)\big), \tau_k\big(\mathbf{t}(s_\alpha w)\big)\big)^{-1} = 1_{\mathcal{A}}.$$

Thus, $\sigma_n(s_\alpha, s_\alpha w) = 1_{\mathcal{A}}$. Also:

$$\sigma_n(s_\alpha, w) = \sigma_n(s_\alpha, s_\alpha^2 w)\, \sigma_n(s_\alpha, s_\alpha w) = \sigma_n(s_\alpha, s_\alpha)\, \sigma_n(1_G, w) = 1_{\mathcal{A}}$$

by Lemma 1. This completes the proof. ∎

Given the 2-cocycle $\sigma_n \in Z^2(G;\mathcal{A})$, the central extension $\widetilde{G}$ of G by $\mathcal{A}$ can be constructed as follows. As a *set*, $\widetilde{G} := G \times \mathcal{A}$, with multiplication defined by:

$$(g,a)\cdot(g',a') := \big(gg', aa'\sigma_n(g,g')\big), \qquad g,g' \in G,\ a,a' \in \mathcal{A}.$$

Let $\mathfrak{s}_n : G \to \widetilde{G}$ be the section $g \mapsto (g, 1_{\mathcal{A}})$. Then:

$$\sigma_n(g,g') = \mathfrak{s}_n(g)\, \mathfrak{s}_n(g')\, \mathfrak{s}_n(gg')^{-1}, \qquad g,g' \in G.$$

With this notation, define $\tilde{s}_\alpha := \mathfrak{s}_n(s_\alpha)$ for all $\alpha \in \Phi$. Note that $1_{\widetilde{G}} := \mathfrak{s}_n(1_G)$ is the identity in $\widetilde{G}$.

Let $F[\Delta]$ be the free group generated by $\{s_\alpha \mid \alpha \in \Phi\}$. Using the ordering on Δ, we order the elements of $F[\Delta]$ *lexicographically.* For every $w \in W$, a *reduced expression* for w is an expression of the form $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ with $\ell = \ell(w)$. Regarding each reduced expression as a word in $F[\Delta]$, we will define the *canonical expression* for w to be the *largest* reduced expression with respect to the lexicographic ordering on $F[\Delta]$. Alternatively, we can define the canonical expression inductively as follows. If $w = 1_G$, then the canonical expression for w is 1_G. Next, suppose that we have already defined the canonical expression for all $w \in W$ with $\ell(w) < \ell$, $\ell \geq 1$. If $w \in W$, $\ell(w) = \ell$, let α_1 be the largest element of Δ such that $w^{-1}\alpha_1 < 0$. Note that $\alpha_1 = \check{\alpha}_k$, where k is the largest integer i such that $\tau_i\big(\mathbf{t}(w)\big) = \mathrm{sign}_i(w) = -1$. Then $w = s_{\alpha_1} w'$ with $w' \in W$, $\ell(w') = \ell - 1$. If the canonical expression for w' is $s_{\alpha_2} \dots s_{\alpha_\ell}$, then we define the canonical expression for w to be $s_{\alpha_1} \dots s_{\alpha_\ell}$. It can be shown that the two definitions are equivalent.

Corollary 3. *Let $w \in W$, and let $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ be its canonical expression. Then $\mathfrak{s}_n(w) = \tilde{s}_{\alpha_1} \dots \tilde{s}_{\alpha_\ell}$.*

Proof : This follows immediately from Proposition 2. ☐

Lemma 4. *We have:*

$$\begin{aligned} \tilde{s}_\alpha^2 &= 1_{\widetilde{G}} & \qquad & \alpha \in \Phi, \\ \tilde{s}_\alpha\, \tilde{s}_\beta &= c(-1,-1)\, \tilde{s}_\beta\, \tilde{s}_\alpha & & \alpha, \beta \in \Phi,\ \langle \alpha, \beta \rangle = 0, \\ \tilde{s}_\alpha\, \tilde{s}_\beta\, \tilde{s}_\alpha &= c(-1,-1)\, \tilde{s}_\beta\, \tilde{s}_\alpha\, \tilde{s}_\beta & & \alpha, \beta \in \Phi,\ \langle \alpha, \beta \rangle = -1. \end{aligned}$$

Proof : The first two relations follow from Lemma 1. Now suppose that $\alpha, \beta \in \Delta$, $\langle \alpha, \beta \rangle = -1$, $\alpha \succ \beta$. Since $s_\alpha s_\beta s_\alpha$ is the canonical expression for $w := s_\alpha s_\beta s_\alpha$, $\mathfrak{s}_n(w) = \tilde{s}_\alpha \tilde{s}_\beta \tilde{s}_\alpha$ by Corollary 3. On the other hand:

$$\mathfrak{s}_n(w) = \mathfrak{s}_n(s_\beta s_\alpha s_\beta) = \tilde{s}_\beta \tilde{s}_\alpha \tilde{s}_\beta\, \sigma_n(s_\beta, s_\alpha s_\beta)\, \sigma_n(s_\alpha, s_\beta).$$

As $\sigma_n(s_\alpha, s_\beta) = 1_{\mathcal{A}}$ by Lemma 1, it suffices to show that $\sigma_n(s_\beta, s_\alpha s_\beta) = c(-1,-1)$.

If $\alpha = \check{\alpha}_k$, then $\beta = \check{\alpha}_{k-1}$, and:

$$\tau_i\big(\mathbf{t}(s_\beta)\big) = \begin{cases} -1 & \text{if } i = k-1, \\ 1 & \text{otherwise,} \end{cases}$$
$$\tau_i\big(\mathbf{t}(s_\alpha s_\beta)\big) = \begin{cases} -1 & \text{if } i = k-1 \text{ or } i = k, \\ 1 & \text{otherwise.} \end{cases}$$

Factor $s_\alpha s_\beta = \mathbf{t}(s_\alpha s_\beta)\eta$ with $\eta \in \mathfrak{M}$. Since $\ell(w_\beta \eta) = \ell(w_\beta) + \ell(\eta)$:

$$\sigma_n(s_\beta, s_\alpha s_\beta) = \sigma_n\big(\mathbf{t}(s_\beta)w_\beta, \mathbf{t}(s_\alpha s_\beta)\eta\big) = \sigma_n\big(\mathbf{t}(s_\beta), {}^{w_\beta}\mathbf{t}(s_\alpha s_\beta)\big)\,\sigma_n\big(w_\beta, \mathbf{t}(s_\alpha s_\beta)\big)$$

by §3 Lemma 10. We have:

$$\begin{aligned}
\sigma_n\big(\mathbf{t}(s_\beta), {}^{w_\beta}\mathbf{t}(s_\alpha s_\beta)\big) &= \prod_{i<j} c\big(\tau_i\big(\mathbf{t}(s_\beta)\big), \tau_{w_\beta^{-1}j}\big(\mathbf{t}(s_\alpha s_\beta)\big)\big)^{-1} \\
&= \prod_{j>k-1} c\big(-1, \tau_{w_\beta^{-1}j}\big(\mathbf{t}(s_\alpha s_\beta)\big)\big)^{-1} \\
&= c\big(-1, \tau_{k-1}\big(\mathbf{t}(s_\alpha s_\beta)\big)\big)^{-1} \prod_{j>k} c\big(-1, \tau_j\big(\mathbf{t}(s_\alpha s_\beta)\big)\big)^{-1} \\
&= c(-1,-1),
\end{aligned}$$

and:

$$\sigma_n\big(w_\beta, \mathbf{t}(s_\alpha s_\beta)\big) = c\big(-\tau_k\big(\mathbf{t}(s_\alpha s_\beta)\big), \tau_{k-1}\big(\mathbf{t}(s_\alpha s_\beta)\big)\big)^{-1} = 1_{\mathcal{A}}.$$

This proves the lemma in this case. Finally, if $\alpha, \beta \in \Delta$, $\langle \alpha, \beta \rangle = -1$, $\beta \succ \alpha$, then we have shown that $\tilde{s}_\beta \tilde{s}_\alpha \tilde{s}_\beta = c(-1,-1)\, \tilde{s}_\alpha \tilde{s}_\beta \tilde{s}_\alpha$, hence $\tilde{s}_\alpha \tilde{s}_\beta \tilde{s}_\alpha = c(-1,-1)\, \tilde{s}_\beta \tilde{s}_\alpha \tilde{s}_\beta$. This completes the proof. □

In order to compute $\sigma_n(w, w')$ for arbitrary $w, w' \in W$, we first determine the canonical expressions for w, w', and ww':

$$w = s_{\alpha_1} \dots s_{\alpha_k},$$
$$w' = s_{\beta_1} \dots s_{\beta_l},$$
$$ww' = s_{\gamma_1} \dots s_{\gamma_m}.$$

By Corollary 3, we have:

$$\sigma_n(w, w') = \mathfrak{s}_n(w)\,\mathfrak{s}_n(w')\,\mathfrak{s}_n(ww')^{-1} = \tilde{s}_{\alpha_1}\ldots\tilde{s}_{\alpha_k}\tilde{s}_{\beta_1}\ldots\tilde{s}_{\beta_k}(\tilde{s}_{\gamma_1}\ldots\tilde{s}_{\gamma_m})^{-1}.$$

The relations of Lemma 4 are then used to compute the cocycle.

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