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## On $m^{\text{th}}$ -autocommutator subgroup of finite abelian groups

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**Abstract.** Let  $G$  be a group and  $Aut(G)$  be the group of automorphisms of  $G$ . For any natural number  $m$ , the  $m^{\text{th}}$ -autocommutator subgroup of  $G$  is defined as:

$$K_m(G) = \langle [g, \alpha_1, \dots, \alpha_m] \mid g \in G, \alpha_1, \dots, \alpha_m \in Aut(G) \rangle.$$

In this paper, we obtain the  $m^{\text{th}}$ -autocommutator subgroup of all finite abelian groups.

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### 1. Introduction

Let  $G$  be a group and  $Aut(G)$  denote the group of automorphisms of  $G$ . As in [3], if  $g \in G$  and  $\alpha \in Aut(G)$ , then the element  $[g, \alpha] = g^{-1}\alpha(g)$  is an *autocommutator* of  $g$  and  $\alpha$ . Hence, following [5] one may define the *autocommutator of weight  $m+1$*  ( $m \geq 2$ ) inductively as:

$$[g, \alpha_1, \alpha_2, \dots, \alpha_m] = [[g, \alpha_1, \dots, \alpha_{m-1}], \alpha_m],$$

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for all  $\alpha_1, \alpha_2, \dots, \alpha_m \in \text{Aut}(G)$ .  
 Now for any natural number  $m$

$$K_m(G) = [G, \underbrace{\text{Aut}(G), \dots, \text{Aut}(G)}_{m\text{-times}}] = \langle [g, \alpha_1, \alpha_2, \dots, \alpha_m] \mid g \in G, \alpha_1, \dots, \alpha_m \in \text{Aut}(G) \rangle,$$

which is called the  $m^{\text{th}}$ -autocommutator subgroup of  $G$ .

Throughout this paper we adopt additive notation for all abelian groups. To be brief,  $([k]_n, [k']_m)$  of group  $\mathbb{Z}_n \oplus \mathbb{Z}_m$  will be indicated as  $(k, k')$ , where  $k \in \{0, 1, 2, \dots, n - 1\}$  and  $k' \in \{0, 1, 2, \dots, m - 1\}$ .

- Example 1.1** i) Let  $n$  be a natural number. Then  $K_m(\mathbb{Z}_{2^n}) = 2^m\mathbb{Z}_{2^n}$ , for any natural number  $m$ .  
 ii) Let  $G = D_{2n}$ , dihedral group of order  $2n$ . Then one can check that  $K_m(G) \cong 2^{m-1}\mathbb{Z}_n$ , for any natural number  $m$ .

**Proof.** i) It is obvious by Lemma 2.2 of [5].  
 ii) We have  $D_{2n} = \langle x, y \mid x^n = y^2 = (xy)^2 = 1 \rangle$  and  $\text{Aut}(D_{2n}) = \{ \alpha_{i,t} \mid 0 \leq i \leq n - 1, 1 \leq t \leq n - 1, (t, n) = 1 \}$  such that  $\alpha_{i,t}(x) = x^t$  and  $\alpha_{i,t}(y) = x^i y$ . So if  $n$  is even, then by induction on  $m$ , we have  $K_m(D_{2n}) = \langle x^{2^{m-1}} \rangle \cong 2^{m-1}\mathbb{Z}_n$ . If  $n$  is odd, then we have  $K_m(D_{2n}) = \langle x \rangle \cong \mathbb{Z}_n \cong 2^{m-1}\mathbb{Z}_n$ . ■

In [5] some properties of autocommutator subgroups of a finite abelian group are studied. The under example shows the  $m^{\text{th}}$ -autocommutator subgroup of a finite abelian group incorrectly concluded in the Theorem 2.5 of [5].

**Example 1.2** Let  $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4$ . Then by Theorem 2.5 of [5],  $K_2(G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . But if we define the automorphisms  $\alpha$  and  $\beta$  of the  $G$ , given by  $\alpha(a, b) = (a, a + b)$  and  $\beta(a, b) = (a + 2b, b)$  for all  $(a, b) \in G$ , then we have  $[(1, 0), \alpha, \beta] = (2, 0)$  and hence,  $K_2(G)$  has an element of order 4. So  $K_2(G) \neq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ .

In [6], we obtained the  $K_m(\oplus_{i=1}^k \mathbb{Z}_{2^{n_i}})$  with  $n_1 > n_2 > \dots > n_k$  using a function which recursively defined in terms of the  $n_1, \dots, n_k$ 's. In this paper we obtain the  $m^{\text{th}}$ -autocommutator subgroup of all finite abelian groups.

## 2. Preliminary Results

We begin with some useful results that will be used in the proof of our main results.

**Lemma 2.1** ([6]) i) Let  $H$  and  $T$  be two arbitrary groups. Then for any natural number  $m$ ,

$$K_m(H) \times K_m(T) \subseteq K_m(H \times T).$$

ii) Let  $H$  and  $T$  be finite groups such that  $(|H|, |T|) = 1$ . Then for any natural number  $m$ ,

$$K_m(H) \times K_m(T) = K_m(H \times T).$$

**Lemma 2.2** ([5], Lemma 2.2) If  $G$  is a finite cyclic group, then

$$K_m(G) = 2^m G, \text{ for any natural number } m.$$

**Corollary 2.3** If  $G$  is a finite abelian group of odd order, then

$$K_m(G) = G, \text{ for any natural number } m.$$

**Proof.** It is obvious by Lemma 2.1 and Lemma 2.2. ■

Recall for any natural number  $n$ , if  $G = \mathbb{Z}_{2^n}$ , then  $Aut(G)$  consists of all automorphisms  $\alpha_i : g \mapsto ig$ , where  $1 \leq i < 2^n$  and  $i$  is an odd number. We know that the automorphism groups of finitely generated abelian groups are well-understood ( see [4]). Now we have

**Lemma 2.4** Let  $G = \bigoplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$  such that  $n_1 > n_2 > \dots > n_k$ . Also let  $\epsilon_i = (0, \dots, 0, \underbrace{1}_i, 0, \dots, 0)$ , for  $i = 1, \dots, k$ . Then for all  $\alpha \in Aut(G)$  we have

$$\begin{aligned} \alpha(\epsilon_1) &= (m_{11}, m_{12}, \dots, m_{1k}) \\ \alpha(\epsilon_2) &= (2^{n_1-n_2}m_{21}, m_{22}, \dots, m_{2k}) \\ \alpha(\epsilon_3) &= (2^{n_1-n_3}m_{31}, 2^{n_2-n_3}m_{32}, m_{33}, \dots, m_{3k}) \\ &\vdots \\ \alpha(\epsilon_k) &= (2^{n_1-n_k}m_{k1}, 2^{n_2-n_k}m_{k2}, \dots, 2^{n_{k-1}-n_k}m_{k(k-1)}, m_{kk}) \end{aligned}$$

where  $m_{ij} \in \mathbb{Z}$  for all  $i, j$  and for all  $i$ ,  $m_{ii}$  is odd.

**Proof.** We know that for  $i = 1, \dots, k$ , we have  $|\epsilon_i| = 2^{n_i}$ . Hence,

$$\begin{aligned} \alpha(\epsilon_1) &= (m_{11}, m_{12}, \dots, m_{1k}) \\ \alpha(\epsilon_2) &= (2^{n_1-n_2}m_{21}, m_{22}, m_{23}, \dots, m_{2k}) \\ \alpha(\epsilon_3) &= (2^{n_1-n_3}m_{31}, 2^{n_2-n_3}m_{32}, m_{33}, m_{34}, \dots, m_{3k}) \\ &\vdots \\ \alpha(\epsilon_k) &= (2^{n_1-n_k}m_{k1}, 2^{n_2-n_k}m_{k2}, \dots, 2^{n_{k-1}-n_k}m_{k(k-1)}, m_{kk}) \end{aligned}$$

for some  $m_{ij} \in \{0, 1, 2, 3, \dots, 2^{n_j} - 1\}$ . Now it is sufficient to prove  $m_{tt}$  is an odd number, for any natural number  $t$ . We use of induction on  $t$ . If  $t = 1$ , then clearly  $m_{11}$  is an odd number. Let  $t > 1$  and assume  $m_{t't'}$  is an odd number, for any natural number  $t'$  such that  $t' < t$ . Then we prove  $m_{tt}$  is an odd number. Assume that  $m_{tt}$  is an even number. Then for any natural number  $r$  such that  $1 \leq r \leq k$  if  $t < r < k$ , then put  $c_r = 0$  and if  $r = t$ , then put  $c_r = 2^{n_t-1}$ . For  $1 \leq r \leq t - 1$  set  $I_r = \{i \in \mathbb{Z} \mid 0 \leq i < r, c_{t-i} \neq 0\}$  and put

$$c_{t-r} = \begin{cases} 0 & \text{if } \sum_{s \in I_r} m_{(t-s)(t-r)} \text{ is an even number,} \\ 2^{n_{t-r}-1} & \text{if } \sum_{s \in I_r} m_{(t-s)(t-r)} \text{ is an odd number.} \end{cases}$$

Now  $(c_1, \dots, c_k) \neq 0$ , but  $\alpha((c_1, \dots, c_k)) = 0$ , which is a contradiction. Hence,  $m_{tt}$  is an odd number and this completes the proof. ■

**Lemma 2.5** ([1]) For all natural numbers  $k, n_1, n_2, \dots, n_k$  such that  $n_1 > n_2 \geq \dots \geq n_k$ ,

$$K_1(\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}}) = 2\mathbb{Z}_{2^{n_1}} \oplus \mathbb{Z}_{2^{n_2}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}}.$$

**Lemma 2.6** ([7]) Suppose that  $G = \bigoplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$  with  $k > 1$  and  $n_1 = n_2 \geq n_3 \geq \dots \geq n_k$ . Then,

$$K_m(G) = G, \text{ for any natural number } m.$$

In [6], we obtained the  $K_m(\oplus_{i=1}^k \mathbb{Z}_{2^{n_i}})$  with  $n_1 > n_2 > \dots > n_k$  using a function which recursively defined in terms of the  $n_1, \dots, n_k$ 's.

**Definition 2.7** We define for  $i = 1, 2, \dots, k$ ,  $T_{0,i} = 0$  and for  $m = 1, 2, \dots$ , that  $T_{m,0} = \infty$ . Now for  $m \geq 0$  and  $i = 1, 2, \dots, k$ , we have

$$T_{m+1,i} = \min\{T_{m,i-1}, T_{m,i} + 1, n_i - n_{i+1} + T_{m,i+1}, \dots, n_i - n_k + T_{m,k}\}.$$

**Theorem 2.8** ([6]) Suppose that  $G = \oplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$  with  $n_1 > n_2 > \dots > n_k$ . Then,

$$K_m(G) = \oplus_{i=1}^k 2^{T_{m,i}} \mathbb{Z}_{2^{n_i}},$$

for any natural number  $m$ .

The floor function of  $x$ , also called the greatest integer function, gives the largest integer less than or equal to  $x$ . In this paper we use of the symbol  $\lfloor x \rfloor$ .

**Definition 2.9** Let  $k, m, n_1, n_2, \dots, n_k$ , be natural numbers such that  $n_1 > n_2 > \dots > n_k$ . Then put  $T_{m1}^1 = m$  and for  $k > 1$  put

$$T_{mj}^k = \begin{cases} \min\{m - j + 1, A_{mj}^k, B_{mj}^k\} & 1 \leq j \leq \min\{m, k - 1\} \\ \min\{m - j + 1, A_{mj}^k\} & j = k \leq m \\ 0 & m < j \leq k \end{cases}$$

where for  $1 \leq j \leq \min\{m, k\}$

$$A_{mj}^k = \min\{(\lfloor \frac{m-j+1}{2} \rfloor)(n_i - n_{i+1}) \mid 1 \leq i \leq \min\{j, k - 1\}\} + \frac{(-1)^{m+j+1}}{2},$$

and for  $1 \leq j \leq \min\{m, k - 1\}$

$$B_{mj}^k = \min\{n_j - n_{j+i} + (\lfloor \frac{m-j+1}{2} \rfloor - i)(n_{j+i} - n_{j+i+1}) \mid 0 \leq i \leq \min\{\lfloor \frac{m-j+1}{2} \rfloor, k - j - 1\}\} + \frac{(-1)^{m+j+1}}{2}.$$

Also for  $k > 1$  and  $1 \leq j \leq \min\{m + 1, k - 1\}$ , put

$$M_{mj} = (\lfloor \frac{m-j+2}{2} \rfloor)(n_j - n_{j+1}) + \frac{(-1)^{m+j+1+1}}{2}.$$

In order to prove main result, we need to prove some technical lemmas.

**Lemma 2.10** i) If  $k > 1$  and  $1 \leq j \leq \min\{m + 1, k\}$ , then

$$A_{(m+1)j}^k = \begin{cases} \min\{A_{m(j-1)}^k, M_{mj}\} & 2 \leq j \leq k - 1, \\ A_{m(j-1)}^k & j = k, \\ M_{mj} & j = 1. \end{cases}$$

ii) If  $k > 1$  and  $1 \leq j \leq \min\{m + 1, k - 1\}$ , then

$$B_{(m+1)j}^k = \begin{cases} \min\{n_j - n_{j+1} + B_{m(j+1)}^k, M_{mj}\} & 1 \leq j \leq \min\{m - 1, k - 2\}, \\ M_{mj} & \text{Otherwise.} \end{cases}$$

**Proof.** It is easy to check that the result is true. ■

**Lemma 2.11** If  $k > 1$  and  $1 \leq j \leq k - 1$ , then

$$T_{mj}^k \geq T_{m(j+1)}^k.$$

**Proof.** If  $j \geq m$ , then the result is obvious. If  $j < m$  and  $j = k - 1$ , then it is easy to check  $T_{m(k-1)}^k \geq T_{mk}^k$ . Now let  $j < m$  and  $j < k - 1$ . Then  $T_{mj}^k = \min\{m - j + 1, A_{mj}^k, B_{mj}^k\}$ . If

$T_{mj}^k = m - j + 1$ , then result is obvious. If  $T_{mj}^k = A_{mj}^k = (\lfloor \frac{m-j+1}{2} \rfloor)(n_{i_0} - n_{i_0+1}) + \frac{(-1)^{m+j+1}}{2}$ , for some  $1 \leq i_0 \leq j$  then  $T_{mj}^k \geq (\lfloor \frac{m-j}{2} \rfloor)(n_{i_0} - n_{i_0+1}) + \frac{(-1)^{m+j+1+1}}{2} \geq A_{m(j+1)}^k \geq T_{m(j+1)}^k$ .

If  $T_{mj}^k = B_{mj}^k = n_j - n_{j+i_0} + (\lfloor \frac{m-j+1}{2} \rfloor - i_0)(n_{j+i_0} - n_{j+i_0+1}) + \frac{(-1)^{m+j+1}}{2}$ , for some  $0 \leq i_0 \leq \min\{\lfloor \frac{m-j+1}{2} \rfloor, k - j - 1\}$ , then we have,

if  $i_0 = 0$  then

$$\begin{aligned} T_{mj}^k &= (\lfloor \frac{m-j+1}{2} \rfloor)(n_j - n_{j+1}) + \frac{(-1)^{m+j+1}}{2} \\ &\geq (\lfloor \frac{m-j}{2} \rfloor)(n_j - n_{j+1}) + \frac{(-1)^{m+j+1+1}}{2} \\ &\geq A_{m(j+1)}^k \\ &\geq T_{m(j+1)}^k. \end{aligned}$$

Now if  $i_0 \geq 1$ , then we have one of the following cases.

(case 1)  $n_{j+i_0} - n_{j+i_0+1} = 1$ . Hence,

$$\begin{aligned} T_{mj}^k &\geq 1 + n_{j+1} - n_{j+i_0} + \lfloor \frac{m-j+1}{2} \rfloor - i_0 + \frac{(-1)^{m+j+1}}{2} \\ &\geq 1 + n_{j+1} - n_{j+i_0} + \lfloor \frac{m-j}{2} \rfloor - i_0 + \frac{(-1)^{m+j+1+1}}{2} \\ &= n_{j+1} - n_{j+i_0} + (\lfloor \frac{m-j}{2} \rfloor - (i_0 - 1))(n_{j+i_0} - n_{j+i_0+1}) + \frac{(-1)^{m+j+1+1}}{2} \\ &\geq B_{m(j+1)}^k \\ &\geq T_{m(j+1)}^k. \end{aligned}$$

(case 2)  $n_j - n_{j+i_0} < 2i_0$ . Hence,  $n_{j+s} - n_{j+s+1} = 1$ , for some  $0 \leq s \leq i_0 - 1$ . Now we have

(case 2.1) If  $s = 0$ , then  $n_j - n_{j+1} = 1$ . Hence,

$$\begin{aligned} T_{mj}^k &\geq i_0 + (\lfloor \frac{m-j+1}{2} \rfloor - i_0) + \frac{(-1)^{m+j+1}}{2} \\ &\geq \lfloor \frac{m-j}{2} \rfloor + \frac{(-1)^{m+j+1+1}}{2} \\ &= \lfloor \frac{m-j}{2} \rfloor (n_j - n_{j+1}) + \frac{(-1)^{m+j+1+1}}{2} \\ &\geq A_{m(j+1)}^k \\ &\geq T_{m(j+1)}^k. \end{aligned}$$

(case 2.2) If  $s \geq 1$ , then

$$\begin{aligned} T_{mj}^k &= n_j - n_{j+1} + n_{j+1} - n_{j+s} + n_{j+s} - n_{j+i_0} + (\lfloor \frac{m-j+1}{2} \rfloor - i_0)(n_{j+i_0} - n_{j+i_0+1}) \\ &\quad + \frac{(-1)^{m+j+1}}{2} \geq 1 + n_{j+1} - n_{j+s} + i_0 - s + \lfloor \frac{m-j+1}{2} \rfloor - i_0 + \frac{(-1)^{m+j+1}}{2} \\ &\geq n_{j+1} - n_{j+s} + \lfloor \frac{m-j}{2} \rfloor - s + 1 + \frac{(-1)^{m+j+1+1}}{2} \\ &= n_{j+1} - n_{j+s} + (\lfloor \frac{m-j}{2} \rfloor - (s - 1))(n_{j+s} - n_{j+s+1}) + \frac{(-1)^{m+j+1+1}}{2} \\ &\geq B_{m(j+1)}^k \\ &\geq T_{m(j+1)}^k. \end{aligned}$$

(case 3)  $n_j - n_{j+i_0} \geq 2i_0$  and  $n_{j+i_0} - n_{j+i_0+1} \geq 2$ . Hence,

$$T_{mj}^k \geq 2i_0 + (\lfloor \frac{m-j+1}{2} \rfloor - i_0) \times 2 + \frac{(-1)^{m+j+1}}{2} = m - j + 1 > T_{m(j+1)}^k.$$

These imply the result. ■

The following lemmas are proved in a similar way with the previous lemma and are left to reader.

**Lemma 2.12** i) If  $k > 2$  and  $1 \leq j \leq k - 2$ , then

$$T_{m(j+2)}^k + n_{j+1} - n_{j+2} \geq T_{m(j+1)}^k.$$

ii) If  $k > 1$  and  $1 \leq j \leq \min\{m, k - 1\}$ , then

$$n_j - n_{j+1} + m - j \geq \min\{m - j + 2, M_{mj}\}.$$

iii) If  $k > 1$  and  $1 \leq j \leq \min\{m, k - 1\}$ , then

$$M_{mj} \geq n_j - n_{j+1} + T_{m(j+1)}^k.$$

**Lemma 2.13** For  $k > 1$  and  $2 \leq j \leq \min\{m + 1, k\}$ ,

i) if  $2 \leq j \leq \min\{m - 1, k - 2\}$ , then

$$B_{m(j-1)}^k \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}, n_j - n_{j+1} + B_{m(j+1)}^k\}.$$

ii) If  $j = k - 1$  or  $m \leq j \leq k - 2$ , then

$$B_{m(j-1)}^k \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}\}.$$

iii) If  $j = k$ , then

$$B_{m(j-1)}^k \geq A_{m(j-1)}^k.$$

**Lemma 2.14** For  $k > 1$ ,  $m > 1$  and  $1 \leq j \leq \min\{m - 1, k\}$ ,

i) if  $2 \leq j \leq k - 2$ , then

$$T_{mj}^k + 1 \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}, n_j - n_{j+1} + B_{m(j+1)}^k\}.$$

ii) If  $j = k - 1$  such that  $k > 2$ , then

$$T_{mj}^k + 1 \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}\}.$$

iii) If  $j = k$ , then

$$T_{mj}^k + 1 \geq \min\{m - j + 2, A_{m(j-1)}^k\}.$$

iv) If  $j = 1$  such that  $k = 2$ , then

$$T_{mj}^k + 1 \geq \min\{m - j + 2, M_{mj}\}.$$

v) If  $j = 1$  such that  $k > 2$ , then

$$T_{mj}^k + 1 \geq \min\{m - j + 2, M_{mj}, n_j - n_{j+1} + B_{m(j+1)}^k\}.$$

**Lemma 2.15** For  $k > 1$ ,  $m > 1$  and  $1 \leq j \leq \min\{m - 1, k - 1\}$ ,

i) if  $2 \leq j \leq k - 2$ , then

$$n_j - n_{j+1} + A_{m(j+1)}^k \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}, n_j - n_{j+1} + B_{m(j+1)}^k\}.$$

ii) If  $j = k - 1$  such that  $k > 2$  then

$$n_j - n_{j+1} + A_{m(j+1)}^k \geq \min\{m - j + 2, A_{m(j-1)}^k, M_{mj}\}.$$

iii) If  $j = 1$  and  $k = 2$ , then

$$n_j - n_{j+1} + A_{m(j+1)}^k \geq \min\{m - j + 2, M_{mj}\}.$$

iv) If  $j = 1$  and  $k > 2$ , then

$$n_j - n_{j+1} + A_{m(j+1)}^k \geq \min\{m - j + 2, M_{mj}, n_j - n_{j+1} + B_{m(j+1)}^k\}.$$

Finally we will prove the following corollary, that is useful in the proof of the main result.

**Corollary 2.16** If  $k > 1$  and  $1 \leq j \leq k$ , then

$$T_{(m+1)j}^k = \begin{cases} \min\{T_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + T_{m(j+1)}^k\} & 2 \leq j \leq k - 1, \\ \min\{T_{mj}^k + 1, n_j - n_{j+1} + T_{m(j+1)}^k\} & j = 1, \\ \min\{T_{m(j-1)}^k, T_{mj}^k + 1\} & j = k. \end{cases}$$

**Proof.** We prove the result for case  $2 \leq j \leq \min\{m - 1, k - 2\}$ .

$$\begin{aligned}
 T_{(m+1)j}^k &= \min\{m - j + 2, A_{(m+1)j}^k, B_{(m+1)j}^k\} \text{ (by Definition 2.9)} \\
 &= \min\{m - j + 2, A_{m(j-1)}^k, n_j - n_{j+1} + B_{m(j+1)}^k, M_{mj}\} \text{ (by Lemma 2.10)} \\
 &= \min\{m - j + 2, A_{m(j-1)}^k, n_j - n_{j+1} + B_{m(j+1)}^k, n_j - n_{j+1} + m - j, M_{mj}\} \\
 &\quad \text{(by Lemma 2.12(ii))} \\
 &= \min\{m - j + 2, A_{m(j-1)}^k, B_{m(j-1)}^k, n_j - n_{j+1} + B_{m(j+1)}^k, n_j - n_{j+1} + m - j, \\
 &\quad M_{mj}\} \text{ (by Lemma 2.13(i))} \\
 &= \min\{m - j + 2, A_{m(j-1)}^k, B_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + B_{m(j+1)}^k, \\
 &\quad n_j - n_{j+1} + m - j, M_{mj}\} \text{ (by Lemma 2.14(i))} \\
 &= \min\{m - j + 2, A_{m(j-1)}^k, B_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + A_{m(j+1)}^k, \\
 &\quad n_j - n_{j+1} + B_{m(j+1)}^k, n_j - n_{j+1} + m - j, M_{mj}\} \text{ (by Lemma 2.15(i))} \\
 &= \min\{T_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + T_{m(j+1)}^k, M_{mj}\} \text{ (by Definition 2.9)} \\
 &= \min\{T_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + T_{m(j+1)}^k\} \text{ (by Lemma 2.12(iii))}.
 \end{aligned}$$

The other cases are proved in a similar way. ■

### 3. Main Results

Let  $G$  be a finite abelian group and  $m$  be a natural number. Then  $G = H \oplus T$ , where  $T$  is a finite abelian group of odd order and  $H$  is trivial group or a finite abelian 2-group. Hence,  $K_m(G) = K_m(H) \oplus T$ , by Lemma 2.1(ii) and Corollary 2.3. Therefore without loss of generality, we restrict attention to the case where  $G$  is a finite abelian 2-group.

**Theorem 3.1** Let  $k, n_1, \dots, n_k$  are natural numbers and  $G = \bigoplus_{j=1}^k \mathbb{Z}_{2^{n_j}}$ . Then using the notations in the Definition 2.9,

i) if  $n_1 > n_2 > \dots > n_k$ , then

$$K_m(G) = \bigoplus_{i=1}^k 2^{T_{m_j}^k} \mathbb{Z}_{2^{n_j}}.$$

ii) If  $k > 1$  and  $n_1 > n_2 > \dots > n_{t-1} > n_t = n_{t+1} \geq \dots \geq n_k$ , for some natural number  $1 \leq t < k$ , then

$$K_m(G) = \begin{cases} G & t = 1, \\ (\bigoplus_{j=1}^{t-1} 2^{T_{m_j}^k} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^t} \oplus \dots \oplus \mathbb{Z}_{2^k} & t > 1, \end{cases}$$

where  $T_{m_j}^t = \min\{T_{m_j}^{t-1}, n_j - n_t\}$ .

**Proof.** i) If  $k = 1$ , then the result is true by Lemma 2.2. Now for  $k > 1$  we use of induction on  $m$ . If  $m = 1$ , then the result is true by Lemma 2.5. Let  $m > 1$  and assume the result to be true for  $m$ , the induction hypothesis. Let  $(a_1, \dots, a_k) \in G$  and  $\alpha_1, \dots, \alpha_{m+1} \in \text{Aut}(G)$ . Then due to the induction hypothesis  $[(a_1, \dots, a_k), \alpha_1, \dots, \alpha_{m+1}] = [(b_1, \dots, b_k), \alpha_{m+1}]$  such that  $b_j = 2^{T_{m_j}^k} b'_j$  for some  $b'_j \in \mathbb{Z}_{2^{n_j}}$ . Hence, by Lemma 2.4,  $[(b_1, \dots, b_k), \alpha_{m+1}] = (c_1, \dots, c_k)$ , where  $c_j = m_{1j} b_1 + m_{2j} b_2 + \dots + m_{(j-1)j} b_{j-1} + (-1 + m_{jj}) b_j + 2^{n_j - n_{j+1}} m_{(j+1)j} b_{j+1} + 2^{n_j - n_{j+2}} m_{(j+2)j} b_{j+2} + \dots + 2^{n_j - n_k} m_{kj} b_k$ . Now if  $2 \leq j \leq k-1$ , then by Lemma 2.11 and Lemma 2.12(i), we have  $c_j = 2^{\min\{T_{m(j-1)}^k, T_{mj}^k + 1, n_j - n_{j+1} + T_{m(j+1)}^k\}} c'_j$  for some  $c'_j \in \mathbb{Z}_{2^{n_j}}$  and hence,  $c_j = 2^{T_{(m+1)j}^k} c'_j$  by Corollary 2.16. Similarly, if  $j = 1$  or  $j = k$ , then  $c_j = 2^{T_{(m+1)j}^k} c'_j$ , for some  $c'_j \in \mathbb{Z}_{2^{n_j}}$ . Hence,

$$K_{m+1}(G) \subseteq \bigoplus_{j=1}^k 2^{T_{(m+1)j}^k} \mathbb{Z}_{2^{n_j}}.$$

Now, for the reverse conclusion if  $T_{(m+1)j}^k \geq n_j$  then  $2^{T_{(m+1)j}^k} \mathbb{Z}_{2^{n_j}} = \langle 0 \rangle$ . Hence, for  $1 \leq j \leq k$  let  $T_{(m+1)j}^k < n_j$ . Now if  $1 < j < k$  then by Corollary 2.16 we have one of the following cases.

(Case 1) If  $T_{(m+1)j}^k = T_{m(j-1)}^k$ , then we define the automorphism  $\alpha$  of  $G$ , given by  $\alpha(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_k) = (a_1, \dots, a_{j-1}, a_{j-1} + a_j, a_{j+1}, \dots, a_k)$ . Hence,  $(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) = [(0, \dots, 0, \underbrace{2^{T_{m(j-1)}^k}}_{j-1}, 0, \dots, 0), \alpha]$ . Now by induction hypothesis  $[(0, \dots, 0, \underbrace{2^{T_{m(j-1)}^k}}_{j-1}, 0, \dots, 0), \alpha] \in [K_m(G), A]$ . So

$$(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) \in K_{m+1}(G).$$

(Case 2) If  $T_{(m+1)j}^k = T_{mj}^k + 1$ , then we define the automorphism  $\alpha$  of  $G$ , given by  $\alpha(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_k) = (a_1, \dots, a_{j-1}, 3a_j, a_{j+1}, \dots, a_k)$ . Hence,  $(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) = [(0, \dots, 0, \underbrace{2^{T_{mj}^k}}_j, 0, \dots, 0), \alpha]$ . Now by induction hypothesis  $[(0, \dots, 0, \underbrace{2^{T_{mj}^k}}_j, 0, \dots, 0), \alpha] \in [K_m(G), A]$ . So

$$(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) \in K_{m+1}(G).$$

(Case 3) If  $T_{(m+1)j}^k = n_j - n_{j+1} + T_{m(j+1)}^k$ , then we define the automorphism  $\alpha$  of  $G$ , given by  $\alpha(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_k) = (a_1, \dots, a_{j-1}, a_j + 2^{n_j - n_{j+1}} a_{j+1}, a_{j+1}, \dots, a_k)$ . Hence,  $(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) = [(0, \dots, 0, \underbrace{2^{T_{m(j+1)}^k}}_{j+1}, 0, \dots, 0), \alpha]$ .

Now by induction hypothesis  $[(0, \dots, 0, \underbrace{2^{T_{m(j+1)}^k}}_{j+1}, 0, \dots, 0), \alpha] \in [K_m(G), A]$ .

So  $(0, \dots, 0, \underbrace{2^{T_{(m+1)j}^k}}_j, 0, \dots, 0) \in K_{m+1}(G)$ .

Similarly if  $j = 1$  or  $j = k$ , then  $(2^{T_{(m+1)1}^k}, 0, \dots, 0) \in K_{m+1}(G)$  and  $(0, \dots, 0, 2^{T_{(m+1)k}^k}) \in K_{m+1}(G)$ . Hence,

$$\bigoplus_{j=1}^k 2^{T_{(m+1)j}^k} \mathbb{Z}_{2^{n_j}} \subseteq K_{m+1}(G).$$

So the result is true by induction.

ii) If  $t = 1$ , then the result is true, by Lemma 2.6. If  $t > 1$ , then we prove the result by induction on  $m$ . If  $m = 1$ , then by Lemma 2.5 the result is true. Let  $m > 1$  and assume the result to be true for  $m$ , the induction hypothesis. Also let  $(a_1, \dots, a_k) \in G$  and  $\alpha_1, \dots, \alpha_{m+1} \in \text{Aut}(G)$ . Then by induction hypothesis  $[(a_1, \dots, a_k), \alpha_1, \dots, \alpha_{m+1}] = [(2^{T'_{m1}} b_1, \dots, 2^{T'_{m(t-1)}} b_{t-1}, b_t, \dots, b_k), \alpha_{m+1}]$  for some  $b_j \in \mathbb{Z}_{2^{n_j}}$ . Now by Lemma 2.4 and a similar way to the part (i) we conclude  $[(2^{T'_{m1}} b_1, \dots, 2^{T'_{m(t-1)}} b_{t-1}, b_t, \dots, b_k), \alpha_{m+1}] = (2^{T'_{(m+1)1}} c_1, \dots, 2^{T'_{(m+1)(t-1)}} c_{t-1}, c_t, \dots, c_k)$  for some  $c_j \in \mathbb{Z}_{2^{n_j}}$ . Hence,

$$K_{m+1}(G) \subseteq (\bigoplus_{j=1}^{t-1} 2^{T'_{(m+1)j}} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}}.$$

Now, for the reverse conclusion by Lemma 2.1(i) and Lemma 2.6 we have,  $K_{m+1}(\mathbb{Z}_{2^{n_1}} \oplus \dots \oplus \mathbb{Z}_{2^{n_{t-1}}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$ . Hence, by the part (i) we have  $2^{T_{(m+1)1}^{t-1}} \mathbb{Z}_{2^{n_1}} \oplus \dots \oplus 2^{T_{(m+1)(t-1)}^{t-1}} \mathbb{Z}_{2^{n_{t-1}}} \oplus \mathbb{Z}_{2^{n_t}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$ . Therefore if  $1 \leq j < t$



such that  $T'_{(m+1)j} = T^{t-1}_{(m+1)j}$ , then

$(0, \dots, 0, \underbrace{2^{T'_{(m+1)j}}}_{j}, 0, \dots, 0) \in K_{m+1}(G)$ . Now if  $1 \leq j < t$  such that  $T'_{(m+1)j} =$

$n_j - n_t$ , then we define the automorphism  $\alpha$  of the group  $G$ , given by

$\alpha(a_1, \dots, a_k) = (a_1, \dots, a_{j-1}, a_j + 2^{n_j - n_t} a_t, a_{j+1}, \dots, a_k)$ . Hence,

$(0, \dots, 0, \underbrace{2^{n_j - n_t}}_j, 0, \dots, 0) = [(0, \dots, 0, \underbrace{1}_t, 0, \dots, 0), \alpha]$ . Now by induction hypothesis

$[(0, \dots, 0, \underbrace{1}_t, 0, \dots, 0), \alpha] \in [K_m(G), A]$ . So

$(0, \dots, 0, \underbrace{2^{n_j - n_t}}_j, 0, \dots, 0) \in K_{m+1}(G)$ . These imply that

$$(\oplus_{j=1}^{t-1} 2^{T'_{(m+1)j}} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{n_t}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}} \subseteq K_{m+1}(G)$$

and lead to the result. ■

**Corollary 3.2** Let  $k, m, n_1, n_2, \dots, n_k$ , be natural numbers such that  $k > 1, n_1 > n_2 > \dots > n_k$  and  $n_1 - n_2 = 1$ . Then

$$K_m(\mathbb{Z}_{2^{n_1}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}}) = \begin{cases} \oplus_{j=1}^k 2^{\lfloor \frac{m-j+2}{2} \rfloor} \mathbb{Z}_{2^{n_j}} & m \geq k, \\ (\oplus_{j=1}^m 2^{\lfloor \frac{m-j+2}{2} \rfloor} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{m+1}} \oplus \dots \oplus \mathbb{Z}_{2^k} & m < k. \end{cases}$$

**Proof.** It is evident  $T_{mj}^k = \lfloor \frac{m-j+1}{2} \rfloor + \frac{(-1)^{m+j+1}}{2} = \lfloor \frac{m-j+2}{2} \rfloor$ , for  $1 \leq j \leq \min\{m, k\}$ . ■

**Corollary 3.3** Let  $k, m, n_1, n_2, \dots, n_k$ , be natural numbers such that  $k > 1$  and  $n_s - n_{s+1} \geq 2$ , for any natural number  $1 \leq s < k$ . Then

$$K_m(\mathbb{Z}_{2^{n_1}} \oplus \dots \oplus \mathbb{Z}_{2^{n_k}}) = \begin{cases} \oplus_{j=1}^k 2^{m-j+1} \mathbb{Z}_{2^{n_j}} & m \geq k, \\ (\oplus_{j=1}^m 2^{m-j+1} \mathbb{Z}_{2^{n_j}}) \oplus \mathbb{Z}_{2^{m+1}} \oplus \dots \oplus \mathbb{Z}_{2^k} & m < k. \end{cases}$$

**Proof.** It is easy to check  $T_{mj}^k = m - j + 1$ , for  $1 \leq j \leq \min\{m, k\}$ . ■

**Corollary 3.4** Let  $G = \oplus_{i=1}^k \mathbb{Z}_{2^{n_i}}$  such that

i)  $n_1 > n_2 > \dots > n_k$ . Then

$$K_{2n_1-1}(G) = \langle 0 \rangle.$$

ii)  $n_1 > n_2 > \dots > n_{t-1} > n_t = n_{t+1} \geq \dots \geq n_k$ , for some natural number  $1 \leq t < k$ . Then

$$K_m(G) \neq \langle 0 \rangle, \text{ for any natural number } m.$$

**Proof.** i) Let  $m = 2n_1 - 1$ . Then  $m \geq k$ . Hence  $m - j + 1 \geq n_j$  and  $A_{mj}^k \geq \lfloor \frac{m-j+1}{2} \rfloor + \frac{(-1)^{m+j+1}}{2} = \lfloor \frac{m-j+2}{2} \rfloor \geq n_j$ , for  $1 \leq j \leq k$ . Also  $B_{mj}^k \geq i + \lfloor \frac{m-j+1}{2} \rfloor - i + \frac{(-1)^{m+j+1}}{2} = \lfloor \frac{m-j+2}{2} \rfloor \geq n_j$ , for  $1 \leq j \leq k - 1$ . Therefore  $T_{mj}^k \geq n_j$ , for  $1 \leq j \leq k$  and hence,  $2^{T_{mj}^k} \mathbb{Z}_{2^{n_j}} = \langle 0 \rangle$ .

ii) It is obvious by Theorem 3.1(ii). ■

At last we conclude the autocommutator subgroups of all finite abelian groups of orders 32 and 24.

**Example 3.5** Let  $G$  be a finite abelian group of order 32. Then

$G$	$K_1(G)$	$K_2(G)$	$K_3(G)$	$K_4(G)$	$K_n(G)(n \geq 5)$
$\mathbb{Z}_{32}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_8$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 0 \rangle$
$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 0 \rangle$	$\langle 0 \rangle$
$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\langle 0 \rangle$
$\mathbb{Z}_8 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_2$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$

**Example 3.6** Let  $G$  be a finite abelian group of order 24. then

$G$	$K_1(G)$	$K_2(G)$	$K_n(G)(n \geq 3)$
$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3$
$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3$

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