STABILITY IN SEVERAL MEASURES AND A DIFFERENTIAL INEQUALITY FOR A PARTIAL INTEGRAL EQUATION

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1. Introduction. Stability in two measures can be traced back at least to Krasovskii [12; p. 155] (originally done in 1956) where he considers a system of functional differential equations

$$(K) \qquad x'(t) = F(t, x_t), \quad F(t, 0) = 0.$$

Here $(C, \|\cdot\|)$ is the Banach space of continuous functions $\varphi : [-h, 0] \to R^n$, $h > 0$, with the supremum norm and $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$. To specify a solution of K it is required that there be given a continuous initial function $\varphi : [t_0 - h, t_0] \to R^n$. If $F : [0, \infty) \times C \to R^n$ is continuous and takes bounded sets into bounded sets and if $t_0 \geq 0$, then there is a solution $x(t; t_0, \varphi)$ of (K) with $x_{t_0}(t_0, \varphi) = \varphi$ and defined on some interval $[t_0, t_0 + \beta)$; if the solution remains bounded, $\beta = \infty$.

The standard definition of stability states that: the zero solution of (K) is stable if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\varphi \in C, \|\varphi\| < \delta, t \geq t_0]$ imply that $\|x_t(t_0, \varphi)\| < \varepsilon$. The stability is entirely in terms of the supremum norm. The definition is continued to uniform stability and uniform asymptotic stability, all in terms of the supremum norm; those definitions and relations are extensively discussed in numerous places, including [7–10], [13–19], [21–22], [24]. Krasovskii noticed that, for smooth F, uniform asymptotic stability could be characterized by Liapunov functionals. Here, W_i

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shall always denote a wedge, which is a continuous increasing function with $W_i(0) = 0$. Krasovskii proved that the zero solution of (K) is uniformly asymptotically stable if and only if there is a continuous scalar functional $V(t, \varphi)$ and wedges W_i with

$$(i) \qquad W_1(\|\varphi\|) \le V(t, \varphi) \le W_2(\|\varphi\|)$$

and

$$(ii) \qquad V'_{(K)}(t, x_t) \le -W_3(\|\varphi\|).$$

But Krasovskii seems to have never found a V for a particular system which satisfied (i) and (ii). He noted that useful Liapunov functionals had a prototype

$$V(t, \varphi) = \varphi^2(0) + \int_{-h}^{0} \varphi^2(s)\, ds.$$

When he stated a theorem for such Liapunov functionals, he inadvertently introduced the fruitful notion of stability in two measures. He proved that if there is a continuous scalar functional $V(t, \varphi)$ and wedges W_i with

$$(i^*) \qquad W_1(|\varphi(0)|) \le V(t, \varphi) \le W_2(|\varphi(0)| + |||\varphi|||)$$

and

$$(ii^*) \qquad V'_{(K)}(t, x_t) \le -W_3(|x(t)|)$$

(here, $|||\cdot|||$ is the L^2-norm), then $x = 0$ is asymptotically stable (we showed [3] that the conclusion is uniform asymptotic stability). In fact, a simple consequence of these relations is that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \ge 0, \varphi \in C, |\varphi(0)| + |||\varphi||| < \delta,\ t \ge t_0]$ imply that $|x(t; t_0, \varphi)| < \varepsilon$. This is stability in the two measures $|\cdot|$ and $|\cdot| + |||\cdot|||$.

Not only do these measures allow much larger initial functions, producing solutions which remain bounded by ε, but the attempts to replace $|\cdot| + |||\cdot|||$ by $\|\cdot\|$ for forty years have recently been proved fruitless by Kato [11] and Makay [22]. Not only was (i^*) a good practical choice over (i), but (i) can not assure uniform asymptotic stability.

The idea was formalized by Movchan [24] for equations without delay and enjoyed some attention. For example, Hatvani ([7] – [10]) dealt with partial stability by means of Liapunov functions which failed to be decrescent or radially unbounded. But a sequence of papers by Lakshmikantham and Liu ([13–17]) culminated in a substantial monograph [18] for ordinary and finite delay equations, followed by a section on infinite delay in the monograph by Lakshmikantham-Zhang-Wen [19].

The general idea is quite parallel to that described above by Krasovskii; one considers two measures $h_0(t,\varphi)$ and $h_1(t,\varphi)$, together with a Liapunov functional V satisfying

$$W_1(h_1(t,\varphi)) \le V(t,\varphi) \le W_2(h_0(t,\varphi)).$$

Briefly, in such an arrangement, h_0 is said to be finer than h_1. If, for example, $V' \le 0$, then an initial function φ which is small in the measure h_0 will give rise to a solution which remains small in the measure of h_1.

The relationships in (i) and (i^*) are natural properties of typical Liapunov functionals used for decades with (K), and corresponding relations are natural in partial stability and boundedness when a Liapunov function fails to be either decrescent or radially unbounded.

But the basic thesis of this paper is that the Sobolev norms which arise in the study of differential and integral equations involving partial derivatives turn out to be very natural and much richer as a source of applications for the idea of stability in two measures. In this paper we illustrate some of those relations for partial integral equations of the form

$$u(t,x) = f(t,x) + \int_{t-h}^{t} D(t,s)u_{xx}(s,x)\,ds, u(t,0) = u(t,1) = 0,$$

where h is a positive constant.

We focus here on linear equations only because existence theory is so accessible. Our work rests on two Liapunov functionals, including

$$V(t,u(\cdot)) = \int_{t-h}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\,dv \right)^2 dx\,ds.$$

Nonlinear problems are considered in the same way merely by replacing u_{xx} throughout by $g(u_x)_x$ where $xg(x) > 0$ if $x \neq 0$.

2. Several measures. We are interested in stability, boundedness, and asymptotic behavior of solutions of

$$u(t,x) = f(t,x) + \int_{t-h}^{t} D(t,s)u_{xx}(s,x)\,ds\,, u(t,0) = u(t,1) = 0, \tag{2.1}$$

where f and D are at least continuous and some of the following hold when $t-h \le s \le t$:

$$f(t,0) = f(t,1) = 0, \tag{2.2}$$

$$D(t,s) \ge 0, D_s(t,s) \ge 0, D_{st}(t,s) \le 0, D_t(t,s) \le 0, and\, D(t,t-h) = 0, \tag{2.3}$$

$$\exists P > 0, Q > 0 \text{ with } D(t,t) \le P, P\int_{t-h}^{t} D_s(t,s)(t-s)\,ds \le Q, \tag{2.4}$$

$$\int_0^\infty \int_0^1 [f_x(t,x) + f_{xx}^2(t,x) + f_{xxx}^2(t,x)]\,dx\,dt < \infty, \tag{2.5}$$

$$F(t) := \int_{t-h}^{t} D_s(t,s) \int_0^1 \left(\int_s^t f_{xx}(v,x)\,dv \right)^2 dx\,ds \to 0 \text{ as } t \to \infty, \tag{2.6}$$

$$\exists k(t) \ge 0 \text{ with } \int_0^\infty k(t)\,dt = \infty \text{ and } D_{st}(t,s) \le -k(t)D_s(t,s). \tag{2.7}$$

The general problem is to show that a solution $u(t,x)$ gets close to $f(t,x)$ in some sense.

2.a. Initial functions. To specify a solution of (2.1) it is required that there be given a continuous function $\varphi : [t_0 - h, t_0] \times [0,1] \to R$, φ_{xx} continuous and $\varphi(t,0) = \varphi(t,1) = 0$. The task then is to prove that there is a solution $u(t,x)$ of (2.1) on an interval $[t_0, t_0 + \alpha)$ for some $\alpha > 0$ with $u(t,x) = \varphi(t,x)$ for $t_0 - h \le t \le t_0$. Such existence results are known under a variety of conditions. For example, if $t_0 = 0$ we can write (2.1) as

$$u(t,x) = f(t,x) + \int_{t-h}^{0} D(t,s)\varphi_{xx}(s,x)\,ds + \int_0^t D(t,s)u_{xx}(s,x)\,ds$$

and apply an existence theorem of Grimmer [6] (see Section 5, Theorems 5.1 and 5.2) directly if we assume that

$$\begin{gathered} D(t,t) > 0, \text{ both } D(t,t) \text{ and } D_t(t,s) \text{ are continuously differentiable in } t, \\ \text{while } D_t(t,s) \text{ is continuously differentiable in } s. \end{gathered} \tag{2.3a}$$

We obtain a solution for $0 \le t \le h$ which we continue by the method of steps. While it is convenient, (2.3a) asks too much; and there are several other ways to obtain solutions. We illustrate one way by separating variables when $f = 0$ in Section 2.b.

But we assume neither (23a) nor $f \equiv 0$ in general; rather, our stability considerations concern sets $\Omega(t_0)$ and $C(t_0)$ of initial functions for which there are solutions on $[t_0, \infty)$.

It is clear that the solution $u(t, x; t_0, \varphi)$ will have a discontinuity at t_0 unless

$$\varphi(t_0, x) = f(t_0, x) + \int_{t_0-h}^{t_0} D(t_0, s)\varphi_{xx}(s, x)\, ds. \tag{2.8}$$

This is parallel to a problem encountered by El'sgol'ts [5; pp. 35] for delay differential equations. He points out that if the initial function is not the natural one for the system, then a discontinuity in the derivative of the solution appears at t_0 but the solution is continuous and smooth for $t > t_0$. The discontinuity in the derivative causes few difficulties unless, for example, there is a need to integrate by parts (as El'sgol'ts does [5; p. 31]) or unless one is constructing a smooth set for the application of a fixed point theorem.

We have problems differentiating V unless (2.8) holds. Thus, our results are of two types. First, when $f(t, x) = 0$ we can show that (2.8) will hold for a function φ^* near a given continuous φ; hence, we assume (2.8) holds and obtain results on stability very close to classical results for differential equations. Next, when $f(t, x)$ is not zero we use a new Liapunov function $W(t, u)$ for $t_0 \le t \le t_0 + h$ to bound the solution for such t; then we proceed with $V(t, u)$ for $t > t_0 + h$ which can now be differentiated. Our bounds are all in terms of Sobolev norms; the interesting property is that when we lose (2.8), then we require an increase in the Sobolev dimension for the stability.

2.b. Existence and (2.8). Our first results concern the case $f(t, x) \equiv 0$. This illustrates stability in several measures very close to stability theory for differential equations since (2.1) then has an equilibrium solution. In addition, it will require a weaker measure for stability than that required when f is not zero. When f is zero we can separate variables to obtain global solutions under conditions much weaker than (2.3a) and we can also show how (2.8) can be satisfied.

In (2.1), let $f(t,x) = 0$, $u(t,x) = X(x)T(t)$ and obtain

$$T(t) \Big/ \int_{t-h}^{t} D(t,s)T(s)\,ds = X''(x)/X(x) = -\lambda^2$$

where $' = d/dx$ and $-\lambda^2$ is the separation constant. This yields

$$X'' + \lambda^2 X = 0, \quad X(0) = X(1) = 0,$$

$$T(t) = -\lambda^2 \int_{t-h}^{t} D(t,s)T(s)\,ds.$$

We take $\lambda^2 = n^2\pi^2$ so $X_n(x) = b_n \sin n\pi x$ and to satisfy (2.8) we need (taking $t_0 = 0$)

$$T_n(0) = -n^2\pi^2 \int_{-h}^{0} D(0,s)T_n(s)\,ds.$$

To solve the equation in $T(t)$ we need an initial function $\psi_n : [-h, 0] \to R$ to obtain a solution $T_n(t) = T_n(t; 0, \psi_n)$ with $T_n(t) = \psi_n(t)$ for $-h \le t < 0$. We showed [4] that for any continuous ψ_n and any $\epsilon > 0$ there is a t_1 in $(-h, 0)$ and as close to 0 as we please, together with a continuous ψ_n^* with $\psi_n^*(t) = \psi_n(t)$ on $[-h, t_1]$ and differing by less than ϵ at 0, while

$$\psi_n^*(0) = -n^2\pi^2 \int_{-h}^{0} D(0,s)\psi_n^*(s)\,ds;$$

moreover, if $T_n^* = T_n^*(t; 0, \psi_n^*)$, then it is bounded and converges to zero as $t \to \infty$. While discussion of infinite sums take more space than we wish to use here, a finite sum

$$U^*(t,x) = \sum b_n T_n^*(t) \sin n\pi x$$

will satisfy (2.1) and (2.8).

2.c. Stability. For a given $t_0 \in R$, denote by

$$\Omega(t_0) \tag{2.9}$$

the set of continuous $\varphi : [t_0 - h, t_0] \times [0,1] \to R$ for which (2.8) holds, $\varphi(t,0) = \varphi(t,1) = \varphi_{xx}(t,0) = \varphi_{xx}(t,1) = 0$, and there is a solution $u(t,x)$ of (2.1) satisfying (2.2), agreeing with φ on $[t_0 - h, t_0]$, existing on $[t_0, \infty)$, and having $u_{xx}(t,x)$ continuous. Denote by

$$C(t_0) \tag{2.9*}$$

those φ for which all of these conditions hold, except possibly (2.8).

For $\varphi \in \Omega(t_0)$ and for $u(t,x) = u(t,x;t_0,\varphi)$ define

$$V(t,u(\cdot)) = \int_{t-h}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds \tag{2.10}$$

which gives rise to the following norms for functions $u(t,x)$ satisfying $u(t,0) = u(t,1) = 0$.

It is known that

$$\int_0^1 u^2(t,x)\, dx \leq \int_0^1 u_x^2(t,x)\, dx \leq \int_0^1 u_{xx}^2(t,x)\, dx$$

for functions with those boundary conditions so that the following norms generate the usual Sobolev topologies for u, u_x, and u_{xx}; but u_{xxx} does not satisfy such inequalities:

$$|u|^2_{H^0(t)} = \int_0^1 u^2(t,x)\, dx,$$

$$|u|^2_{H^1(t)} = \int_0^1 u_x^2(t,x)\, dx,$$

$$|u|^2_{H^2(t)} = \int_0^1 u_{xx}^2(t,x)\, dx,$$

$$|u|^2_{H^3(t)} = |u|^2_{H^2} + \int_0^1 u_{xxx}^2(t,x)\, dx,$$

$$|u|^2_{B^0(t)} = \int_{t-h}^{t} \int_0^1 u^2(s,x)\, dx\, ds,$$

$$|u|^2_{B^1(t)} = \int_{t-h}^{t} \int_0^1 u_x^2(s,x)\, dx\, ds,$$

$$|u|^2_{B^2(t)} = \int_{t-h}^{t} \int_0^1 u_{xx}^2(s,x)\, dx\, ds,$$

$$|u|^2_{B^3(t)} = \int_{t-h}^{t} \int_0^1 [u_{xx}^2(s,x) + u_{xxx}^2(s,x)]\, dx\, ds.$$

In the terminology of stability in several measures, H^2 is finer than H^1 which is finer than H^0. One of the interesting properties here is that, while B^2 is generally not comparable to H^0, along solutions of (2.1) we have B^2 finer than H^0.

2.d. The case f = 0. Generally, our work focuses on $f(t,x)$ not identically zero; indeed, if we consider

$$u(t,x) = f(t,x) + \int_0^t D(t,s) u_{xx}(s,x)\, ds,$$

a nonzero solution on $[0, \infty)$ will generally exist only if $f(t,x)$ is not identically zero. But for (2.1) the case $f(t,x)$ identically zero is not only nontrivial, but it provides an introduction to stability which is close to classical theory for differential equations.

CONVENTION. If $\varphi \in \Omega(t_0)$, then $|\varphi|_{B^2}$ means $|\varphi|_{B^2(t_0)}$.

Def. 0. Let $f(t,x) = 0$ and $u = u(t,x;t_0,\varphi)$.

(a) The solution $u = 0$ of (2.1) is said to be (B^2, H^0)-uniformly stable if $[\forall \varepsilon > 0, \forall t_0 \in R]\, \exists \delta > 0$ such that $[t \geq t_0, \varphi \in \Omega(t_0), |\varphi|_{B^2} < \delta] \Rightarrow |u|_{H^0(t)} < \varepsilon$. If δ depends on t_0, then $u = 0$ is (B^2, H^0)-stable.

(b) The solution $u = 0$ of (2.1) is said to be $((B^2, H^0), H^0)$-uniformly asymptotically stable if it is $(B^2, H^0) - US$ and if there is a $\gamma > 0$ and $\forall \mu > 0\, \exists T > 0$ such that $[t_0 \in R, \varphi \in \Omega(t_0), |\varphi|_{B^2} < \gamma,\, t \geq t_0 + T] \Rightarrow |u|_{H^0(t)} < \mu$.

(c) The solution $u = 0$ of (2.1) is said to be $((B^2, H^0), B^1)$-asymptotically stable if it is (B^2, H^0)-stable and if $\forall t_0 \in R\; \exists \gamma > 0$ such that $[\varphi \in \Omega(t_0), |\varphi|_{B^2} < \gamma] \Rightarrow |u|_{B^1(t)} \to 0$ as $t \to \infty$.

Note in (2.9) that $\varphi \in \Omega(t_0)$ implies $u(t,x;t_0,\varphi)$ is continuable.

Theorem O. *Let (2.3) hold, except possibly $D_t(t,s) \leq 0$, and define $\Omega(t_0)$ in (2.9). Suppose that $f(t,x) = 0$. Then the zero solution of (2.1) is $(B^2, H^0) - US$ and $((B^2, H^0), B^1) - AS$. If there is a $K > 0$ with $D_{st}(t,s) \leq -K D_s(t,s)$, then $u = 0$ is $((B^2, H^0), H^0) - UAS$.*

Proof. In the proof of Theorem 1 we will show that for V defined in (2.10) and $u = u(t,x;t_0,\varphi)$ with $f(t,x) = 0$, then there are positive constants P and Q with

$$(i) \qquad |u|^2_{H^0(t)} \leq PV(t,u(\cdot)) \leq 2Q|u|^2_{B^2(t)}$$

and

$$(ii) \qquad V'(t,u(\cdot)) \leq -2|u|^2_{H^1(t)} + \int_{t-h}^{t} D_{st}(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds$$

and $D_{st} \leq 0$ yields

$$(ii^*) \qquad V'(t,u(\cdot)) \leq -2|u|^2_{H^1(t)}.$$

The calculations for $f \neq 0$ contain those for $f = 0$ and so are delayed until later.

For US, let $\varepsilon > 0$ be given and choose $\delta^2 = \varepsilon^2/Q$ so that if $t_0 \in R$, if $\varphi \in \Omega(t_0)$, and if $|\varphi|^2_{B^2} < \delta^2$, then (i) and (ii*) yield

$$|u|^2_{H^0(t)} \leq PV(t, u(\cdot)) \leq PV(t_0, \varphi) \leq Q|\varphi|^2_{B^2} < Q\delta^2 = \varepsilon^2,$$

as required.

For $((B^2, H^0), B^1) - AS$, if $t_0 \in R$, $\varphi \in \Omega(t_0)$, then an integration of (ii*) yields

$$0 \leq V(t, u(\cdot)) \leq V(t_0, \varphi) - 2\int_{t_0}^{t} |u|^2_{H^1(s)}\, ds$$

so that $\int_{t-h}^{t} |u|^2_{H^1(s)}\, ds \to 0$ as $t \to \infty$.

If $D_{st} \leq -KD_s$, then $V' \leq -KV$ so that

$$\begin{aligned} |u|^2_{H^0(t)} &\leq PV(t, u(\cdot)) \leq PV(t_0, \varphi)e^{-K(t-t_0)} \\ &\leq Q|\varphi|^2_{B^2}e^{-K(t-t_0)}, \end{aligned}$$

proving the UAS.

Example O. Let $D(t, s) = d(t - s)$ so that for $0 \leq t \leq h$ we ask that $d(t) \geq 0$, $d'(t) \leq 0$, $d''(t) \geq 0$, $d(h) = 0$ and (2.3) will be satisfied.

(a) If $n > 1$ and $d(t) = (h - t)^n$, then for $0 \leq t \leq h$ we have $d'(t) = -n(h - t)^{n-1} \leq 0$, $d''(t) = n(n - 1)(h - t)^{n-2} \geq 0$, and $d''(t)/d'(t) = -(n - 1)/(h - t) \leq -(n - 1)/h =: -K$ and we will have $V' \leq -KV$ so (2.1) is $((B^2, H^0), H^0) - UAS$.

(b) Let $d(t) = 1 + \cos(t + \frac{\pi}{2})$, $h = \pi/2$, so that $d'(t) = -\sin(t + \frac{\pi}{2})$ and $d''(t) = -\cos(t + \frac{\pi}{2})$. Then $d''(t)/d'(t) = \cot(t + \frac{\pi}{2})$ and we can not satisfy $V' \leq -KV$ for $K > 0$. Thus, we conclude only $(B^2, H^0) - US$ and $((B^2, H^0), B^1) - AS$.

2.e. The case $f \neq 0$. If f is not identically zero then the idea is that the solution of (2.1) may converge to f in some way. Recall that $C(t_0)$ is defined with (2.9*).

CONVENTION. If $\varphi \in C(t_0)$, then $|\varphi - f|_{B^3} = |\varphi - f|_{B^3(t_0)}$.

DEF. 1.

(a) Solutions of (2.1) are said to be (B^3, H^0)-uniformly bounded relative to f if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[t_0 \in R,\ \varphi \in C(t_0),\ |\varphi - f|_{B^3} < B_1,\ t \geq t_0]$ imply that $|u - f|_{H^0(t)} < B_2$.

(b) The function f is said to be eventually (B^3, H^0)-uniformly stable if for each $\varepsilon > 0$ there is a $\delta > 0$ and $T > 0$ such that $[t_0 \geq T, \varphi \in C(t_0),\ |\varphi - f|_{B^3} < \delta, t \geq t_0]$ imply that $|u - f|_{H^0(t)} < \varepsilon$.

(c) The function f is said to be B^1-globally attractive if $[t_0 \in R, \varphi \in C(t_0)]$ imply that $|u - f|_{B^1(t)} \to 0$ as $t \to \infty$.

(d) The function f is said to be H^0-globally attractive if $[t_0 \in R, \varphi \in C(t_0)]$ imply that $|u - f|_{H^0(t)} \to 0$ as $t \to \infty$.

We will formulate theorems yielding such properties and there is one more of interest here. In the theory of differential equations a Liapunov function frequently does not have quite enough properties to yield a given result, but simple examination of the equation supplies the missing information. The classical example is the Barbashin-Marachkov-Krasovskii-LaSalle-Yoshizawa idea ([1], [12], [20], [23], [25]) that if the equation $x' = p(t, x)$ is bounded for x bounded then every bounded solution approaches a set where the derivative of the Liapunov function is zero. We offer an interesting counterpart here. We show that if $|u|_{H^1(t)}$ is bounded, then $|u - f|_{H^0(t)} \to 0$ as $t \to \infty$.

If u is a solution of (2.1) with initial function φ satisfying (2.8), then (2.10) can be differentiated for $t \geq t_0$ and stability properties will result. But if (2.8) fails then we will need a growth estimate on V for $t_0 \leq t \leq t_0 + h$. Such an estimate is obtained by writing (2.1) as

$$\text{(2.1}a\text{)} \qquad \begin{aligned} u(t,x) &= f(t,x) + \int_{t-h}^{t_0} D(t,s)\varphi_{xx}(s,x)\,ds + \int_{t_0}^{t} D(t,s)u_{xx}(s,x)\,ds \\ &=: G(t,x,\varphi) + \int_{t_0}^{t} D(t,s)u_{xx}(s,x)\,ds \end{aligned}$$

and employing the Liapunov functional

$$
\begin{aligned}
W(t,u) &= \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\,dv \right)^2 dx\,ds \\
&\quad + \int_0^1 D(t,t_0) \left(\int_{t_0}^t u_{xx}(v,x)\,dv \right)^2 dx. \qquad (2.11)
\end{aligned}
$$

Theorem 1. *Let u be continuous with u(t,0) = u(t,1) = 0 on* $[t_1 - h, \infty)$*, satisfy (2.1) on* $[t_1, \infty)$*, and let (2.2) - (2.6) hold. Then for V defined in (2.10) and* $t \geq t_1$ *we have*

(i) $$|u - f|^2_{H^0(t)} \leq PV(t,u) \leq 2Q|u - f|^2_{B^2(t)} + 2F(t) \quad \text{(see (2.6))},$$

(ii) $$V'(t,u) \leq -|u - f|^2_{H^1(t)} - |u|^2_{H^1(t)} + |f|^2_{H^1(t)},$$

and

(iii) $$|u - f|^2_{H^0(t)} \leq P|u - f|_{H^1(t)} h^{1/2} |u|_{B^1(t)}.$$

If, in addition, (2.7) holds, then

(iv) $$V'(t,u) \leq -k(t)V + |f|^2_{H^1(t)} \text{ and } |u - f|_{H^0(t)} \to 0 \text{ as } t \to \infty.$$

Theorem 2. *Let* $t_0 \in R$*,* $\varphi \in C(t_0)$*,* $u = u(t,x;t_0,\varphi)$ *satisfy (2.1) for* $t_0 \leq t \leq t_0 + h$*. Suppose that (2.2) – (2.6) hold and that* $V(t,u)$ *and* $W(t,u)$ *are defined by (2.10) and (2.11). Let* $\Delta = \max D(t,s)$ *for* $t_0 \leq t \leq t_0 + h$ *and* $t_0 - h \leq s \leq t_0$*. Then for* $t \in [t_0, t_0 + h]$ *we have*

(i) $$V(t,u) \leq 2V(t_0,\varphi) + 2W(t,u)$$

(ii) $$W'(t,u) \leq -|u - G|^2_{H^1(t)} + +|G|^2_{H^1(t)} \text{ for } G \text{ defined in (2.1a)},$$

(iii) $$W(t,u) \leq 4\Delta^2 h^2 |\varphi - f|_{B^3} + 2|f|^2_{B^1(t_0+h)} + 4\Delta^2 h^2 |f|_{B^3(t_0+h)}$$

and

(iv) $$|u - f|^2_{H^0} \leq 4\Delta^2 |\varphi - f|_{B^2} + 4\Delta^2 |f|_{B^2(t_0+h)} + 8\Delta W(t,u)$$

COR. If (2.2) – (2.6) hold then f is (B^3, H^0)-uniformly bounded, eventually (B^3, H^0)-uniformly stable, and it is globally B^1-attractive. If, in addition, (2.7) holds, then f is H^0-globally attractive. If there is a continuous φ with $|u-f|_{H^1(t)}$ bounded on some $[t_2, \infty)$, then $|u-f|_{H^0(t)} \to 0$ as $t \to \infty$.

Proof of Theorem 1. We first compute the derivative of V along a solution $u(t,x)$ of (2.1) which is continuous on $[t-h, t]$. We have

$$\begin{aligned} V' = &-\int_0^1 D_s(t, t-h)\Big(\int_{t-h}^t u_{xx}(v,x)dv\Big)^2 dx \\ &+ \int_0^1 \int_{t-h}^t D_{st}(t,s)\Big(\int_s^t u_{xx}(v,x)dv\Big)^2 ds\, dx \\ &+ 2\int_0^1 u_{xx}(t,x) \int_{t-h}^t D_s(t,s) \int_s^t u_{xx}(v,x)\, dv\, ds\, dx \end{aligned}$$

and the last term can be integrated by parts to obtain

$$\begin{aligned} &2\int_0^1 u_{xx}(t,x)\Big[D(t,s)\int_s^t u_{xx}(v,x)\,dv\Big|_{t-h}^t + \int_{t-h}^t D(t,s)u_{xx}(s,x)\,ds\Big]\,dx \\ &= 2\int_0^1 u_{xx}(t,x)\int_{t-h}^t D(t,s)u_{xx}(s,x)\,ds\,dx \ \text{(since } D(t,t-h)=0) \\ &= 2\int_0^1 u_{xx}(t,x)[u(t,x)-f(t,x)]\,dx \ \text{(from (2.1))} \\ &= 2u_x(t,x)\Big[u(t,x)-f(t,x)\Big]_{x=0}^{x=1} - 2\int_0^1 u_x(t,x)[u_x(t,x)-f_x(t,x)]\,dx \end{aligned}$$

or

$$\begin{aligned} V' \le &\int_0^1 \int_{t-h}^t D_{st}(t,s)\Big(\int_s^t u_{xx}(v,x)\,dv\Big)^2 ds\,dx \\ &- \int_0^1 [u_x(t,x)-f_x(t,x)]^2\,dx \\ &- \int_0^1 u_x^2(t,x)\,dx + \int_0^1 f_x^2(t,x)\,dx \end{aligned} \tag{2.12}$$

from which we obtain (ii).

Next, we get a lower bound on V. From (2.1) we have

$$\int_0^1 (u(t,x)-f(t,x))^2\,dx = \int_0^1 \left(\int_{t-h}^t D(t,s)u_{xx}(s,x)\,ds\right)^2 dx$$
$$= \int_0^1 \left(-D(t,s)\int_s^t u_{xx}(v,x)\,dv\Big|_{t-h}^t + \int_{t-h}^t D_s(t,s)\int_s^t u_{xx}(v,x)\,dv\,ds\right)^2 dx$$
$$= \int_0^1 \left(\int_{t-h}^t \sqrt{D_s(t,s)}\sqrt{D_s(t,s)}\int_s^t u_{xx}(v,x)\,dv\,ds\right)^2 dx$$
$$\le \int_0^1 \int_{t-h}^t D_s(t,s)\,ds \int_{t-h}^t D_s(t,s)\left(\int_s^t u_{xx}(v,x)\,dv\right)^2 ds\,dx$$
$$= D(t,t)V(t,u(\cdot))$$

or

$$\int_0^1 (u(t,x)-f(t,x))^2\,dx \le PV(t,u(\cdot)).$$

Finally, we get an upper bound on V. We have

$$V(t,u(\cdot)) \le \int_0^1 \int_{t-h}^t 2D_s(t,s)\left(\int_s^t [u_{xx}(v,x)-f_{xx}(v,x)]\,dv\right)^2 ds\,dx$$
$$+ 2\int_0^1 \int_{t-h}^t D_s(t,s)\left(\int_s^t f_{xx}(v,x)\,dv\right)^2 ds\,dx$$
$$\le 2\int_0^1 \int_{t-h}^t D_s(t,s)(t-s)\int_s^t [u_{xx}(v,x)-f_{xx}(v,x)]^2\,dv\,ds\,dx$$
$$+ 2\int_0^1 \int_{t-h}^t D_s(t,s)\left(\int_s^t f_{xx}(v,x)\,dv\right)^2 ds\,dx.$$

In view of (2.4) and (2.6) we write

$$\int_0^1 [u(t,x)-f(t,x)]^2\,dx \le PV(t,u) \tag{2.13}$$
$$\le 2Q\int_0^1 \int_{t-h}^t [u_{xx}(v,x)-f_{xx}(v,x)]^2\,dv\,dx + 2F(t)$$

from which (i) follows.

To get (iii) we multiply by $u-f$, integrate from 0 to 1, and obtain

$$\int_0^1 [u(t,x)-f(t,x)]^2\,dx = \int_0^1 \int_{t-h}^t D(t,s)u_{xx}(s,x)[u(t,x)-f(t,x)]\,ds\,dx$$
$$\le \left[\int_0^1 (u_x(t,x)-f_x(t,x))^2\,dx\right]^{1/2} \int_{t-h}^t D(t,s)\left(\int_0^1 u_x^2(s,x)\,dx\right)^{1/2} ds$$
$$\le |u-f|_{H^1(t)}Ph^{1/2}|u|_{B^1(t)}$$

proving (iii).

If (2.7) holds, then the inequality in (iv) is true; since (ii) yields $|f|^2_{H^1(t)} \in L^1[t_0+h, \infty)$, the conclusion of (iv) holds and Theorem 1 is true.

Proof of Theorem 2. If $t_0 \in R$, $\varphi \in C(t_0)$, $u = u(t, x; t_0, \varphi)$, and W is defined by (2.11) and V by (2.10), then for $t_0 \le t \le t_0 + h$ we have

$$V(t, u(\cdot)) = \int_{t-h}^{t_0} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds$$
$$+ \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds.$$

Since $D_s(t,s) \ge 0$ and $D_{st}(t,s) \le 0$ it follows that

$$\begin{aligned}
V(t,u) &= \int_{t-h}^{t_0} D_s(t,s) \int_0^1 \left(\int_s^{t_0} \varphi_{xx}(v,x)\, dv + \int_{t_0}^{t} u_{xx}(v,x)\, dv \right)^2 dxds \\
&+ \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds \\
&\le 2 \int_{t-h}^{t_0} D_s(t,s) \int_0^1 \left[\left(\int_s^{t_0} \varphi_{xx}(v,x)\, dv \right)^2 + \left(\int_{t_0}^{t} u_{xx}(v,x)\, dv \right)^2 \right] dx\, ds \\
&+ \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{zz}(v,x)\, dv \right)^2 dx\, ds \\
&\le 2 \int_{t_0-h}^{t_0} D_s(t_0,s) \int_0^1 \left(\int_s^{t_0} \varphi_{xx}(v,x)\, dv \right)^2 dx\, ds \\
&\quad \text{(since } D_s(t,s) \ge 0) \\
&+ 2 \int_{t-h}^{t_0} D_s(t,s) \int_0^1 \left(\int_{t_0}^{t} u_{xx}(v,x)\, dv \right)^2 dx\, ds \\
&+ \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds \\
&\le 2V(t_0,\varphi) + 2[D(t,t_0) - D(t,t-h)] \int_0^1 \left(\int_{t_0}^{t} u_{xx}(v,x)\, dv \right)^2 dx \\
&+ \int_{t_0}^{t} D_s(t,s) \int_0^1 \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 dx\, ds \\
&\le 2V(t_0,\varphi) + 2W(t,u)
\end{aligned}$$

proving (i).

Next,

$$W'(t,u) \le \int_0^1 2u_{xx}(t,x) \int_{t_0}^t D_s(t,s) \int_s^t u_{xx}(v,x)\,dv\,ds\,dx$$
$$+ \int_0^1 D_t(t,t_0)\Big(\int_{t_0}^t u_{xx}(v,x)dv\Big)^2\,dx$$
$$+ 2\int_0^1 D(t,t_0)u_{xx}(t,x)\int_{t_0}^t u_{xx}(v,x)\,dv\,dx.$$

Integrating the first term on the right by parts yields

$$2\int_0^1 u_{xx}(t,x)[D(t,s)\int_s^t u_{xx}(v,x)\,dv\Big|_{s=t_0}^{s=t}$$
$$+ \int_{t_0}^t D(t,s)u_{xx}(s,x)\,ds]\,dx$$
$$= 2\int_0^1 u_{xx}(t,x)[-D(t,t_0)\int_{t_0}^t u_{xx}(v,x)\,dv + u(t,x) - G(t,x,\varphi)]\,dx$$

from (2.1a) which defined G. Hence, (consult the material with (2.9))

$$W'(t,u) \le 2\int_0^1 u_{xx}(t,x)[u(t,x) - G(t,x,\varphi)]\,dx$$
$$= -2\int_0^1 u_x(t,x)[u_x(t,x) - G_x(t,x,\varphi)]\,dx$$
$$\le -\int_0^1 [u_x(t,x) - G_x(t,x,\varphi)]^2\,dx + \int_0^1 G_x^2(t,x,\varphi)\,dx$$

proving (ii).

Now $t_0 \le t \le t_0 + h$ implies that (integrating (ii))

$$W(t,u) \le \int_{t_0}^{t_0+h}\int_0^1 G_x^2(t,x,\varphi)\,dx\,dt$$
$$= \int_{t_0}^{t_0+h}\int_0^1 [f_x(t,x) + \int_{t-h}^{t_0} D(t,s)\varphi_{xxx}(s,x)\,ds]^2\,dx\,dt$$
$$\le 2\int_{t_0}^{t_0+h}\int_0^1 f_x^2(t,x)\,dx\,dt + 2\int_0^1\int_{t_0}^{t_0+h}\int_{t-h}^{t_0} D^2(t,s)\,ds\int_{t-h}^{t} \varphi_{xxx}^2(s,x)\,dt\,dx$$
$$\le 2|f|^2_{B^1(t_0+h)} + 2\Delta^2 h^2|\varphi|^2_{B^3}$$

from which (iii) will follow when we write

$$|\varphi|^2_{B^3} \le 2\Big(|\varphi - f|^2_{B^3} + |f|^2_{B^3}\Big).$$

To obtain (iv) from (2.1) we have

$$
\begin{aligned}
&\int_0^1 (u(t,x) - f(t,x))^2\, dx \\
&\quad = \int_0^1 \left(\int_{t-h}^{t_0} D(t,s)\varphi_{xx}(s,x)\, ds + \int_{t_0}^t D(t,s) u_{xx}(s,x)\, ds \right)^2 dx \\
&\quad \le 2\int_0^1 \left(\int_{t-h}^{t_0} D(t,s)\varphi_{xx}(s,x)\, ds \right)^2 dx + 2\int_0^1 \left(\int_{t_0}^t D(t,s) u_{xx}(s,x)\, ds \right)^2 dx \\
&\quad \le 2\Delta^2 |\varphi|^2_{B^2} \\
&\quad + 2\int_0^1 [-D(t,s)\int_s^t u_{xx}(v,x)\, dv \Big|_{s=t_0}^{s=t} + \int_{t_0}^t D_s(t,s) \int_s^t u_{xx}(v,x)\, dv\, ds]^2\, dx \\
&\quad \le 2\Delta^2 |\varphi|^2_{B^2} + 4\int_0^1 \left(D(t,t_0) \int_{t_0}^t u_{xx}(v,x)\, dv \right)^2 dx \\
&\quad + 4\int_0^1 \int_{t_0}^t D_s(t,s)\, ds \int_{t_0}^t D_s(t,s) \left(\int_s^t u_{xx}(v,x)\, dv \right)^2 ds\, dx \\
&\quad \le 2\Delta^2 |\varphi|^2_{B^2} + 8\Delta W(t,u)
\end{aligned}
$$

which will yield (iv) and prove Theorem 2.

Proof of Cor. Let (2.2) – (2.6) hold, $B_1 > 0$ be given. We must find $B_2 > 0$ such that $[t_0 \in R,\ \varphi \in C(t_0),\ |\varphi - f|_{B^3} < B_1,\ t \ge t_0]$ imply that $|u - f|_{H^0(t)} < B_2$. From (iv) and (iii) of Theorem 2, if $t_0 \le t \le t_0 + h$ then

$$
\begin{aligned}
|u - f|^2_{H^0(t)} &\le 4\Delta^2 B_1 + 4\Delta^2 |f|_{B^2(t_0+h)} \\
&+ 8\Delta \left(4\Delta^2 h^2 B_1 + 2|f|^2_{B^1(t_0+h)} + 4\Delta^2 h^2 |f|_{B^3(t_0+h)} \right) \\
&=: \overline{B_2}
\end{aligned}
$$

Next, from (i) and (iii) of Theorem 2 we have

$$
\begin{aligned}
&V(t_0 + h, u) \le 2V(t_0, \varphi) + 2W(t_0 + h, u) \\
&\le \frac{2}{P}\left[2Q|\varphi - f|^2_{B^2} + 2F(t_0) \right] + 2\Big[4\Delta^2 h^2 B_1 + 2|f|^2_{B^1(t_0+h)} \\
&\qquad + 4\Delta^2 h^2 |f|_{B^3(t_0+h)} \Big] =: \widetilde{B_2}.
\end{aligned}
$$

Then from (ii) and (i) of Theorem 1 for $t \geq t_0 + h$ we have

$$|u - f|^2_{H^0} \leq PV(t,u) \leq P\left[V(t_0 + h) + \int_{t_0+h}^{t} |f|^2_{H^1(s)}\right] ds$$
$$\leq P\left[\widetilde{B_2} + \int_{t_0+h}^{\infty} |f|^2_{H^1(s)}\right] ds =: B_2^*.$$

Hence, $B_2 = \max[\overline{B_2}, \widetilde{B_2}, B_2^*]$ satisfies the uniform boundedness requirement.

Since $F(t) \to 0$ as $t \to \infty$ and (2.5) holds, a completely parallel proof yields the eventual stability.

For any solution we see from Theorem 1 (ii) that $\int_{t_0+h}^{t} |u - f|^2_{H^1(s)}\, ds < \infty$ and so $|u - f|^2_{B^1(t)} \to 0$ as $t \to \infty$.

The H^0-attractivity is clear.

If $|u - f|_{H^1}$ is bounded, then from Theorem 1(ii) we have $|u|^2_{B^1(t)} \to 0$ as $t \to \infty$ and so by Theorem 1(iii) we see that $|u - f|^2_{H^0(t)} \to 0$ as $t \to \infty$. This proves the corollary.

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