The Transmission Loss in Squeeze-Film Flow of Newtonian and Some Viscoelastic Fluids

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It is shown that the difference between the forces acting on the top and bottom plates of a squeeze-film bearing, or the transmission loss, is entirely an inertia phenomenon and arises due to the acceleration of the top plate (the bottom plate is stationary). Its magnitude is given by half of the mass of the fluid film multiplied by the top plate acceleration. This is valid for Newtonian and some Oldroyd-B fluid provided that edge effects are negligible and no bifurcation of solution occurs.

1 Introduction

We consider the flow of a viscoelastic fluid being squeezed between two parallel plates; the top plate is moving with a prescribed velocity $\dot{h}(t)$ and the bottom plate is stationary. The flow is of considerable interest in that it models the action of a lubricant in a bearing under unsteady load conditions. It is also relevant to the interpretation of various plastometers and to some polymer processing operations such as compression moulding.

Early works in this field were concerned with Newtonian fluids in creeping flow using the lubrication approximation which dated back to Stefan [1] and Reynolds [2]. In later works inertia was included either as a perturbation correction to the basic Newtonian creeping flow [3–5] or by solving the Navier-Stokes equations numerically [6, 7]. Extensions to non-Newtonian fluids have been considered by many authors, including Tanner [8], Kramer [9], Leider and Bird [10], Brindley et al. [11], Tichy and Winer [12], and Phan-Thien et al. [13–15]. In particular, the latter authors showed that there are exact solutions to the squeeze-film flow of Oldroyd-B fluids. These exact solutions show that the load required to squeeze an Oldroyd-B fluid with a prescribed velocity is less than that required to squeeze a Newtonian fluid of the same velocity. Experimental studies on squeeze-film flow of viscoelastic fluids are somewhat contradictory but for some dilute suspensions of polymer in highly-viscous solvents, the experimental data appear to support the theoretical findings; for more details the reader is referred to the review [15]. Here, we are concerned with a different aspect of the problem. We wish to show that the difference between the forces acting on the top plate and on the bottom plate arises due to the fluid inertia and is given by $-1/2(\rho\pi R^2 h(t))\ddot{h}(t)$, where R is the radius of the bearing, ρ is the density of the fluid, and $h(t)$ is the film thickness. This simple formula holds for an Oldroyd-type fluid which includes the Newtonian fluid, and the Oldroyd-B fluid provided that edge effects can be neglected and no bifurcation of solution occurs. The Oldroyd-B fluid has been known to be a good model for a dilute suspension of polymer in a highly viscous solvent (the so-called Boger fluid [15]) and, therefore, the findings reported here can be subjected to experimental verification.

2 Analysis

We consider the flow generated by squeezing a viscoelastic fluid between two circular plates of infinite radius. In reality, what is required is that the radius R of the plates should be considerably greater than the film thickness $h(t)$ and we implicitly assume that edge effects are negligible. This assumption can be justified for Newtonian fluids where edge effects influence about one film thickness upstream from the exit. For viscoelastic fluids there is no theoretical estimate of the edge effects although our numerical works [16] indicate that these effects only influence about one film thickness upstream from the exit.

2.1 Constitutive Relations. We are concerned with an Oldroyd-type fluid in which the total stress is given by

$$\boldsymbol{\sigma} = -P\mathbf{1} + 2\eta_s\mathbf{D} + \mathbf{S}, \quad \eta_s \neq 0 \tag{1}$$

where P is the hydrostatic pressure which arise from the incompressibility constraint, $2\eta_s\mathbf{D}$ is the Newtonian contribution from the solvent, η_s is the solvent viscosity, $\mathbf{L} = (\nabla\mathbf{u})^T$ is the velocity gradient tensor, $\mathbf{D} = 1/2((\mathbf{L} + \mathbf{L}^T))$ is the strain rate tensor and $\mathbf{S}$ obeys

$$\mathbf{S} + \lambda\{\partial_t\mathbf{S} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T\} = 2\eta_p\mathbf{D}. \tag{2}$$

In (2) λ is the relaxation time, η_p is the polymer viscosity. When (1) and (2) are combined, one has the familiar Oldroyd-B model [17]. In a simple shearing flow with shear rate γ, the model predicts a constant viscosity $\eta = \eta_p + \eta_s$, a first normal-stress difference $N_1 = 2\eta_p\lambda\dot{\gamma}^2$ and a zero second normal-stress difference.

The analysis reported below is also applicable for the following constitutive relation

$$\mathbf{S} + \lambda\{\partial_t\mathbf{S} + \mathbf{u}\cdot\nabla\mathbf{S} - \mathbf{LS} - \mathbf{SL}^T\} = 2\eta_p\left\{\mathbf{D} - \mu\lambda\mathbf{D}^2 + \frac{1}{2}\mu(tr\mathbf{D}^2)\mathbf{1}\right\} \tag{3}$$

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which includes the Oldroyd-B model as a special case. In a simple shear flow this latter model predicts the same viscosity and first normal-stress difference as the Oldroyd-B model. In addition, the second normal-stress difference is predicted to be $N_2 = -1/4(\mu N_1)$. From experimental evidence, μ is positive and of order unity. Although this latter model provides more flexibility from an experimental point of view, we find it more convenient to present the analysis for the Oldroyd-B case.

2.2 Preliminaries. We record here some general results that are valid for any fluids. One starts with the conservation laws:

$$\nabla \cdot \mathbf{u} = 0, \ \mathbf{x} \epsilon V \tag{4}$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla \cdot \sigma, \ \mathbf{x} \epsilon V, \tag{5}$$

where $\mathbf{u}$ is the velocity vector and $V = \{\mathbf{x} = (r, \theta, z)^T: 0 \le r < R,\ 0 < z < h(t),\ 0 \le \theta < 2\pi\}$ is the flow domain. Its bounding surface is $\partial V = \{\mathbf{x}: r = R,\ z = 0,\ z = h(t)\}$. This bounding surface consists of the bottom plate $\partial V_b = \{\mathbf{x}: z = 0\}$, the top plate $\partial V_t = \{\mathbf{x}: z = h(t)\}$ and the perimeter surface $\partial V_p = \{\mathbf{x}: r = R\}$. Note that the top plate is moving with a velocity $\dot{h}(t)$ while the bottom plate is stationary. A volume integral (over V) of the conservation of momentum is taken. With the help of (4) and the divergence theorem we have

$$\rho \int_V \partial_t \mathbf{u} dV + \rho \int_{\partial V} \mathbf{u}\mathbf{u} \cdot \mathbf{n} dS = \int_{\partial V} \sigma \cdot \mathbf{n} dS, \tag{6}$$

where $\mathbf{n}$ is the outward normal unit vector on ∂V. Along the z-direction, (6) takes the form

$$\int_{\partial V_t} \sigma_{zz} dS - \int_{\partial V_b} \sigma_{zz} dS + \int_{\partial V_p} \sigma_{rz} dS$$

$$= \rho \left\{ \int_V (\partial_t w) dV + \pi R^2 \dot{h}(t)^2 + \int_{\partial V_p} uw dS \right\}, \tag{7}$$

where $\{u, 0, w\}$ are the velocity components in polar coordinates. Since the forces exerted on the top and bottom plate are given, respectively, by

$$F_t = -\int_{\partial V_t} \sigma_{zz} dS, \ F_b = -\int_{\partial V_b} \sigma_{zz} dS,$$

equation (7) becomes

$$\Delta F - \int_{\partial V_p} \sigma_{rz} dS = -\rho \left\{ \int_V (\partial_t w) dV + \pi R^2 \dot{h}(t)^2 + \int_{\partial V_p} uw dS \right\}, \tag{8}$$

where $\Delta F = F_t - F_b$ is the difference between the two forces, sometimes referred to as transmission loss.

2.3 Newtonian Fluid. We normalize z by $h(t)$ and write

$$\xi = z/h(t), \tag{9}$$

then the exact kinematics for Newtonian fluids are given by [7, 13],

$$u = -\frac{1}{2} \frac{r}{h} \dot{h} f'(\xi, t), \quad w = \dot{h} f(\xi, t), \tag{10}$$

where $f(\xi, t)$ is a function of ξ and time t, the super dot denotes a time derivative and the prime denotes a derivative with respect to ξ. The boundary conditions for f are that

$$f(0, t) = f'(0, t) = 0; f(1, t) = 1, f'(1, t) = 0. \tag{11}$$

On the boundary ∂V_p where $r = R$ the shear stress is given by

$$\sigma_{rz} = -\frac{1}{2} \eta \frac{\dot{h}}{h^2} R f'',$$

where η is the fluid viscosity. Consequently, the integral of σ_{rz} over ∂V_p vanishes owing to the boundary conditions imposed on f'. In a similar manner the integral of uw over ∂V_p is evaluated to $-1/2(\pi R^2 \dot{h}^2)$. Thus, (8) reduces to

$$\Delta F = -\rho \pi R^2 h \left\{ \frac{1}{h} \int_0^h \dot{w} dz + \frac{1}{2} \frac{\dot{h}^2}{h} \right\}$$

$$= -\rho \pi R^2 h \left\{ \frac{d}{dt} \bar{w} + \frac{\dot{h}}{h} \bar{w} - \frac{1}{2} \frac{\dot{h}^2}{h} \right\}, \tag{12}$$

where $\bar{w} = 1/h \int_0^h w(z, t)\, dz$ is the average of w over the film thickness. In going from the first to the second equation of (12) one notes that the upper limit of the integral is itself a function of time.

Now, *from the conservation of linear momentum and the* kinematics (9), one has

$$f^{IV} - \frac{\rho h \dot{h}}{\eta} \left\{ \frac{h}{\dot{h}} \dot{f}'' + \left(\frac{h \ddot{h}}{\dot{h}^2} - 2 \right) f'' + (f - \xi) f''' \right\} = 0 \tag{13}$$

If we define g by $f = g + 1/2$, then the equation for g is antisymmetric about $\xi = 1/2$; its boundary conditions are also antisymmetric, i.e., $g(0) = -1/2$, $g(1) = 1/2$. Thus, provided that the initial condition for g is also antisymmetric about the midplane and that there is no bifurcation of solution, then this antisymmetry about the midplane is preserved through time. (That g is antisymmetric or f' is symmetric about the midplane at least up to a Reynolds number of 96 is amply demonstrated in the numerical work of Hamza and MacDonald [7].) Therefore, the average of f is simply 1/2 and one has, from (12),

$$\Delta F = -\frac{1}{2} \rho \pi R^2 h(t) \ddot{h}(t) \tag{14}$$

This simple formula indicates that the transmission loss is entirely an inertia phenomenon and arises due to the acceleration of the top plate.

2.4 Oldroyd-B Fluid. If we neglect edge effects, then the exact kinematics are again given by (10). In this case, the Maxwellian stress $\mathbf{S}$ is solved by

$$S_{rr} = -\eta \frac{\dot{h}}{h} \left(R_1(\xi, t) + \frac{r^2}{h^2} R_2(\xi, t) \right),$$

$$S_{rz} = -\eta \frac{\dot{h}}{h^2} r T(\xi, t),$$

$$S_{zz} = -\eta \frac{\dot{h}}{h} Z(\xi, t)$$

$$S_{\theta\theta} = -\eta \frac{\dot{h}}{h} R_1(\xi, t)$$

where $\eta = \eta_s + \eta_p$ and R_1, R_2, T, and Z (here referred to as "stress components") obey

$$R_1 + \lambda \frac{\dot{h}}{h} \left\{ \frac{h}{\dot{h}} \partial_t R_1 + \left(\frac{\ddot{h} h}{\dot{h}^2} - 1 \right) R_1 + (f - \xi) R_1' + f' R_1 \right\} = \beta f', \tag{15}$$

$$R_2 + \lambda \frac{\dot{h}}{h} \left\{ \frac{h}{\dot{h}} \partial_t R_2 + \left(\frac{\ddot{h} h}{\dot{h}^2} - 3 \right) R_2 + (f - \xi) R_2' + f'' T \right\} = 0, \tag{16}$$

$$T + \lambda \frac{\dot{h}}{h} \left\{ \frac{h}{\dot{h}} \partial_t T + \left(\frac{\ddot{h} h}{\dot{h}^2} - 2 \right) T + (f - \xi) T' - f' T + \frac{1}{2} f'' Z \right\} = \frac{1}{2} \beta f'', \tag{17}$$

$$Z + \lambda \frac{\dot h}{h}\left\{\frac{h}{\dot h}\,\partial_t Z + \left(\frac{\ddot h h}{\dot h^2} - 1\right) Z \right.$$

$$\left. + (f-\xi)Z' - 2f'Z\right\} = -2\beta f', \qquad (18)$$

in which $\beta = \eta_p/\eta$. We assume that $\beta \neq 1$. Conservation of linear momentum requires

$$3R_2' + T'' + \frac{1}{2}\,(1-\beta)f^{\text{iv}} - \frac{1}{2}\,\frac{\rho h \dot h}{\eta}\left\{\frac{h}{\dot h}\,\partial_t f'' \right.$$

$$\left. + \left(\frac{h\ddot h}{\dot h^2} - 2\right) f'' + (f-\xi) f'''\right\} = 0, \qquad (19)$$

which is subjected to the boundary conditions (11).

Again, with the assumption that g is antisymmetric about the midplane $\xi = 1/2$, where $f = g + 1/2$, one finds that Z and R_1 are symmetric about $\xi = 1/2$. It follows that T and R_2 are, respectively, antisymmetric and symmetric about $\xi = 1/2$. This symmetry property is consistent with equation (19) and has been demonstrated numerically in [14-15]. Consequently, the integral of σ_{rz} over the perimeter surface is zero and the z-average of f is 1/2. Thus, the force difference is still given by equation (14).

This analysis can obviously be repeated for the constitutive relation (3). In two-dimensional flows it can be shown that $\mathbf{D}^2 = 1/2\ (tr\mathbf{D}^2)\mathbf{1}$ and (3) is identical to (2). In axisymmetric flows, the extra terms in (3) give rise to some additional terms (see the related paper [18]); however, these additional terms do not affect the symmetry property of the governing equations and relation (14) remains valid for this particular fluid.

3 Concluding Remarks

There are at least two circumstances under which the above analysis is not valid:

(i) The solution may bifurcate at some finite Reynolds number. Across these bifurcation points or limit points, the stability of the solution and the flow pattern often change and it is not expected that the antisymmetry of g be preserved. In such cases, the average of f may not be equal to 1/2. However, no bifurcation of the squeeze-film flow has been discovered, either for a Newtonian or the Oldroyd-B fluid.

(ii) When $\eta_s = 0$ one has a Maxwell fluid and the governing equations allow for the development and propagation of vortex sheets (velocity discontinuities). Again, in this case, the antisymmetry of g cannot be expected and indeed has not been observed numerically [13]. Most dilute polymer solutions, which are most commonly-used lubricants, have $\eta_s \neq 0$ (their β values are of the order $0.1-0.2$). Our estimate for the transmission loss should be applicable to these materials.

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