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Unique Continuation Property and Decay for the Korteweg-de-Vries-Burgers Equation With Localized Damping

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Abstract: In this paper, we prove the unique continuation property (UPC) and decay about the Korteweg-de-Burgers (KdVB) equation

$$u_t - u_{xx} + u_{xxx} + uu_x + a(x)u = 0$$

in a bounded interval with a localized damping term. We will show that the UPC holds with the condition of $u_x(0,t) = 0$ and $u \equiv 0$ in $\omega \times (0,T)$, where ω is a nonempty open subset of $(0,L)$, if a localized damping acting on a moving internal is applied in KdVB equation. And we prove the exponential decay of the KdVB equation with some boundary and initial condition.

Keywords: unique continuation property ;Korteweg-de-Vries-Burgers equation; exponential decay

1 Introduction

In this article, we are concerned with the Korteweg-de-Vries-Burgers Equation (KdVB) in a bounded interval with a localized damping term $a(x)u$,

$$\begin{cases} u_t - u_{xx} + u_{xxx} + uu_x + a(x)u = 0, \text{ in } (0,L)\times(0,T) \\ u(0,t) = u(L,t) = 0, t \in (0,T) \\ u_x(L,t) = 0, t \in (0,T) \\ u(0) = u_0, \text{on (0,L)} \end{cases} \tag{1}$$

All along the paper we assume that the function $a(x)$ satisfies

$$a \in L^\infty(0,L), a(x) \geq a_0 > 0 \quad a.e.in \ \omega, \tag{2}$$

where ω is a nonempty open subset of $(0,L)$. The original KdVB equation is

$$u_t - \gamma u_{xx} + u_{xxx} + uu_x = 0,$$

which is reduced from the generalized Korteweg-de-Vries-Burgers(GKdVB) equation

$$u_t - \gamma u_{xx} + \mu u_{xxx} + u^\alpha u_x = 0$$

mentioned in [1] as a model for the propagation of one-dimensional, unidirectional small amplitude long waves in nonlinear media. In GKdVB equation, $\mu, \gamma > 0$ and α is a positive integer is considered, the independent variable x represents the medium of propagation, t is proportional to elapsed time, and $u(x,t)$ is a velocity at the point x at time t. And

If $\nu = 0, \alpha = 1$, the GKdVB equation (1) reduces to the Korteweg-de-Vries(KdV) equation, which is a nonlinear dispersive partial differential equation that presents a model of propagation of small amplitude along water in a uniform channel[4-5].

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If $\mu = 0, \alpha = 1$, the KdVB equation (1) reduces to the Burgers equation which models turbulent liquid flow through a channel.[6-8].

Recently, some researchers obtained many results about the UCP of differential equation. In [9] for the KdV equation, Kenig, Ponce, and Vega considered conditions, only at two different times, on the support of a solution u, proving that if this solution is supported in an interval $(-\infty, B)$ at $t = 0$ and $t = 1$, then $u = 0$.A similar result was obtained in [10] for the generalized KdV equation. Reinhard Recke proved a UCP for solutions of the wave equation in [11]. In [12],for a general class of dispersive equations, Saut and Scheurer proved that if a solution $u = u(x,t), x \in R^n, t \in R$, in such class,vanishes in an open set $\Omega \in R^n \times R$, then it vanished in all the horizontal components of Ω. Rosier and Bingyu Zhang applied the moment approach to prove the UCP of the BBM equation and extended it to certain BBM-like equation in [13].The UCP for the Ostrovsky equation with negative dispersion was proved in [14] by Pedro Isaza. In [16], Ademir Fernando Pazoto proved the exponential decay for the energy of solutions of the Korteweg-de Vries equation in a bounded interval with a localized damping term.

In this paper, we investigate the Unique Continuation Property (UCP) of KdVB equation and its applications to control problem for (1). Recall that UCP holds in some class X of function if, given any nonempty open set $\omega \subset \mathrm{T}$, the only solution $u \in X$ of (1) fulfilling $u_(x,t) = 0$ for$(x,t) \in \omega \times (0,T)$ is the trivial solution $u \equiv 0$. This property is very important in control theory, and the study of unique continuation property is usually proved by applying some Carleman estimated, as it is equivalent to the approximate controllability for linear PDE, and it also is involved in the classical uniqueness/compactness approach in the proof of the stability for PDF with a localized damping. Here,we apply the method in Menzala [15] which combines energy estimates, multipliers and compactness arguments the problem is reduced to prove the unique continuation of weak solutions. Multiplying the equation (1) by u ,

$$uu_t - uu_{xx} + uu_{xxx} + uu_x + a(x)u^2 = 0$$

integrating the above equation in $(0, L)$, and set

$$E(t) = \frac{1}{2}\int_0^L |u(x,t)|^2\,dx \tag{3}$$

we get

$$\frac{dE(t)}{dt} = -\int_0^L a(x)\,|u(x,t)|^2\,dx - \int_0^L |u_x(x,t)|^2\,dx - \frac{1}{2}\,|u_x(0,t)|^2 \tag{4}$$

This indicates that the term $a(x)u$ in the equation is a feedback damping mechanism. Consequently, as an energy function, $E(t)$ is a decreasing function and a rate of decay of solutions is expected. For the boundary value problem (1) under consideration, according to the above dissipation law (4),when $a = 0$, the energy is dissipated through the extreme $x = 0$. On the other hand,when $a \equiv 1$, it is straightforward to see from (4) that the energy decays uniformly exponentially as $t \to \infty$. The same holds when $a(x) \geq a_0 > 0, a.e.$ in $(0, L)$.

The problem of stabilization when the damping is effective only on a bounded subset of the interval $(0, L)$ is much more subtle and, in view of (3), the problem of the exponential decay of $E(t)$ can be stated in the following equivalent form: To find $T > 0$ and $C > 0$ such that

$$E(0) \leq C\int_0^T [\int_0^L (a(x)u^2(x,t) + u_x^2(x,t))dx + u_x^2(0,t)]dt \tag{5}$$

holds for every finite energy solution of (1). From (4) and (5), combining the semigroup theory, we have $E(t) \leq \lambda E(0)$ with $0 < \lambda < 1$, i.e. the exponential decay of $E(t)$ is derived.

This article focus on analyzing this problem. The case for KdV equation with the damping term is in [15] , in this case, the damping term $a(x)u$ is active simultaneously in a neighborhood of both extremes of the interval $(0, L)$ was addressed. It was proved that for all $T > 0$ and any solution with initial data satisfying $E(0) \leq R$, simultaneously, the following inequality holds

$$E(0) \leq C\int_0^T [\int_0^L a(x)u^2(x,t)dx + u_x^2(0,t)]dt$$

with a constant $C = C(R,T)$. Consequently it was shown that, for any $R > 0$, there exist positive constants $C(R)$ and $\alpha(R)$ satisfying

$$E(t) \leq C(R)E(0)e^{-\alpha(R)t}, \forall t > 0 \tag{6}$$

provided $E(0) \leq R$. The proof in [15] follows closely the multiplier techniques developed in [18] for the analysis of controllability properties. However, when using multipliers, the nonlinearity produces extra terms in [15] were handled by compactness. In fact, proceeding as in [15] the problem of obtaining (5) is reduced to showing that the unique solution of KdV equation, such that,$a(x)u = 0$ everywhere and $u_x(0,t) = 0$ for all time t, has to be the trivial one. This problem may be viewed as a unique continuation one since $au = 0$ implies $u = 0$ in $\{a > 0\} \times (0,T)$.

However, the existing unique continuation results (see [12]) do not apply directly since the solutions we are dealing with are weak, with initial data in $L^2(0,L)$ and the regularity conditions required to derive Calerman inequalities are not fulfilled. We point out that such inequalities are those reminiscent of the classical Carleman estimates in which the lower order terms (with bounded coefficients or even with unbounded coefficients under suitable integrability conditions) of the equation can be controlled in some weighted norms by the principal part of the operator (see for instance [19-20]). Consequently, this article may be considered as a new contribution in the subject of proving unique continuation properties of weak solutions of partial differential equations, this time in the context of KdVB equations. As far as we know, the situation we are considering here has not been addressed in the literature yet since, to our knowledge, the existing results on unique continuation for KdV-like equations ([15][20]) require the solution u to be in $L^\infty(0,T;H^s(0,L))$ with $s > \frac{2}{3}$.

Let us now describe our strategy of proof in some more detail. We first differentiate the equation in (1) with respect to t and analyze the regularity of $y = u_t$, which is a solution of

$$\begin{cases} y_t - y_{xx} + y_{xxx} + (u(x,t)y)_x + a(x)y = 0, \text{ in } (0,L)\times(0,T) \\ y(0,t) = y(L,t) = 0, t \in (0,T) \\ y_x(L,t) = 0, t \in (0,T) \\ y(0) = y_0, \text{on (0,L)} \end{cases} \tag{7}$$

where $u \in L^2(0,T;H_0^1(0,L)) \cap L^\infty(0,T;L^2(0,L))$ is the weak solution of (1) and $y_0 = y(x,0) = u_t(x,0)$ in $H^(- 3)(0,L)$. Of cause,since $u \equiv 0$ in $\omega \times (0,T)$,$y \equiv 0$ on $\omega \times (0,T)$ as well,ω being the subinterval where the damping potential a is effective. The above model (7) can be viewed as a linearized KdVB equation. Therefore, inspired by the work of Rosier and Zhang [20-21], we argue as in the linear case, combining multiplier techniques and the so called "compactness-uniqueness" argument (see [22]), which is useful to handle the extra terms that the "potential" $u(x,t)$ produces in the inequality. And then we can prove the fact that $y \equiv 0$ in $\omega \times (0,T)$ implies the extra regularity property $y \in L^2(0,T;H_0^1(0,L)) \cap L^\infty(0,T;L^2(0,L))$ which yields enough regularity on to apply the unique continuation results obtained in [12] by means of Carleman inequalities.

The paper is organized as follows. In Section 2, we state our the main results of this work and prove our main result, i.e., the UCP of weak solutions and the exponential decay of (1) .

2 Main result and proof

Firstly, for the sake of completeness, we state the well-posedness result for problem (1) obtained in [17]:
Theorem 2.1*(see[17])For any give $u_0 \in L^2(0,L)$, the problem (1) has a unique global mild solution. And, the following enery identity holds for all $T \geq 0$:*

$$\frac{1}{2}\|u(t)\|_{L^2(R)}^2 + \int_0^T \|u_x(x,t)\|_{L^2(R)}^2 dt + \int_0^T \int_R a(x)|u(x,t)|^2 dxdt = \frac{1}{2}\|u_0\|_{L^2(R)}^2 . \tag{8}$$

Furthermore, let $T \geq 0$ and $a(x) \in H^1(0,L)$,for every $u_0 \in H^s(0,L), 0 \leq s \leq 3$, the nonlinear problem (1) admits a unique solution u, which belongs to the class $\mathbb{B}_{s,T}$,there exists a continuous function $C : R^+ \times (0,\infty) \to R^+$, nondecreasing in its first variable, such that

$$\|u\|_{\mathbb{B}_{s,T}} \leq C(\|u_0\|_2, T)\|u_0\|_{H^s(R)} \tag{9}$$

in which $\mathbb{B}_T := C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$,
with the norm $\|u\|_{\mathbb{B}} = \|u\|_{C([0,T];L^2(0,L))} + \|u_x\|_{L^2(0,T;L^2(0,L))}$
The main results of this paper can be summarized as follows:
Theorem 2.2 *Let u be the solution of problem (1) obtained in Theorem 2.1 and ω and $a(x)$ as in (2), $0 < T < \infty$. If $u_x(0,t) = 0$ in $\omega \times (0,T)$, then*

$$u \in L^2(0,T;H^3(0,L)) \cap H^1(0,T;L^2(0,L))$$

Consequently, the UCP holds and, therefore, $u \equiv 0$.
For proving Theorem2.2, we show the following lemmas:
Lemma 2.1 *Let u be the solution of problem (1) obtained in Theorem 2.1. Then, problem (7) has an unique mild solution*

$y \in L^2(0,T;H_0^1(0,L)) \cap L^\infty(0,T;L^2(0,L))$ whenever $y_0 \in L^2(0,L)$.

Proof. Firstly, we consider the following system

$$\begin{cases} y_t - y_{xx} + y_{xxx} + a(x)y = 0, \text{ in } (0,L)\times(0,T) \\ y(0,t) = y(L,t) = 0, t \in (0,T) \\ y_x(L,t) = 0, t \in (0,T) \\ y(0) = y_0, on(0,L) \end{cases} \quad (10)$$

Setting

$$A = -\partial_x^3 + \partial_x^2 - aI, D(A) = H^3(R)$$

According to[17],A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(R), T > 0$,and (10) has aunique mild solution $y \in C([0,T];L^2(R))$,and $y(t) = S(t)u_0$.Furthermore, we multiply the equation in (10) by y and xy, and integrate over $(0,L)\times(0,T)$,we obtain

$$\|y(\cdot,T)\|^2_{L^2(0,L)} = \|y_0\|^2_{L^2(0,L)} - \int_0^T |y_x(0,t)|^2 dt - 2\int_0^T \int_0^L [a(x)|y|^2 + |y_x|^2]dxdt$$

and

$$\int_0^T \int_0^L y_x^2 dxdt + \frac{1}{3}\int_0^L xy^2(x,T)dx + \frac{2}{3}\int_0^T \int_0^L xa(x)y^2 dxdt + \frac{2}{3}\int_0^T \int_0^L xy_x^2 dxdt = \frac{1}{3}\int_0^L xy_0(x)dx$$

Then,we may deduce that $S(\cdot)$ satisfies the following properties

$$\begin{array}{l} \|S(t)y_0\|_{L^2(0,L)} \leq \|v_0\|_{L^2(0,L)} \\ \|S(\bullet)y_0\|_{L^2(0,T;H_0^1(0,L))} \leq C(L+T)\|v_0\|_{L^2(0,L)}. \end{array} \quad (11)$$

for all $0 \leq t \leq T$ and $v_0 \in L^2(0,L)$. According to [17], the unique mild solution of (7) has the following

$$y(t) = S(t)y_0 + \int_0^t S(t-s)[u(x,s)y]_x ds = \Phi[y](t) \quad (12)$$

Thus, the problem of existence and uniqueness for (7) is reduced to finding a fixed point of Φ. To do that, we show that Φ is a contraction from a suitable ball $\mathbb{B}_R$ of $\mathbb{B}_T := C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ into itself when $T > 0$ is small enough (both R and T depend on the size of the initial data y_0 in $L^2(0,L)$ and of the potential $u = u(x,t)$ in $C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$. This shows that system (7) has an unique mild solution for $0 \leq t < T$, with T small. Thus, for concluding the proof of Lemma 2.1, it is sufficient to prove that this solution exists globally. We first multiply the 2equation in (7) by y and integrate by parts over $(0,L)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + \int_0^L y_x^2 dx + \frac{1}{2}y_x^2(0,t) + \int_0^L y^2 a(x)dx = \int_0^L yy_x u dx. \quad (13)$$

Applying the Cauchy-Schwarz and Holder's inequalities in (19), we have

$$\begin{array}{l} \int_0^L y^2 dx + \int_0^T \int_0^L y_x^2 dxdt \\ \leq \int_0^L y_0^2 dx + \int_0^T \int_0^L |uy_x y|\, dxdt \\ \leq \int_0^L y_0^2 dx + (\int_0^T \|uy\|^2_{L^2(0,L)} dt)^{\frac{1}{2}} \int_0^T \|y_x\|^2_{L^2(0,L)} dt)^{\frac{1}{2}} \\ \leq \int_0^L y_0^2 dx + \int_0^T \|u\|^2_{L^\infty(0,L)} \|y\|^2_{L^2(0,L)} dt + \int_0^T \|y_x\|^2_{L^2(0,L)} dt. \end{array} \quad (14)$$

Simplify (14), we deduce that

$$\begin{array}{l} \int_0^L y^2 dx \leq \int_0^L y_0^2 dx + \int_0^T \|u\|^2_{L^\infty(0,L)} \|y\|^2_{L^2(0,L)} dt \\ \Rightarrow \|y\|_{L^\infty(0,T;L^2(0,L))} \leq \int_0^L y_0^2 dx + \int_0^T \|u\|^2_{L^\infty(0,L)} \|y\|^2_{L^2(0,L)} dt = C_1. \end{array} \quad (15)$$

Here C_1 is a positive constant. On the other side, multiply equation in (7) by xy and integrate by parts over $(0,L)\times(0,T)$,using the boundary condition. Then according to Poincare's inequality, we deduce that

$$\int_0^T \int_0^L y_x^2(x)dxdt \leq C_2 \left\{ \int_0^L y_0^2(x)dx + \int_0^T (1 + \|u\|^2_{L^\infty(0,L)}) \|y\|^2_{L^2(0,L)} dt \right\} \quad (16)$$

For some positive constant $C_2 = C_2(T, \|u_0\|_{L^2(0,L)}, \|y_0\|_{L^2(0,L)}) > 0$.Combining (15),(16) and Theorem 2.1, we have

$$\|y\|_{L^2(0,T;H_0^1(0,L))} \le C \tag{17}$$

Here $C = \max\{C_1, C_2\}$. This concludes the proof of Lemma 2.1. ■

Lemma 2.2 *(see in [17]) There exists a positive constant $C = C(T, \|u_0\|_{L^2(0,L)})$ such that*

$$\begin{aligned} \|y_0\|^2_{L^2(0,L)} &= \|y(t)\|^2_{L^2(0,L)} + 2\int_0^T \|y_x(t)\|^2_{L^2(0,L)}\,dt + 2\int_0^T\int_0^L a(x)\,|y|^2\,dxdt \\ &\le C\left\{\int_0^T y_x^2(0,t)dt + \int_0^T\int_0^L a(x)\,|y|^2\,dxdt\right\} + \|y_0\|^2_{H^{-3}(0,L)} \end{aligned} \tag{18}$$

holds for every solution y of (7) as in Lemma 2.1.

We are now in conditions to prove Theorem 2.2.

Proof. Let $u_0 \in L^2(0,L)$.According to the definition of y,we have

$$y_0 = y(x,0) = u_t(x,0) = -u_{0,xxx} - u_0u_{0,x} + u_{0,xx} - a(x)u_0 \in H^3(0,L)$$

If $u_x(0,t) = 0$ and $a(x)u$ vanishes, then $y_x(0,t) = 0$ and $a(x)y \equiv 0$ as well. Consequently, if the damping potential $a = a(x), y \equiv 0$, in $\omega \times (0,T)$, according to (2) and Lemma 2.2, we obtained that $y_0 \in L^2(0,L)$, combining Lemma2.1 and system (1), we get

$$u_t = v \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L)) \tag{19}$$

and

$$\begin{cases} u_{xxx} = -u_t - u_x - uu_x - a(x)u \text{ in } (0,L)\times(0,T) \\ u(0,t) = u(L,t) = 0 \ t \in (0,T) \end{cases} \tag{20}$$

Then, combining (19) ,(20) and Theorem2.1, we have a conclusion that $u \in L^2(0,T;H^3(0,L)) \cap H^1(0,T;L^2(0,L))$ and using the UPC in (12),we have $u \equiv 0$. ■

Our main result on exponential decay is as follows:

Theorem 2.3 *For any L, any damping potential a satisfying (2) and $M > 0$, there exist $c = c(M) > 0$ and $\mu = \mu(M)$ such that*

$$E(t) \le c\,\|u_0\|^2_{L^2(0,L)}\,e^{-\mu t} \tag{21}$$

holds for all $t \ge 0$ and any solution of (1) with $u_0 \in L^2(0,L)$ such that $\|u_0\|_{L^2(0,L)} \le M$

Proof of Theorem 2.3: As mentioned in the introduction, once Theorem 2.2 is known, Theorem 2.3 holds immediately applying the methods in [17].

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