On Disconnected Graph with Large Reconstruction Number

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Abstract

The *reconstruction number rn(G)* of graph G is the minimum number of vertex-deleted subgraphs of G required in order to identify G up to isomporphism. Myrvold and Molina have shown that if G is disconnected and not all components are isomorphic then $rn(G) = 3$, whereas, if all components are isomorphic and have c vertices each, then $rn(G)$ can be as large as $c + 2$. In this paper we propose and initiate the study of the gap between $rn(G) = 3$ and $rn(G) = c + 2$. Myrvold showed that if G consists of p copies of K_c, then $rn(G) = c + 2$. We show that, in fact, this is the only class of disconnected graphs with this value of $rn(G)$. We also show that if $rn(G) \geq c + 1$ (where c is still the number of vertices in any component), then, again, G can only be copies of K_c. It then follows that there exist no disconnected graphs G with c vertices in each component and $rn(G) = c + 1$. This poses the problem of obtaining for a given c, the largest value of $t = t(c)$ such that there exists a disconnected graph with all components of order c, isomorphic and not equal to K_c and is such that $rn(G) = t$.

1. Introduction

In this paper, all graphs considered are *simple, finite* and *undirected*. The vertex set of a graph is denoted by $V(G)$ and the edge set by $E(G)$. Two vertices u and v are said to be *adjacent*, denoted as $u \sim v$, if there is an edge $\{u,v\}$ joining them. The edge $\{u,v\}$ is usually abbreviated to uv. The number of vertices in a graph, denoted by $|V(G)|$, is the *order* of G. The *degree* of a vertex v of a graph G, denoted by $deg(v)$, is the number of edges of G incident to v.

A graph is *regular* if all its vertices have the same degree. A graph G of order n is *quasi-regular* if it has a vertex u of degree $n - 1$ such that $G - u$ is regular. Note that, if a quasi-regular graph is regular, then it must be complete. If H, K are graphs, then $H \cup K$ consists of the graph with vertex-set $V(H) \cup V(K)$ and edge-set $E(H) \cup E(K)$; pH denotes the graph consisting of the union of p isomorphic copies of H.

A *vertex-deleted subgraph* of a graph G is a subgraph $G - v$ obtained by deleting from G the vertex v and all the edges incident to it. The *deck* of a graph G, denoted by $D(G)$, is the collection of all unlabelled vertex-deleted subgraphs of G, and the elements of $D(G)$ are referred to as *cards*. Note that if G contains isomorphic vertex-deleted subgraphs, then such subgraphs would be repeated in $D(G)$ and therefore $D(G)$ is a multiset.

A collection S of graphs $H_1, \ldots, H_n$ is said to be a *legitimate deck* if there is a graph G with n vertices such that $S = D(G)$. Otherwise S is said to be an *illegitimate deck*. A collection S of graphs $H_1, \ldots, H_k$, each with $n - 1$ vertices, and $k < n$, is said to be a *legitimate subdeck* if there is a graph G with n vertices and $v_1, \ldots, v_k \in V(G)$ such that $H_i \approx G - v_i$, $i = 1, \ldots, k$. If there is no such graph G, then $\in S$ is said to be an *illegitimate subdeck*.

The *reconstruction number* $rn(G)$ of G is the minimum number of vertex-deleted subgraphs of G required in order to identify G up to isomorphism. This number was defined by Harary and Plantholt in [2], and was later referred to as the ally-reconstruction number by Myrvold in [4,5]. We refer the reader to [1,6] for excellent survey papers on graph reconstruction.

In [3,4,5], Myrvold and Molina showed that if a disconnected graph G has at least two nonisomorphic components then $rn(G) = 3$. Therefore we shall henceforth assume that G is a disconnected graph with all components isomorphic and having c vertices each, that is, $G = pH$, $|V(H)| = c$.

To motivate the definitions and lemmas given in the next sections, we now briefly show that if $rn(G) = c + 2$ then $H = K_c$. Let $\{H_1, H_2, \ldots, H_c\}$ be the deck of G and suppose $rn(G) = c + 2$. Let S be the subdeck of G containing $H_1 \cup (p-1)H, H_2 \cup (p-1)H, \ldots, H_c \cup (p-1)H$ and any other vertex deleted subgraph of G. Suppose, without loss of generality, that H_1 is connected. Then, since $rn(G) = c + 2$, this deck does not reconstruct G. So let G' be constructed by adding a new vertex to $H_1 \cup (p-1)H$, such that G' also contains S as a subdeck. As described in some more detail below, this can only be done if either $G' = F \cup (p-1)H$ where $F - u \approx H_1$ and $F \not\approx H$ or $G' = H_1 \cup F \cup (p-2)H$ where $F - u \approx H$. But in the first case, all the graphs in S must be obtained by deleting vertices from F, and this is impossible since $|V(F)| = c$ and S has $c + 1$ subgraphs. In the second case, all graphs in S must be obtained by deleting vertices from F. This implies two things: that each H_i is isomorphic to H_1, therefore H is regular, and since each $F - v$, for all $v \in V(F)$, is isomorphic to H, therefore F is also regular. But $H = F - u$, therefore H is a complete graph as required.

When the assumption is that $rn(G) = c + 1$, the deck S we have to work with contains only c subgraphs, and the arguments do not then follow so easily. The rest of the paper deals with this situation.

Any graph theoretic notation not explicitly defined in this paper can be found in [7].

2. Regular and quasi-regular graphs

We first prove, in this section, a few simple results about regular and quasi-regular graphs which will be required later. The proof of the next theorem, which we omit, follows easily from the fact that the degree sequence of a graph is reconstructible from its deck.

Theorem 1 *Regular and quasi-regular graphs are reconstructible.*

Notation A degree sequence represented by ($d_1^{e_1}$, $d_2^{e_2}$, … , $d_k^{e_k}$) means that there exists e_i vertices of degree d_i.

Lemma 1 *Let H be a connected simple graph where* $|V(H)| = n$. *If* $H - v_1$, $H - v_2$, …, $H - v_{n-1}$ *are all isomorphic then either H is regular or quasi-regular.*

Proof Suppose H is a graph with n vertices $v_1, v_2, \ldots, v_n$.
Since $H - v_1, H - v_2, \ldots, H - v_{n-1}$ are all isomorphic,

$$deg(v_1) = deg(v_2) = deg(v_3) = \ldots = deg(v_{n-1}) = r.$$

Case(i) Assume that v_n is adjacent to at most $(n - 2)$ vertices in H.

Without loss of generality, let $v_n \sim v_i$, $v_n \nsim v_j$ and $H - v_i \approx H\text{-}v_j$.

The neighbours of v_i must have degree r and thus
degree sequence in $H - v_i$: $[(r-1)^{r-1}, (deg(v_n) - 1), r^{n-1-r}]$ (1)
Similarly the neighbours of v_j have degree r and thus
degree sequence in $H - v_j$: $[(r-1)^r, deg(v_n), r^{n-2-r}]$ (2)
Since $H - v_i \approx H - v_j$, the degree sequence in $H - v_i$ and $H - v_j$ must be the same. Comparing (1) and (2) we deduce that $deg(v_n) = r$ and $deg(v_n) - 1 = r - 1$. As a result, $deg(v_k) = r$ for all values of $k = 1, 2, 3, \ldots, n$. Hence H is regular.

Case(ii) Assume that v_n is adjacent to all vertices in H.

It is obvious that $deg(v_n) = n - 1$. But $H - v_1 \approx H - v_2 \approx \ldots \approx H - v_{n-1}$.
This means that $deg(v_1) = deg(v_2) = \ldots = deg(v_{n-1}) = r$.
Hence H is quasi-regular.

Lemma 2 *Let H be a connected regular graph of order c . Let u be a vertex not in H and suppose that F is a graph such that* $F - u \approx H$. *Suppose that F is either regular or quasi-regular and that there is a vertex* $v(\neq u)$ *in F, such that* $F - v \approx H$. *Then* $H = K_c$*, the complete graph on c vertices*

Proof Let H be a connected regular graph of order c with m edges.
Then

$$deg(v_1) = deg(v_2) = \ldots = deg(v_c) = r$$

Using the Handshaking Lemma [7], we have $cr = 2m$ (1)

Case(i) Suppose F is regular. Then

$$deg(v_1) = deg(v_2) = \ldots = deg(v_c) = deg(u) = r + 1.$$

By the Handshaking Lemma,

$$(c + 1)(r + 1) = 2(m + deg(u))$$

thus $$cr + r + c + 1 = 2(m + r + 1)$$

giving $$cr + c = 2m + r + 1 \qquad (2)$$

Substituting (1) in (2), we have $$2m + c = 2m + r + 1.$$

Hence $$r = c - 1$$

Therefore H is regular of order c and $\deg(v_k) = c - 1$, for all values of $k = 1, 2, \ldots, c$. This means that $H = K_c$.

Case(ii) Suppose F is quasi-regular.
As $F - v \approx H$ where H is regular of order r, then u has degree r in $F - v$. Hence u has degree $r + 1$ in F. But $deg(u) = |V(H)| = c$.
This means that $r + 1 = c$ and therefore $r = c - 1$. Hence H is regular of degree $c - 1$, that is, $H = K_c$.

3. Illegitimate decks

In the following lemma we show that a necesssary condition for a deck to be legitimate is that the sum of the number of edges of all subgraphs is a multiple of $n - 2$ where $n = |V(G)|$.

Lemma 3 *Let $G_1, \ldots, G_n$ be a family of graphs such that each has $n-1$ vertices. Then a necessary condition for this family to be a legitimate deck is that $\sum_{i=1}^{n} |E(G_i)|$ is a multiple of $n - 2$.*

Proof Suppose $G_1, G_2, \ldots, G_n$ is a legitimate deck of a graph G.
Each edge uv appears on $n - 2$ cards (all cards except $G - u$, $G - v$).

Lemma 4 *Let H be a graph of order c having two vertices u,v such that $deg(u) > deg(v)$ and $deg(u) - deg(v) < c - 2$.
Then $S = D(H) - (H - v) + (H - u)$ is an illegitimate deck.*

Proof As $D(H)$ is legitimate then from Lemma 3

$$\sum_{H_i \in D(H)} |E(H_i)| = p(c - 2) \qquad (1)$$

where $p \in \mathbb{Z}^+$.
Suppose that S is legitimate. Then

$$\sum_{H_j \in S} \left| E\left(H_j \right) \right| = q(c-2) \qquad (2)$$

where $q \in \mathbb{Z}^+$, $p > q$.

Subtracting (2) from (1) gives

$$|E(H-v)| - |E(H-u)| = (p-q)(c-2)$$

$$|E(H)| - deg(v) - |E(H)| + deg(u) = (p-q)(c-2)$$

Hence $deg(u) - deg(v) = (p-q)(c-2)$

$$deg(u) - deg(v) \geq c-2 \quad \text{as} \quad p > q$$

contradicting the hypothesis in the Lemma.

Lemma 5 (Illegitimate deck lemma) *Let H be a connected graph on c vertices which is not regular or quasi-regular. Let D(H) =* $H - v_1, H - v_2, \ldots, H - v_c$.
Then there exists v_i, v_j, $i \neq j$ *such that if* $H - v_i$ *is replaced by* $H - v_j$ *then*
(i) the resulting family of subgraphs is not a legitimate deck, and
(ii) not all subgraphs in the family are isomorphic, and
(iii) at least one of these subgraphs is connected.

Proof (i) As *H* is not regular or a star (since stars are quasi-regular), we can choose vertices v_i, v_j such that $deg(v_i) < deg(v_j)$ and $deg(v_j) - deg(v_i) < c - 2$. Using Lemma 4, the resulting family of subgraph is an illegitimate deck.
(ii) Suppose that, when $H - v_i$ is replaced by $H - v_j$, all the resulting subgraphs are now isomorphic. Then $H - v_k$, $k = 1, 2, \ldots, c$ must be isomorphic except $H - v_i$ since $deg(v_j) > deg(v_i)$. Using Lemma 1, *H* must be quasi-regular, a contradiction.
(iii) As *H* is connected, then at least two cards are connected. Since only one card is replaced by another, which may be disconnected, then at least one of the cards is still connected.

4. The Main Result

The proof of the main result of this paper is based on the next three theorems that we now present.

Theorem A *Let H be a connected graph of order at least 3. Suppose H has k cards* ($k \geq 3$) *such that (1) they reconstruct H (therefore* $rn(H) \leq k$*),*
(2) not all are isomorphic and (3) at least one is connected.
Then $rn(pH) \leq k$.

Proof Let $G = pH$ and $H_1, H_2, \dots, H_k$ be the subgraphs which reconstruct H, and let H_1 be connected.
Consider k cards of G : $H_1 \cup (p-1)H, H_2 \cup (p-1)H, \dots, H_k \cup (p-1)H$. Denote this subdeck of G by S. We claim that S reconstructs G uniquely, giving the required result.

There are four possible ways of reconstructing from $H_1 \cup (p-1)H$. This is achieved by putting a new vertex u back to $H_1 \cup (p-1)H$ to obtain G' in one of the following ways:-
(i) joining u to vertices from more than one component of $H_1 \cup (p-1)H$.
(ii) adding u as an isolated vertex.
(iii) adding u to an H component only.
(iv) adding u to the H_1 component only.
In each case we have to consider what happens if we assume that G' contains the subdeck S and we must show that if $G' \not\approx G$, then this is impossible.

Case(i) Suppose u is adjacent to some vertices in two different components of $H_1 \cup (p-1)H$. Then G' has a component C that is larger than H. There can be at most two cards of G' in which no component larger than H appears: either by deleting u or else by deleting the vertex y when this is the only vertex of H adjacent to u. But this means that G' cannot have S as a subdeck, since $k \geq 3$.

Case (ii) Let u be added as an isolated vertex, and let $G' = F \cup (p-1)H$ where $F = H_1 \cup \{u\}$. Therefore the k subgraphs of G' which form the family S must arise by deleting a vertex from F. Thus there exists vertices $x_1, x_2, \dots, x_k$ such that the graphs $F - x_i \cup (p-1)H$ form the subdeck S. This implies that $F - x_i$, $i = 1,2, \dots, k$ is the subdeck of H: $H_1, H_2, \dots, H_k$. But this is impossible, since $F \not\approx H$ and the subdeck $H_1, H_2, \dots, H_k$ reconstruct H uniquely.

Case (iii) G' is of the type $H_1 \cup F \cup (p-2)H$ where F is a graph such that $F - u \approx H$.

It is obvious that if G' is to contain all the subdeck S then these subgraphs must all arise by deleting a vertex from the component F to give H thus leaving the component H_1 to appear in all the selected k cards. This means that $H_1, H_2, \dots, H_k$ must be all isomorphic, contradicting the fact that not all these cards are isomorphic.

Case (iv) G' is of the type $F \cup (p-1)H$ where F is a graph such that $F - u \approx H_1$.

Suppose, for contradiction, that F is not isomorphic to H. This case is now similar to case (*ii*). Here the k subgraphs of G' which form the family S, must come by deleting a vertex from the component F. This implies that F

and H share the common subdeck $H_1, H_2, \ldots, H_k$. But this is impossible since $F \not\approx H$ and $H_1, H_2, \ldots, H_k$ reconstruct H uniquely.

Therefore the only way left to reconstruct G' is as $F \cup (p-1)H$ where F is isomorphic to H. That is G' must be isomorphic to G.

Theorem B *Let H be a connected graph. Suppose there is a family* $\Im$ *of k cards of H such that*

(1) if a card appears in D(H) r times, then it appears in $\Im$ *at most (r + 1) times*

(2) $\Im$ *is an illegitimate subdeck*

(3) at least one card in $\Im$ *is connected*

(4) not all cards are isomorphic

Then $rn(pH) \leq k$.

Proof Let $G = pH$ and let $H_1, H_2, \ldots, H_k \in \Im$, such that H_1 is connected.

Consider the following k cards of G : $H_1 \cup (p-1)H$, $H_2 \cup (p-1)H$, $\ldots$, $H_k \cup (p-1)$H.

This is clearly a subdeck of G since any subgraph H_i is repeated at most once more than it appears in *D(H)* and G has at least two components isomorphic to H. Let this subdeck of G be denoted by S. Again we claim that S reconstructs G uniquely.

As in Theorem A, there are four possible ways of reconstructing G from $H_1 \cup (p-1)H$. The first case is not considered for the same arguments used in proving Theorem A hold.

Therefore these are the ways of reconstructing from $H_1 \cup (p-1)H$:- either *(i)* by adding the missing vertex to an H component, that is, as $G' = H_1 \cup F \cup (p-2)H$ where F is connected and $F - u \approx H$ or *(ii)* by adding the missing vertex u either as an isolated vertex or joined to H_1, that is, as $G' = F \cup (p-1)H$ where $F - u \approx H_1$ and F can either be connected or equal to $H_1 \cup \{u\}$.

In *Case (i)* assume that G' contains the subdeck S. All these k subgraphs of G' must come by deleting a vertex from F to give H. Therefore H_1 will appear always in these k cards and hence $H_1, H_2, \ldots, H_k$ must be all isomorphic. This contradicts condition (4).

In *Case (ii)* suppose $G' \not\approx G$, therefore F is not isomorphic to H (this will certainly be the case when $F = H_1 \cup \{u\}$). Now all subgraphs of G' which form S must come by deleting a vertex from F. This means that k subgraphs $F - x_i \cup (p-1)H$, $i = 1, 2, \ldots, k$ are the chosen subgraphs $H_1 \cup (p-1)H, \ldots, H_k \cup (p-1)H$. Hence the k cards $F - x_i$ are a subdeck of F which contradicts condition (2). Therefore F must be isomorphic to H, that is $G' \approx G$. Thus S reconstructs G, that is $rn(G) \leq k$.

The next theorem tells us what type of graph can be reconstructed from the subdeck S if we relax the statement of Theorem A by omitting condition (2).

Theorem C *Let H be a connected graph. Suppose H has a subdeck of connected subgraphs* $H_1, H_2, \ldots, H_k$ *that reconstruct H.*
Suppose $rn(pH) > k$ *and suppose* G' *is a graph, not isomorphic to G, which contains the subdeck S of pH given by* $H_1 \cup (p-1)H$, $H_2 \cup (p-1)H$, …, $H_k \cup (p-1)H$. *Then* G' *is of the type* $H_1 \cup F \cup (p-2)H$, *where F is a connected graph such that* $F - u \approx H$ *for some* $u \in V(F)$.

Proof As in Theorem A, there are four ways to reconstruct from *S*. But cases *(i), (ii)* and *(iv)* are not possible by using the same arguments as in Theorem A. Thus the only possible reconstruction is of the type $H_1 \cup F \cup (p-2)H$ as given by the theorem.

Main Theorem *Let H be a connected graph of order c. If G = pH and* $rn(G) \geq c+1$, *then* $H = K_c$.

Proof As $rn(G) \geq c+1$, then any *c* subgraphs of *G* are a subdeck of some nonisomorphic graph. Consider when the component *H* is
(i) not regular and not quasi-regular, or
(ii) regular or quasi-regular

Case(i) H not regular, *H* not quasi-regular

Choose *c* subgraphs of *H* as in the Lemma 5. Since *H* is connected, these *c* subgraphs satisfy all four conditions of theorem B.
Therefore $rn(pH) \leq c$ which contradicts the fact that $rn(G) \geq c+1$.

Case (ii) H regular or quasi-regular

Let $H_1, H_2, \ldots, H_c$ be the full deck of *H* and suppose that H_1 is connected. Let *S* be the subdeck of *G* formed by the *c* subgraphs $H_1 \cup (p-1)H$, $H_2 \cup (p-1)H$, … , $H_c \cup (p-1)H$.

If $H_1, H_2, \ldots, H_c$ are not all isomorphic then $rn(pH) \leq c$ by Theorem A, which contradicts the fact that $rn(pH) \geq c+1$. If all cards are isomorphic then by Theorem C any graph which has *S* as subdeck must be of the type $G' = H_1 \cup F \cup (p-2)H$ where *F* is a connected graph such that $F - u \approx H$ for some $u \in V(F)$.
Suppose G' contains *S*. Obviously all subgraphs of G' which are in *S* must come by deleting a vertex from *F* and so $F - x_i$, $i = 1, 2, \ldots, c$ must be isomorphic to *H*. From Lemma 1, *F* is either regular or quasi-regular. Hence from Lemma 2 both cases will make *H* equal to K_c.

Using the above result and the fact that $G = pK_c$ has reconstruction number $c + 2$, the following corollary holds:

Corollary *There exists no disconnected graph G with c vertices in each component such that* $rn(G) = c + 1$.

5. Conclusion

We have shown that if $G = pH$ for some connected graph H of order c and $p > 1$ and if $rn(G) \geq c + 1$, then $H \approx K_c$. We believe that this result is far from best possible and that the gap between $rn(G) = 3$ when not all components are isomorphic and $rn(G) = c + 2$ needs further investigation. We therefore pose the following questions:

> Given $c > 0$ what is the largest value of $t = t(c)$ such that there exist disconnected graphs $G = pH$ with H connected of order c and $H \neq K_c$, such that $rn(G) = t$? Is there a constant t_o such that if the disconnected graph G has $rn(G) > t_o$ then G must be equal to the disjoint union of copies of K_c?

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