Flow in a Channel With Longitudinal Ribs

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The laminar, viscous flow between parallel plates with evenly spaced longitudinal ribs is solved by an eigenfunction expansion and point-match method. The ribs on both plates may be symmetrically placed or staggered. For a given pressure gradient, the mean velocity is plotted as a function of the geometric parameters. We find the wetted perimeter and the friction factor—Reynolds number product are unsuitable parameters for the flow through ducts of complex geometry.

Introduction

The flow in ducts is a basic topic in fluid mechanics. Steady, parallel, laminar, fully developed, constant property flow has been solved for a variety of cross sections (Blevins, 1984; Shah and Bhatti, 1987). This paper studies the flow between parallel plates which have longitudinal ribs. Such ribs may serve as strengtheners or may be conduits or heating elements.

We shall use an eigenfunction expansion and point-match method. The earliest use of this method for parallel flow in a nonregular boundary was perhaps due to Sparrow and Loeffler (1959) although for certain geometries such a method may not be suitable (Sparrow, 1966). See Shah and London (1978) for a review.

On the other hand, complex regions may be decomposed into contiguous simpler sub-regions. The solutions to each sub-region can then be matched along their common boundary. The idea, in one dimension, is similar to using cubic splines to determine the shape of a constrained elastic rod. The earliest use of patching solutions of two dimensional regions seems to be due to Weil (1951) who studied the Stokes flow into a gap. Other applications include the works of Zarling (1976), Chen (1980), Trogdon and Joseph (1982). The matching processes used, however, were somewhat complicated.

The combination of using eigenfunction expansions for separate sub-regions and using simple point-match along the boundary is well suited for the present problem. As we shall see later, the algorithm is simple and highly efficient.

Formulation

Figure 1(a, b) shows the cross sections of channels with ribs on both plates. The ribs are evenly spaced either symmetrically or staggered. We normalize all lengths by the half period H. Due to symmetry only the L shaped regions in Fig. 1(c, d) need to be considered.

Let the velocity be normalized by H^2G/μ where G is the axial pressure gradient and μ is the viscosity. The governing equation for velocity w is the Poisson equation

$$\nabla^2 w = -1 \tag{1}$$

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The L shaped regions are further sub-divided into two rectangular pieces I and II. For the symmetric case (Fig. 1(c)) the boundary conditions are

$$w_I\left(x, \frac{a}{2}\right)=0, \quad \frac{\partial w_I}{\partial y}(x, 0)=0, \quad \frac{\partial w_I}{\partial x}(0, y)=0 \tag{2}$$

$$w_I(c, y)=0, \quad \frac{a}{2}-b<y<\frac{a}{2} \tag{3}$$

$$w_{II}\left(x, \frac{a}{2}-b\right)=0, \quad \frac{\partial \omega_{II}}{\partial y}(x, 0)=0, \quad \frac{\partial w_{II}}{\partial x}(1, y)=0 \tag{4}$$

Also both w_I and w_{II} and their derivatives match on their common boundary at $x=c$. For the staggered case (Fig. 1(d)) the boundary conditions are

$$w_I(x, a)=0, \quad w_I(x, 0)=0, \quad w_I(0, y)=w_I(0, a-y) \tag{5}$$

$$w_I\left(c-\frac{1}{2}, y\right)=0, \quad a-b<y<a \tag{6}$$

$$w_{II}(x, a-b)=0, \quad w_{II}(x, 0)=0, \quad \frac{\partial w_{II}}{\partial x}\left(\frac{1}{2}, y\right)=0 \tag{7}$$

In addition, velocities and its derivatives should match at $x=c-1/2$.

Symmetric Case

We construct the following solutions which satisfy Eqs. (1), (2), (4)

$$w_I=\frac{1}{2}\left(\frac{a^2}{4}-y^2\right)+\sum_n A_n\cos(\alpha_n y)\,(e^{\alpha_n(x-c)}+e^{-\alpha_n(x+c)} \tag{8}$$

$$w_{II}=\frac{1}{2}\left[\left(\frac{a}{2}-b\right)^2-y^2\right] + \sum_m B_m\cos(\beta_m y)\,(e^{\beta_m(x-2+c)}+e^{-\beta_m(x-c)}) \tag{9}$$

Here A_n, B_m are constant coefficients and

$$\alpha_n\equiv(2n-1)\frac{\pi}{a}, \quad \beta_m\equiv\frac{(2m-1)\pi}{(a-2b)} \tag{10}$$

The other boundary conditions are Eq. (3) and

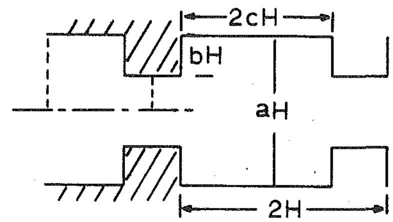


Fig. 1(*a*)

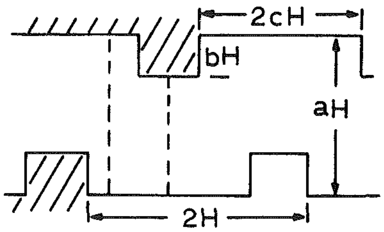


Fig. 1(*b*)

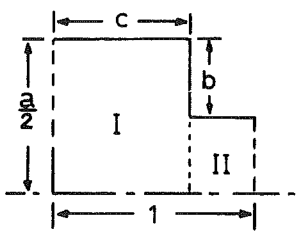


Fig. 1(*c*)

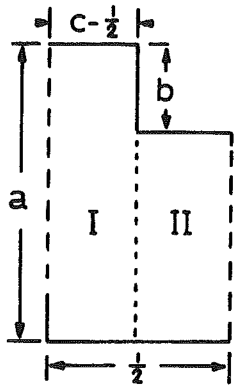


Fig. 1(*d*)

Fig. 1 (*a*, *b*) symmetric and staggered fins. (*c*, *d*) the corresponding computational regions.

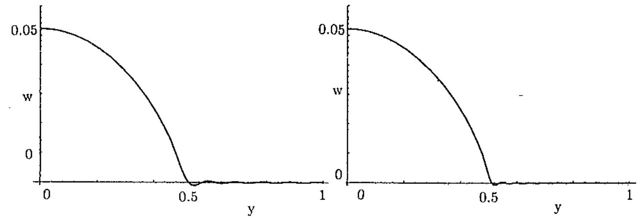


Fig. 2 Velocity along a mixed boundary (Fig. 3, at $x=0.5$). left: $N=20$, right: $N=40$.

$$w_I(c, y) = w_{II}(c, y), \quad 0<y<\frac{a}{2}-b \tag{11}$$

$$\frac{\partial w_I}{\partial x}(c, y) = \frac{\partial w_{II}}{\partial x}(c, y), \quad 0<y<\frac{a}{2}-b \tag{12}$$

We choose N points along the boundary at $x=c$

$$y_i = (i-1)a/(2N), \quad i=1, \ldots, N \tag{13}$$

and truncate A_n to N terms and B_m to M terms where

$$M = \text{Int}[N(1-2b/a)]+1 \tag{14}$$

Equations (3), (11), (12) become

$$\sum_1^N A_n \cos(\alpha_n y_i)(1+e^{-2\alpha_n c}) = \frac{1}{2}\left(y_i^2 - \frac{a^2}{4}\right), \quad i=M+1 \text{ to } N \tag{15}$$

$$\sum_1^N A_n \cos(\alpha_n y_i)(1+e^{-2\alpha_n c}) - \sum_1^M B_m \cos(\beta_m y_i) \times (1+e^{-2\beta_m(1-c)}) = \frac{1}{2}(b^2-ab) \quad i=1 \text{ to } M \tag{16}$$

$$\sum_1^N A_n \alpha_n \cos(\alpha_n y_i)(1-e^{-2\alpha_n c}) + \sum_1^M B_m \beta_m \cos(\beta_m y_i)(1-e^{-2\beta_m(1-c)}) = 0, \quad i=1 \text{ to } M \tag{17}$$

The linear system of $M+N$ equations and $M+N$ unknowns are solved for A_n and B_m. The accuracy of the solution depends on the number of points N. Figure 2 shows the velocity distribution along the boundary at $x=c$ ($a=1$, $b=0.25$, $c=0.5$). The error in w is less than 2 percent (as compared to the exact no-slip condition for $0.5<y<1$) for $N=20$ and decreases to 1 percent for $N=40$. Convergence is fairly fast. The corresponding constant velocity lines are shown in Fig. 3.

The mean velocity is integrated analytically:

Nomenclature

a = normalized distance between plates
A_n = constant coefficients
b = normalized height of ribs
B_m = constant coefficients
c = $1-$normalized width of ribs
C_n = constant coefficients
D_h = hydraulic diameter
$f\text{Re}$ = friction factor—Reynolds number product
G = axial pressure gradient
H = half period of ribs
i = integer index
m = integer index
M = integer
n = integer index
N = integer
u = normalized mean velocity
v = normalized mean velocity
v' = dimensional mean velocity
w = normalized velocity
x = Cartesian axis
y = Cartesian axis
z = $y - a/2$
α_n = eigenvalue Eq. (10)
β_m = eigenvalue Eq. (10) or Eq. (23)
γ_n = eigenvalue Eq. (23)
μ = viscosity

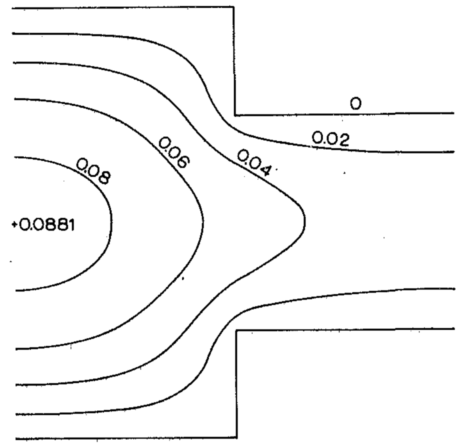


Fig. 3 Constant velocity lines for $a=1$, $b=0.25$, $c=0.5$

$$v=\frac{1}{\left[\frac{a}{2}-b(1-c)\right]}\left[\int_0^c\int_0^{a/2} w_I dydx+\int_c^1\int_0^{a/2-b} w_{II} dydx\right]$$

$$=\frac{1}{\left[\frac{a}{2}-b(1-c)\right]}\left\{\frac{1}{24}[ca^3+(1-c)(a-2b)^3]+\sum_1^N\frac{A_n}{(\alpha_n)^2}(-1)^{n+1}(1-e^{-2\alpha_n c})+\sum_1^M\frac{B_m}{(\beta_m)^2}(-1)^{m+1}[1-e^{-2\beta_m(1-c)}]\right\} \quad (18)$$

Staggered Case

The staggered case is slightly more complicated. We note that the solution to Region I, Fig. 1(*d*), has polar symmetry such that

$$w_I(x, z)=w_I(-x, -z) \quad (19)$$

where

$$z\equiv y-\frac{a}{2} \quad (20)$$

Guided by the boundary conditions, the following solutions are constructed

$$w_I=\frac{1}{2}\left(\frac{a^2}{4}-z^2\right)+\sum A_n\cos(\alpha_n z)[e^{\alpha_n(x-c+1/2)}+e^{-\alpha_n(x+c-1/2)}]+\sum C_n\sin(\gamma_n z)[e^{\gamma_n(x-c+1/2)}-e^{-\gamma_n(x+c-1/2)}] \quad (21)$$

$$w_{II}=\frac{1}{2}y(a-b-y)+\sum B_m\sin(\beta_m y)[e^{\beta_m(x+c-3/2)}+e^{-\beta_m(x-c+1/2)}] \quad (22)$$

Here A_n, C_n, B_m are to be determined and

$$\alpha_n=(2n-1)\frac{\pi}{a},\quad \gamma_n=2n\frac{\pi}{a},\quad \beta_m=\frac{m\pi}{a-b} \quad (23)$$

Similarly, $2N$ points are chosen along $x=c$

$$y_i=ia/(2N+1) \quad (24)$$

$$M=\text{Int}[2N(1-b/a)] \quad (25)$$

The algebraic equations to be solved are obtained from the conditions along the common boundary:

$$\sum_1^N A_n\cos\left[\alpha_n\left(y_i-\frac{a}{2}\right)\right][1+e^{-\alpha_n(2c-1)}]+\sum_1^N C_n\sin\left[\gamma_n\left(y_i-\frac{a}{2}\right)\right][1-e^{-\gamma_n(2c-1)}]=\frac{1}{2}y_i(y_i-a),\quad i=M+1\text{ to }2N \quad (26)$$

$$\sum_1^N A_n\cos\left[\alpha_n\left(y_i-\frac{a}{2}\right)\right][1+e^{-\alpha_n(2c-1)}]+\sum_1^N C_n\sin\left[\gamma_n\left(y_i-\frac{a}{2}\right)\right][1-e^{-\gamma_n(2c-1)}]-\sum_1^M B_m\sin(\beta_m y_i)(1+e^{-2\beta_m(1-c)})=\frac{-b}{2}y_i,\quad i=1\text{ to }M \quad (27)$$

$$\sum_1^N A_n\alpha_n\cos\left[\alpha_n\left(y_i-\frac{a}{2}\right)\right][1-e^{-\alpha_n(2c-1)}]+\sum_1^N C_n\gamma_n\sin\left[\gamma_n\left(y_i-\frac{a}{2}\right)\right][1+e^{-\gamma_n(2c-1)}]+\sum_1^M B_m\beta_m\sin(\beta_m y_i)(1-e^{-2\beta_m(1-c)})=0,\quad i=1\text{ to }M \quad (28)$$

There are $2N+M$ equations and $2N+M$ unknowns. The mean velocity is then

$$v=\frac{1}{\left[\frac{a}{2}-b(1-c)\right]}\left\{\frac{1}{12}\left[\left(c-\frac{1}{2}\right)a^3+(1-c)(a-b)^3\right]+\sum_1^N\frac{2(-1)^{n+1}A_n}{\alpha_n^2}[1-e^{-\alpha_n(2c-1)}]+\sum_1^M\frac{B_m[1-(-1)^m]}{\beta_m^2}[1-e^{-2\beta_m(1-c)}]\right\} \quad (29)$$

Results and Discussion

The conventional parameter to quantify flow rate is the friction factor—Reynolds number product

$$f\text{Re}=\frac{GD_h^2}{2\mu v'}=\frac{2[a-2b(1-c)]^2}{(1+b)^2 v} \quad (30)$$

Here D_h is the hydraulic diameter equal to 4 (area)/(wetted perimeter). Since the presence of ribs greatly affects both flow rate and wetted perimeter, we find fRe, as defined, is an unsuitable parameter since the results would be unwieldy. This fact was also noted by Sparrow and Chukaev (1980) who used finite differences to solve the flow between a ribbed plate and a flat plate. In our results we shall quantify flow rate by the mean velocity normalized by the distance between the plates

$$u=\frac{v'}{(aH)^2G/\mu}=\frac{v}{a^2} \quad (31)$$

Figure 4 shows the mean velocity u for the symmetric case. When the ribs are absent ($b=0$), $u=0.08333$, from Eq. (30) $f\text{Re}=24$ for the flow between parallel flat plates. When $b/a=0.5$, opposite ribs touch and our results agree with published results of the flow through rectangular conduits. The mean velocity increases slightly when b is close to $0.5a$ is due to the elimination of the almost stagnant cross-sectional region between opposite ribs. The total flow still decreases with increased

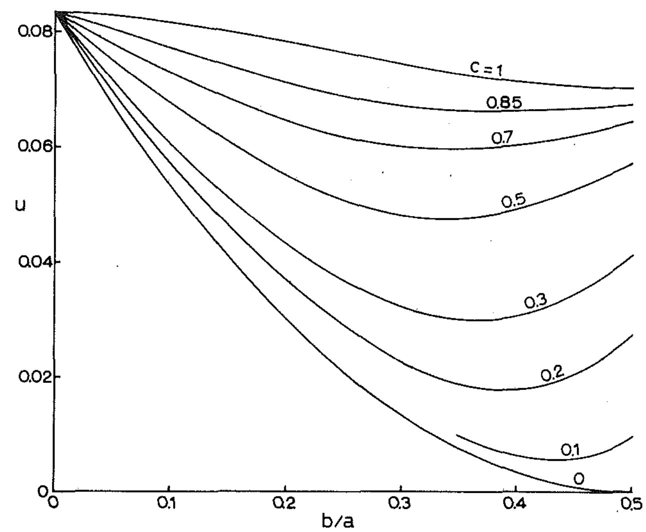


Fig. 4(*a*)

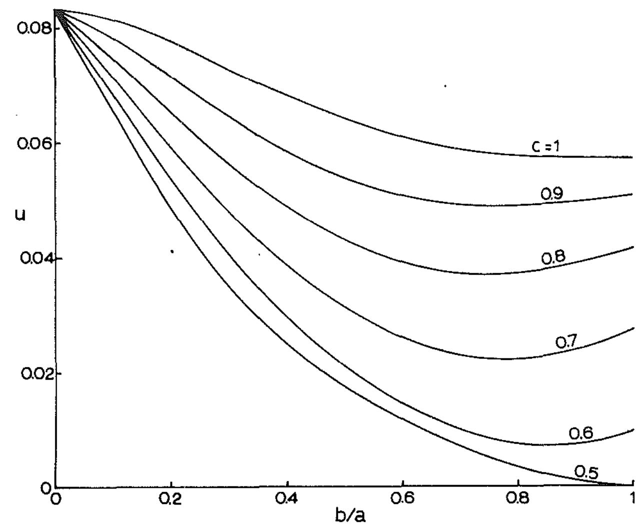


Fig. 5(*a*)

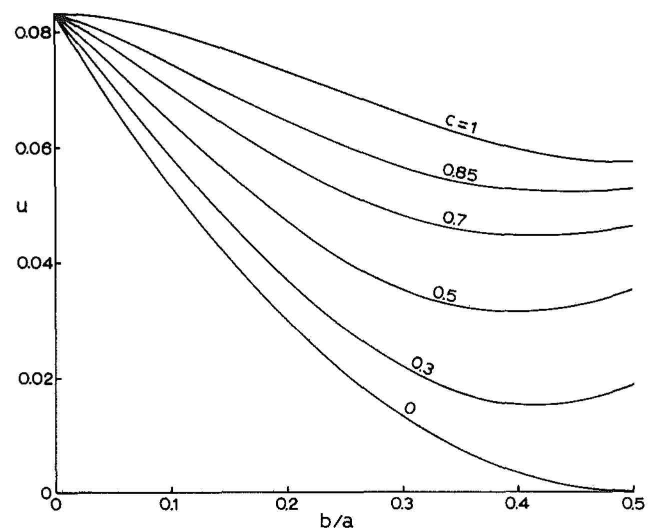
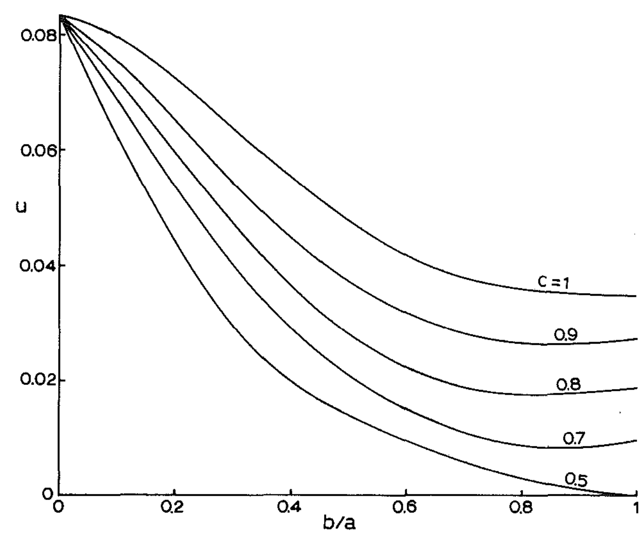


Fig. 4(*b*)



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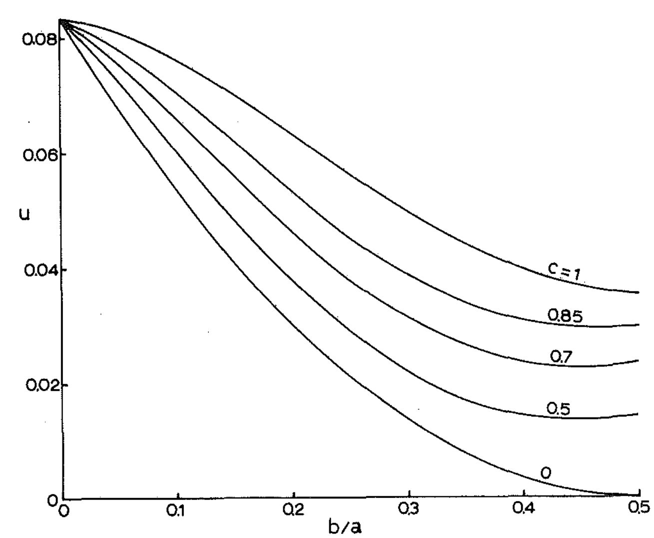

Fig. 4(*c*)

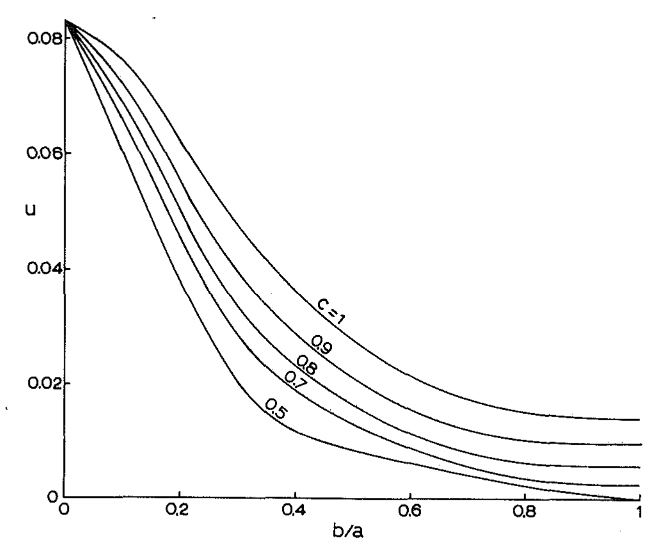

Fig. 5(*c*)

Fig. 4 Mean velocity for the symmetric case (*a*) $a = 0.5$, (*b*) $a = 1$, (*c*) $a = 2$.

Fig. 5 Mean velocity for the staggered case (*a*) $a = 0.5$, (*b*) $a = 1$, (*c*) $a = 2$.

b. Of interest is the $c=1$ case where the ribs become thin fins. The $c=0$ (all fin) case is equivalent to a narrower channel without fins.

Figure 5 shows the results for the staggered case where the ribs can reach the opposite plate ($b=a$) and thus c is limited to the range of 0.5 to 1. For the same values of a, b, c the staggered case yields lower flow than the symmetric case. The dependence of flow with phase shift (which changes neither cross sectional area nor wetted perimeter) was also noted by Wang (1976) for two plates with small wavy corrugations.

Conclusions

Our method used here is quite advantageous in comparison to direct numerical integration. The number of computations is less than the square root of that of finite differences. Also the double integral for flow can be evaluated analytically, eliminating another source of error. The method is simple enough such that other results, for values of the parameters not presented here, can be easily computed.

We find the cross-sectional area and the wetted perimeter alone could not have defined the flow rate. The use of $f\mathrm{Re}$ as the dependent variable is also unsuitable (compare Fig. 3.42 of Shah and Bhatti, 1987). We advocate using the normalized mean velocity to quantify the flow through a duct with complex geometry.

References

Blevins, R. D., 1984, *Applied Fluid Dynamics Handbook*, VanNostrand, New York.

Chen, Y. Z., 1980, "Solutions of Torsion Crack Problems of a Rectangular Bar by Harmonic Function Continuation Technique," *Engineering Fracture Mechanics*, Vol. 13, pp. 193–212.

Shah, R. K., and Bhatti, M. S., 1987, *Laminar Convective Heat Transfer in Ducts, in Handbook of Single-Phase Convective Heat Transfer*, S. Kakac, R. K. Shah, W. Aung, eds., Wiley, New York, Chapter 3.

Shah, R. K., and London, A. L., 1978, *Laminar Flow Forced Convection in Ducts. Advances in Heat Transfer*, Supplement 1, Academic, New York, Chapter 4.

Sparrow, E. M., 1966, Discussion to K. C. Cheng, "Analog Solution of Laminar Heat Transfer in Non Circular Ducts by Moiré and Point-Matching," ASME *Journal of Heat Transfer*, Vol. 88, p. 182.

Sparrow, E. M., and Chukaev, A., 1980, "Forced-Convection Heat Transfer in a Duct having Spanwise-Periodic Rectangular Protuberances," *Numerical Heat Transfer*, Vol. 3, pp. 149–167.

Sparrow, E. M., and Loeffler, A. L., 1959, "Longitudinal Laminar Flow Between Cylinders Arranged in Regular Array," *AIChE Journal*, Vol. 5, pp. 325–330.

Trogdon, S. A., and Joseph, D. D., 1982, "Matched Eigenfunction Expansions for Slow Flow Over a Slot," *Journal of NonNewtonian Fluid Mechanics*, Vol. 10, pp. 185–213.

Wang, C. Y., 1976, "Parallel Flow Between Corrugated Plates," *ASCE Journal of the Engineering Mechanics Division*, Vol. 102, pp. 1088–1090.

Weil, H., 1951, "On the Extrusion of a Very Viscous Liquid," ASME *Journal of Applied Mechanics*, Vol. 18, pp. 267–272.

Zarling, J. P., 1976, "An Analysis of Laminar Flow and Pressure Drop in Complex Shaped Ducts," ASME Journal of Fluids Engineering, Vol. 98, pp. 702–706.