Research Article

Finite-Time Stabilization of Uncertain Switched Positive Linear Systems with Time-Varying Delays

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This paper is concerned with finite-time stabilization (FTS) analysis for a class of uncertain switched positive linear systems with time-varying delays. First, a new definition of finite-time boundedness (FTB) is introduced for switched positive system. This definition can simplify FTS analysis. Taking interval and polytopic uncertainties into account, a robust state feedback controller is built such that the switched positive linear system is finite-time bounded. Finally, an example is employed to illustrate the validities of obtained results.

1. Introduction

As a special kind of hybrid dynamical systems, the switched system consists of finite subsystems and a switching law. The switching law orchestrates the switches between subsystems [1]. The switched positive system means that its output and state are nonnegative whenever the initial condition and input are nonnegative [2]. The switched positive systems are widely applied in practice, such as formation flying [3], communication system [4], viral mutation [5], and vehicle control [6].

In control theory, the stability and stabilization are basic problems for switched system. Many valuable results on these problems have been obtained, such as stability [7–12] and stabilization [13–15]. Most of the existing results focus on Lyapunov asymptotic stability (LAS). As we all know, LAS is defined over infinite-time interval. However, due to large state amplitude in transient process, some asymptotic stable systems may be useless [16]. Thus, FTB and FTS are introduced to analyze the transient performance of system. FTB implies that system state remains within a prescribed bound over a fixed interval. FTS means that designing a stabilization controller guarantees that the system is finite-time bounded. The main concern of FTB is the behavior of system state over a fixed finite time interval. It should be pointed out that there is not necessary relation between FTB

and LAS. In practice, the finite-time bounded systems are employed in many fields, such as networked control systems [17] and network congestion control [18].

Since switched system contains several dynamic subsystems and a switching law, it is difficult to analyze FTB and FTS. Recently, these problems have been extensively studied. Based on the average dwell-time approach, Lin et al. investigated FTB for switched systems with fixed delays [19]. However, the time delay is always time-varying in practice. Then the application of obtained results is limited. Taking time-varying delay into account, M. Xiang and Z. Xiang analyzed FTB and FTS for switched linear systems [20]. In the above-mentioned literatures, the switching instants must be scheduled in advance. Therefore the corresponding switching law is time-dependent. However, the switching instants are determined by system state in some switched system. In this case, the obtained approaches may be useless. In [21], a state-dependent switching law was designed such that the switched linear system was finite-time bounded. In order to simplify the mathematical calculation, Zhong and Chen adopted a new method to analyze the FTB [22]. In fact, the switches between subsystems produce great effect on system state. Thus, for the problems of FTB and FTS, it is significant to investigate this issue. In [23], Xiang and Xiao investigated the switching behavior's impacts on system state. Furthermore, a state feedback controller was built such that the system was

finite-time bounded. In practice, the researchers sometimes care about not only whether the system could be stabilized but also whether the stabilization time meets the requirements. In [24], some criteria were proposed on designing an optimal switching law. Naveed et al. investigated the problem of finite-time H-infinity state estimation for switched systems under asynchronous switching [25], and the developed method is useful for FTB and FTS. There are some other latest literatures [26–29] related to FTB and FTS, and the references are cited therein.

About the mentioned literatures, we need to pay attention to the following facts. First, switched positive system is an important kind of switched system. However, there are few works on its FTB and FTS. Second, most of the obtained results are based on a common assumption that the norm of external disturbance is bounded over infinite time interval. But the external disturbance usually does not satisfy this strict assumption. In this case, the obtained results may be unsuitable for application. Thus, it is very significant to reduce the restriction on external disturbance. Besides, the definition of FTB contains a product term of gain matrix and state vector; therefore complex mathematical calculations are unavoidable. Finally, the state of switched positive system is nonnegative. Then, whether a more concise definition of FTB could be proposed for switched positive system to simplify the analysis of FTS is a problem that naturally arises.

Motivated by the above considerations, we investigate FTS for uncertain switched positive linear systems. The main object of this paper is to design a robust state feedback controller such that the system is finite-time bounded. The main contributions and novelties of this paper are summarized as follows: (1) for switched positive system, a much more concise definition of FTB is introduced to simplify mathematical calculation; (2) a state feedback controller is built for the uncertain switched positive linear system with time-varying delay; (3) different from most of other results, the disturbance is only required to be bounded in this paper; (4) taking the interval and polytopic uncertainties into account, the obtained controller is robust to uncertainties.

The remainder of the paper is organized as follows. In Section 2, some necessary definitions and lemmas are introduced. In Section 3, a robust stabilization controller is designed for uncertain switched positive linear systems with time-varying delays. A numerical example is given in Section 4. The paper is concluded by Section 5.

Notation. R^n (R_+^n) denotes the n-dimensional real (positive) vector space; M represents Metzler matrices whose off-diagonal entries are nonnegative; $A \succ 0$ ($A \succeq 0$, $A \prec 0$) means that all elements of matrix A are positive (nonnegative, negative); let $\underline{m} = [1, \ldots, m]$, where m is positive integer.

2. Preliminaries and Problem Formulation

Consider a switched linear system described as

$$\dot{x}(t) = (A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t)$$
$$+ (B_{\sigma(t)} + \Delta B_{\sigma(t)})x(t - h(t))$$

$$+ F_{\sigma(t)}u(t) + G_{\sigma(t)}w(t)$$

$$0 \le h(t) \le d, \quad 0 \le t,$$

$$x(t) = \varphi(t) \quad t \in [-d, 0), \tag{1}$$

where $x(t) \in R^n$ represents the system state; $\sigma(t)$ is the switching law which is a piecewise continuous function with $\sigma(t) \in \underline{m}$; the switched system is composed of m subsystems; if $\sigma(t) = i$, the ith subsystem is activated; $A_{\sigma(t)}$, $B_{\sigma(t)}$, $F_{\sigma(t)}$, and $G_{\sigma(t)} \succeq 0$ are known constant matrices with appropriate dimension; $\Delta A_{\sigma(t)}$ and $\Delta B_{\sigma(t)}$ represent uncertainties satisfying $(B_{\sigma(t)} + \Delta B_{\sigma(t)}) \succeq 0$; $h(t)$ represents the time-varying delay; $u(t)$ is the control input; $w(t)$ denotes the external disturbance; $\varphi(t) \succeq 0$ is the continuous vector-valued initial function over $t \in [-d, 0)$; and d is a positive constant.

The state feedback controller is given as follows:

$$u(t) = H_{\sigma(t)}x(t), \tag{2}$$

where $H_{\sigma(t)}$ is the gain matrix to be determined.

The main task is to choose appropriate $H_{\sigma(t)}$ such that system (1) is finite-time bounded.

Define

$$\widetilde{A}_{\sigma(t)} = (A_{\sigma(t)} + \Delta A_{\sigma(t)}),$$
$$\widetilde{B}_{\sigma(t)} = (B_{\sigma(t)} + \Delta B_{\sigma(t)}). \tag{3}$$

Applying (2) to (1), the following closed-loop system is obtained:

$$\dot{x}(t) = (\widetilde{A}_{\sigma(t)} + F_{\sigma(t)}H_{\sigma(t)})x(t)$$

$$+ \widetilde{B}_{\sigma(t)}x(t - h(t)) + G_{\sigma(t)}w(t)$$

$$0 \le h(t) \le d, \quad 0 \le t, \tag{4}$$

$$x(t) = \varphi(t) \quad t \in [-d, 0).$$

Assume that system (1) satisfies the following assumptions.

Assumption 1 (see [30]). The external disturbance $w(t)$ satisfies

$$0 \preceq w(t) \preceq \gamma. \tag{5}$$

Remark 2. In most of the existing literatures, the external disturbance should satisfy

$$\int_0^\infty \omega^T(t)\omega(t)dt \le \theta \quad 0 < \theta. \tag{6}$$

Obviously, condition (6) is much stricter than (5). In Assumption 1, the external disturbance is only required to be bounded. In addition, Assumption 1 conforms much more to the practical situation.

Assumption 3. The time-varying delay satisfies

$$\dot{h}(t) \le \rho, \quad 0 \le h(t) \le d, \ \rho \le 1. \tag{7}$$

Assumption 4. The state trajectory is continuous everywhere. Furthermore, the switching number of $\sigma(t)$ is finite over fixed finite-time interval.

　　Next, Lemma 5 is introduced for subsequent analysis.

Lemma 5. *System (3) is positive if and only if $(\widetilde{A}_{\sigma(t)} + F_{\sigma(t)}H_{\sigma(t)}) \in M$, $\widetilde{B}_{\sigma(t)} \succeq 0$, $G_{\sigma(t)} \succeq 0$, and initial states $x_0 \succeq 0$ and $w(t) \succeq 0$ hold for $\sigma(t) \in \underline{m}$ [26].*

　　Definition 6 is proposed as follows.

Definition 6 (see [31]). Forgiven positive constants $C_1 < C_2$, T_f, γ, and a switching signal $\sigma(t)$, if

$$\|x(t_0)\| \le C_1 \implies \|x(t)\| < C_2 \quad \forall t \in [0, T_f], \ \forall w(s) \le \gamma, \tag{8}$$

then system (3) is said to be finite-time bounded with respect to $(C_1, C_2, T_f, \gamma, \sigma(t))$, where $\|x\| = \sum_{i=1}^{n} |x_i|$; x_i is the ith element of vector x. In this paper, we specify $C_1 = \sup_{t \in [-d, 0)} \|x(t)\|$.

Remark 7. According to the definition of FTB in most of other literatures, system (3) should satisfy

$$x^T(t_0)Rx(t_0) \le C_1 \implies x^T(t)Rx(t) < C_2 \quad \forall t \in [0, T_f]. \tag{9}$$

Note the differences between (8) and (9). First, compared with (9), the gain matrix R is removed from (8). This improvement brings in much convenience for subsequence process. Some complex matrix transformations could be avoided. Besides, if we choose the definition (9), it is not convenient to adopt multiply copositive Lyapunov-Krasovskii function to analyze FTB due to the term of $x^T(t)x(t)$. Therefore, $x^T(t)x(t)$ is replaced by $\|x(t)\|$ which can simplify subsequent mathematical analysis.

Remark 8. The essence of FTB lies in that the system state must remain within the prescribed bound over fixed interval. Since both (8) and (9) can guarantee it, the two different definitions are consistent in essence.

Definition 9. For $T \ge t \ge 0$, let $N_{\sigma(t)}(t, T)$ denote the switching number of $\sigma(t)$ over $(t, T]$. If

$$N_{\sigma(t)}(t, T) \le N_0 + \frac{T - t}{\tau_a} \tag{10}$$

holds for $\tau_a \ge 0$ and an integer $N_0 \ge 0$, then τ_a is called an average dwell-time [16].

3. Main Result

Assume that the uncertainties in system (3) are interval or polytopic uncertainties. First, system (3) with interval uncertainties is considered. In this case, we build a robust state feedback controller such that system (3) is finite-time bounded.

Theorem 10. *Assume that $\underline{A}_i \preceq \widetilde{A}_i \preceq \overline{A}_i$, $0 \preceq \underline{B}_i \preceq \widetilde{B}_i \preceq \overline{B}_i$ for $i \in \underline{m}$. $\underline{A}_i$, $\overline{A}_i$, $\underline{B}_i$, and $\overline{B}_i$ are known matrices. For given positive constants C_1, C_2, T_f, γ, λ, and $\beta > 1$, if there exist vectors $v_i \succeq 0$, $\eta_i \succeq 0$, $\zeta_i \succeq 0$, and ξ_i such that*

$$\mathrm{diag}\,\{\psi_1, \ldots, \psi_n, \phi_1, \ldots, \phi_n\} < 0, \tag{11}$$

$$v_i \le \beta v_j, \qquad \eta_i \le \beta \eta_j, \qquad \zeta_i \le \beta \zeta_j \tag{12}$$

$$\text{for } i, j \in [1, 2, \ldots, m],$$

$$(\underline{A}_i + F_i H_i)_{l,p} > 0 \quad l, p \in \underline{n}, \ l \ne p, \tag{13}$$

and the average dwell-time satisfies

$$\tau_a > \frac{T_f \ln \beta}{\ln C_2 + \ln K_4 - \ln(\Phi_1 + \Phi_2)}, \tag{14}$$

then under the controller $u(t) = H_{\sigma(t)}x(t)$ with gain matrix $H_{\sigma(t)}$ satisfying

$$\xi_i = (F_i H_i)^T v_i, \tag{15}$$

system (3) is finite-time bounded with respect to $(C_1, C_2, T_f, \gamma, \sigma(t))$ under the switching law (14).

　　$\psi_j = (\overline{A}_i^T)_j v_i + \xi_{i,j} - \lambda v_{i,j} + \eta_{i,j} + d\zeta_{i,j}$, $\phi_j = (\overline{B}_i^T)_j v_i - (1 - \rho)\eta_{i,j}$, $i \in \underline{m}$, $j \in \underline{n}$; $(\overline{A}_i^T)_j$ *denotes the jth row of matrix $\overline{A}_i^T$; $\xi_{i,j}$, $v_{i,j}$, $\eta_{i,j}$, and $\zeta_{i,j}$ represent the jth element of vectors ξ_i, v_i, η_i, and ζ_i, respectively; $(\underline{A}_i + F_i H_i)_{l,p}$ is the lth row and pth column element of matrix $(\underline{A}_i + F_i H_i)$; $K_0 = \max_{i \in \underline{m}} \lambda(G_i)$, and $\lambda(G_i)$ denotes all eigenvalues of matrix G_i; $K_1 = \max(v_{i,j})$, $K_2 = \max(\eta_{i,j})$, $K_3 = \max(\zeta_{i,j})$, and $K_4 = \min(v_{i,j})$ for $i \in \underline{m}$, $j \in \underline{n}$. In addition, $\Phi_1 = (K_1 + de^{\lambda d}K_2 + d^2 e^{\lambda d}K_3)\sup_{t \in [-d, 0)} \|x(t)\|$; $\Phi_2 = e^{2\lambda T_f}T_f n\gamma K_0 K_1$.*

Remark 11. Due to the existence of interval uncertainties, system matrices $\widetilde{A}_i$ and $\widetilde{B}_i$ are unknown matrices. However, their value ranges usually are known in practice. In fact, $\overline{A}_i$ and $\overline{B}_i$ denote the upper bounds of $\widetilde{A}_i$ and $\widetilde{B}_i$, respectively; $\underline{A}_i$ and $\underline{B}_i$ are the lower bounds of $\widetilde{A}_i$ and $\widetilde{B}_i$, respectively.

Proof. Since (13) holds, $(\underline{A}_i + F_i H_i) \in M$. Noting that $\underline{A}_i \preceq \widetilde{A}_i$, then $(\widetilde{A}_i + F_i H_i) \in M$. Besides, $\widetilde{B}_i \succeq 0$, $G_i \succeq 0$, $\varphi(t) \succeq 0$, and $0 \preceq w(t)$. By Lemma 5, system (3) is a switched positive system.

　　Construct multiply copositive Lyapunov-Krasovskii function for system (3) as follows:

$$V(t) = V_i(t) = V_{i,1}(t) + V_{i,2}(t) + V_{i,3}(t), \tag{16}$$

where

$$V_{i,1}(t) = x^T(t)v_i,$$

$$V_{i,2}(t) = \int_{t-h(t)}^{t} e^{\lambda(t-s)}x^T(s)\eta_i ds, \tag{17}$$

$$V_{i,3}(t) = \int_{-d}^{0} \int_{t+\theta}^{t} e^{\lambda(t-s)}x^T(s)\zeta_i ds d\theta.$$

Taking the derivatives of $V_{i,1}(t)$, $V_{i,2}(t)$, and $V_{i,3}(t)$ with respect to t along the trajectory of system (3) on $t \in [t_k, t_{k+1})$, we have

$$\dot{V}_{i,1}(t) = \lambda V_{i,1}(t) + x^T(t)\left(\widetilde{A}_i + F_i H_i\right)^T v_i$$
$$+ x^T(t - h(t))\widetilde{B}_i^T v_i + w^T(t) G_i^T v_i - \lambda x^T(t) v_i$$
$$\leq \lambda V_{i,1}(t) + x^T(t)\left(\overline{A}_i + F_i H_i\right)^T v_i$$
$$+ x^T(t - h(t))\overline{B}_i^T v_i + w^T(t) G_i^T v_i - \lambda x^T(t) v_i,$$
$$\dot{V}_{i,2}(t) = \lambda V_{i,2}(t) + x^T(t)\eta_i - \left(1 - \dot{h}(t)\right)e^{\lambda h(t)} x^T(t - h(t))\eta_i$$
$$\leq \lambda V_{i,2}(t) + x^T(t)\eta_i - (1 - \rho)x^T(t - h(t))\eta_i,$$
$$\dot{V}_{i,3}(t) = \lambda V_{i,3}(t) + d x^T(t)\zeta_i - \int_{t-d}^{t} e^{\lambda(t-s)} x^T(s)\zeta_i ds$$
$$\leq \lambda V_{i,3}(t) + d x^T(t)\zeta_i - \int_{t-d}^{t} x^T(s)\zeta_i ds. \tag{18}$$

From (18), it follows that

$$\dot{V}_i(t) - \lambda V_i(t) \leq x^T(t)\left(\left(\overline{A}_i + F_i H_i\right)^T v_i - \lambda v_i + \eta_i + d\zeta_i\right)$$
$$+ x^T(t - h(t))\left(\overline{B}_i^T v_i - (1 - \rho)\eta_i\right)$$
$$+ w^T(t) G_i^T v_i - \int_{t-d}^{t} x^T(s)\zeta_i ds. \tag{19}$$

It is derived from (11) that

$$x^T(t)\left[\left(\overline{A}_i + F_i H_i\right)^T v_i - \lambda v_i + \eta_i + d\zeta_i\right] < 0,$$
$$x^T(t - h(t))\left[\overline{B}_i^T v_i - (1 - \rho)\eta_i\right] < 0. \tag{20}$$

Applying (20) to (19), we get

$$\dot{V}_i(t) - \lambda V_i(t) < w^T(t) G_i^T v_i. \tag{21}$$

Since $(d/dt)(e^{-\lambda t} V_i(t)) = e^{-\lambda t}[\dot{V}_i(t) - \lambda V_i(t)]$, (21) is transformed into

$$\frac{d}{dt}\left(e^{-\lambda t} V_i(t)\right) < e^{-\lambda t} w^T(t) G_i^T v_i. \tag{22}$$

Let t_k stand for the instant of the Kth switching and t_{k^-} denote the instant just before t_k. Integrating from t_k to t on both sides of (22), then

$$V_i(t) < e^{\lambda(t-t_k)} V_i(t_k) + \int_{t_k}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds. \tag{23}$$

Noting (12) and the continuity of $x(t)$, then

$$V_i(t) < \beta e^{\lambda(t-t_k)} V_i(t_{k^-}) + \int_{t_k}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds,$$
$$V_i(t_{k^-}) < e^{\lambda(t_k - t_{k-1})} V_i(t_{k-1}) + \int_{t_{k-1}}^{t_k} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds. \tag{24}$$

It is obtained via (24) that

$$V_i(t) < \beta^2 e^{\lambda(t-t_{k-1})} V_i\left(t_{(k-1)^-}\right)$$
$$+ \beta e^{\lambda(t-t_k)} \int_{t_{k-1}}^{t_k} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$+ \int_{t_k}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds. \tag{25}$$

Assume that the switching number of $\sigma(t)$ over $[0, T_f]$ is N. We get

$$V_i(t) < \beta^N e^{\lambda t} V_i(0)$$
$$+ \beta^N e^{\lambda(t-t_1)} \int_{t_0}^{t_1} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$+ \cdots + \beta e^{\lambda(t-t_k)} \int_{t_{k-1}}^{t_k} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$+ \int_{t_k}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$< \beta^N e^{\lambda T_f} V_i(0) + \beta^N e^{\lambda T_f} \int_{t_0}^{t_1} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$+ \cdots + \beta^N e^{\lambda T_f} \int_{t_{k-1}}^{t_k} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$+ \beta^N e^{\lambda T_f} \int_{t_k}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds \tag{26}$$
$$= \beta^N e^{\lambda T_f} V_i(0) + \beta^N e^{\lambda T_f} \int_{t_0}^{t} w^T(s)e^{\lambda(t-s)} G_i^T v_i ds$$
$$< \beta^N e^{\lambda T_f} V_i(0) + \beta^N e^{2\lambda T_f} K_0 \int_{t_0}^{t} w^T(s) v_i ds$$
$$< \beta^N e^{\lambda T_f} V_i(0) + \beta^N e^{2\lambda T_f} K_0 \int_{t_0}^{t} n\gamma K_1 ds$$
$$= \beta^N e^{\lambda T_f} V_i(0) + \beta^N e^{2\lambda T_f} T_f n\gamma K_0 K_1.$$

According to the definition of $V_i(0)$,

$$V(0) = x^T(0) v_{\sigma(0)} + \int_{-h(0)}^{0} e^{-\lambda s} x^T(s)\eta_{\sigma(0)} ds$$
$$+ \int_{-d}^{0} \int_{\theta}^{0} e^{-\lambda s} x^T(s)\zeta_{\sigma(0)} ds d\theta$$
$$\leq x^T(0)\left[K_1^T, \ldots, K_1^T\right]^T$$
$$+ \int_{-d}^{0} e^{-\lambda s} x^T(s)\left[K_2^T, \ldots, K_2^T\right]^T ds$$
$$+ \int_{-d}^{0} \int_{\theta}^{0} e^{-\lambda s} x^T(s)\left[K_3^T, \ldots, K_3^T\right]^T ds d\theta$$

$$\leq x^T(0)\left[K_1^T,\ldots,K_1^T\right]^T$$

$$+ e^{\lambda d}\int_{-d}^{0} x^T(s)\left[K_2^T,\ldots,K_2^T\right]^T ds$$

$$+ e^{\lambda d}\int_{-d}^{0}\int_{\theta}^{0} x^T(s)\left[K_3^T,\ldots,K_3^T\right]^T ds d\theta$$

$$\leq K_1\|x(0)\| + de^{\lambda d}K_2\sup_{t\in[-d,0]}\|x(t)\|$$

$$+ d^2 e^{\lambda d}K_3\sup_{t\in[-d,0]}\|x(t)\|$$

$$\leq \left(K_1 + de^{\lambda d}K_2 + d^2 e^{\lambda d}K_3\right)\sup_{t\in[-d,0]}\|x(t)\|. \tag{27}$$

Substituting (27) into (26) yields

$$V_i(t) < \beta^N e^{\lambda T_f}\left(K_1 + de^{\lambda d}K_2 + d^2 e^{\lambda d}K_3\right)\sup_{t\in[-d,0]}\|x(t)\| \tag{28}$$

$$+ \beta^N e^{2\lambda T_f}T_f n\gamma K_0 K_1.$$

Obviously,

$$x^T(t)\left[K_4^T,\ldots,K_4^T\right]^T < x^T(t)v_i < V_i(t). \tag{29}$$

It is derived from (28) and (29) that

$$\|x(t)\|$$

$$< \left(\beta^N e^{\lambda T_f}\left(K_1 + de^{\lambda d}K_2 + d^2 e^{\lambda d}K_3\right)\sup_{t\in[-d,0]}\|x(t)\|\right.$$

$$\left. + \beta^N e^{2\lambda T_f}T_f n\gamma K_0 K_1\right) \tag{30}$$

$$\times (K_4)^{-1}$$

$$= \frac{\beta^N(\Phi_1 + \Phi_2)}{K_4}.$$

On the other hand, (31) could be derived from (14):

$$\frac{T_f \ln\beta}{\ln C_2 + \ln K_4 - \ln(\Phi_1 + \Phi_2)} < \tau_a,$$

$$\frac{T_f}{\tau_a} < \frac{\ln C_2 + \ln K_4 - \ln(\Phi_1 + \Phi_2)}{\ln\beta},$$

$$N < \frac{\ln C_2 + \ln K_4 - \ln(\Phi_1 + \Phi_2)}{\ln\beta}, \tag{31}$$

$$N\ln\beta < \ln\frac{C_2 K_4}{\Phi_1 + \Phi_2},$$

$$\beta^N\frac{\Phi_1 + \Phi_{24}}{K_4} < C_2.$$

Consequently, $\|x(t)\| < C_2$.

According to Definition 6, the system (3) is finite-time bounded. Thus, the designed controller is the desired controller. This completes the proof of Theorem 10. □

Remark 12. Note (11) and (15). If the ξ_i in (11) is replaced by $(F_i H_i)^T v_i$, then the inequality (11) is not LMIS. In this case, we could not solve (11) via MATLAB. Thus, we should analyze (11) directly. Note that $\psi_j = (\overline{A}_i^T)_j v_i + \xi_{i,j} - \lambda v_{i,j} + \eta_{i,j} + d\zeta_{i,j}$ and $\phi_j = (\overline{B}_i^T)_j v_i - (1-\rho)\eta_{i,j}$. $(\overline{A}_i^T)_j$, $(\overline{B}_i^T)_j$, λ, and d are known; $v_i, \xi_{i,j}, v_{i,j}, \eta_{i,j}$, and $\zeta_{i,j}$ are unknown. We find that ψ_j and ϕ_j do not contain any product term between two unknown variables. Therefore inequality (11) is LMIS. ξ_i and v_i could have been obtained by solving (11). Then substitute them into (15). In this way, H_i is obtained finally.

Remark 13. Compared with other literatures, such as [19, 21, 22], the derivation process of Theorem 10 in this paper is much more concise. The main reason lies in that the definition on FTB in this paper is much more concise compared with other literatures, such as [19, 21, 22]. Thus complex matrix translations are eliminated. Besides, the description on FTB is also very concise in [20, see Definition 5]. However, the physical meaning of Definition 6 in this paper is much clearer.

Next, the attention is focused on designing a state feedback controller such that the system (3) with polytopic uncertainties is finite-time bounded.

Theorem 14. *Assume that* $\Gamma_i = \{\widetilde{A}_i^p, p \in \underline{n}\}$ *and* $\Theta_i = \{\widetilde{B}_i^p, p \in \underline{n}\}$, $i \in \underline{m}$; Φ_i *and* Θ_i *denote the convex hull of the vertices* $\widetilde{A}_i^p$ *and* $\widetilde{B}_i^p$, *respectively.* $\widetilde{A}_i^p$ *and* $\widetilde{B}_i^p \succeq 0$ *are known matrices. For given positive constants* $C_1, C_2, T_f, \gamma, \lambda$, *and* $\beta > 1$, *if there exist vectors* $v_i \succeq 0$, $\eta_i \succeq 0$, $\zeta_i \succeq 0$, *and* ξ_i *such that*

$$\text{diag}\{\psi_1,\ldots,\psi_n,\phi_1,\ldots,\phi_n\} < 0, \tag{32}$$

$$v_i \leq \beta v_j, \qquad \eta_i \leq \beta\eta_j, \qquad \zeta_i \leq \beta\zeta_j, \tag{33}$$

$$\text{for } i,j \in [1,2,\ldots,m],$$

$$\left(\widetilde{A}_i^p + F_i H_i\right)_{l,j} > 0 \quad l,j \in \underline{n}, \ l \neq j, \tag{34}$$

and the average dwell-time satisfies

$$\tau_a > \frac{T_f \ln\beta}{\ln C_2 + \ln K_4 - \ln(\Phi_1 + \Phi_2)}, \tag{35}$$

then under the controller $u(t) = H_{\sigma(t)}x(t)$ *with gain matrix* $H_{\sigma(t)}$ *satisfying*

$$\xi_i = (F_i H_i)^T v_i, \tag{36}$$

system (3) is finite-time bounded with respect to $(C_1, C_2, T_f, \gamma, \sigma(t))$ *under the switching law (35).*

$\psi_j = (\widetilde{A}_i^p)_j^T v_i + \xi_{i,j} - \lambda v_{i,j} + \eta_{i,j} + d\zeta_{i,j}$, $\phi_j = (\widetilde{B}_i^p)_j^T v_i - (1-\rho)\eta_{i,j}$, $i \in \underline{m}$, $j \in \underline{n}$; $(\widetilde{A}_i^p)_j^T$ *denotes the jth row of matrix* $(\widetilde{A}_i^p)^T$; $v_{i,j}, \eta_{i,j}$, *and* $\zeta_{i,j}$ *represent the jth element of vectors* v_i, η_i, *and* ζ_i, *respectively;* $(\widetilde{A}_i^p + F_i H_i)_{l,p}$ *is the lth row and pth column*

element of matrix $(\widetilde{A}_i^p + F_iH_i)$; let $K_0 = \max_{i\in\underline{m}}\lambda(G_i)$; $\lambda(G_i)$ denotes all eigenvalues of matrix G_i; $K_1 = \max(v_{i,j})$, $K_2 = \max(\eta_{i,j})$, $K_3 = \max(\zeta_{i,j})$, and $K_4 = \min(v_{i,j})$ for $i \in \underline{m}$, $j \in \underline{n}$; In addition, $\Phi_1 = (K_1 + de^{\lambda d}K_2 + d^2e^{\lambda d}K_3)\sup_{t\in[-d,0)}\|x(t)\|$; $\Phi_2 = e^{2\lambda T_f}T_fn\gamma K_0K_1$.

Proof. By the property of convex hull, $\widetilde{A}_i = \sum_{p=1}^n \gamma_p\widetilde{A}_i^p$, $0 < \gamma_p < 1$, and $\sum_{p=1}^n \gamma_p = 1$ for $i \in \underline{m}$. Furthermore, $(\widetilde{A}_i^p + F_iH_i)_{l,j} > 0$ $l, j \in \underline{n}$ and $l \neq j$; therefore $(\sum_{p=1}^n \gamma_p\widetilde{A}_i^p + \sum_{p=1}^n \gamma_pF_iH_i)_{l,j} > 0$ which implies that $(\widetilde{A}_i + F_iH_i) \in M$. Besides, $\widetilde{B}_i \succeq 0$, $G_i \succeq 0$, $\varphi(t) \succeq 0$ and $0 \preceq w(t)$. By Lemma 5, system (3) is a switched positive system.

Construct multiply copositive Lyapunov-Krasovskii functions which have the same structure of (16). Taking the derivatives of $V_{i,1}(t)$, $V_{i,2}(t)$, and $V_{i,3}(t)$ with respect to t along the trajectory of system (3) on $t \in [t_k, t_{k+1})$, we get

$$\dot{V}_i(t) - \lambda V_i(t) \le x^T(t)\left(\left(\widetilde{A}_i + F_iH_i\right)^Tv_i - \lambda v_i + \eta_i + d\zeta_i\right)$$
$$+ x^T(t - h(t))\left(\widetilde{B}_iv_i - (1-\rho)\eta_i\right)$$
$$+ w^T(t)G_i^Tv_i - \int_{t-d}^t x^T(s)\zeta_i ds. \tag{37}$$

It is derived from (32) that

$$\left(\gamma_p\widetilde{A}_i^p + \gamma_pF_iH_i\right)^Tv_i + \gamma_p\left(-\lambda v_i + \eta_i + d\zeta_i\right) \prec 0. \tag{38}$$

It follows that

$$\left(\sum_{p=1}^n\gamma_p\widetilde{A}_i^p + \sum_{p=1}^n\gamma_pF_iH_i\right)^Tv_i + \sum_{p=1}^n\gamma_p\left(-\lambda v_i + \eta_i + d\zeta_i\right) \prec 0. \tag{39}$$

Note that $\sum_{p=1}^n\gamma_p = 1$ and $\widetilde{A}_i = \sum_{p=1}^n\gamma_p\widetilde{A}_i^p$; then

$$\left(\widetilde{A}_i + F_iH_i\right)^Tv_i + \left(-\lambda v_i + \eta_i + d\zeta_i\right) \prec 0. \tag{40}$$

Similarly, we get

$$\widetilde{B}_i^Tv_i - (1-\rho)\eta_i \prec 0. \tag{41}$$

Applying (40) and (41) to (37) yields

$$\dot{V}_i(t) - \lambda V_i(t) < w^T(t)G_i^Tv_i. \tag{42}$$

The remainder of the proof is similar to (23)–(31) and omitted. Therefore the system (3) is finite-time bounded under the designed controller. This completes the proof of Theorem 14. □

Remark 15. Inequalities (11), (13), (32), and (34) are LMI. Once inequality (11) or (32) is resolved, the values of ξ_i and v_i are obtained. Substituting $\xi_{i,j}$ and v_i into $\xi_i = (F_iH_i)^Tv_i$, H_i is obtained. Furthermore, if inequality (13) or (34) is satisfied, then H_i is the desired gain matrix. By this way, the desired controller is built such that system (3) is finite-time bounded.

4. Numerical Example

A numerical example is given to illustrate the validities of the obtained results in this section. Consider the following system with interval uncertainties:

$$\dot{x}(t) = (A_{\sigma(t)} + \Delta A_{\sigma(t)})x(t) + (B_{\sigma(t)} + \Delta B_{\sigma(t)})x(t - h(t))$$
$$+ F_{\sigma(t)}u(t) + G_{\sigma(t)}w(t),$$
$$x(t) = \varphi(t) \quad t \in [-d, 0), \tag{43}$$

where

$$A_1 = \begin{bmatrix} -7 & 7 \\ 5 & -7 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.2 & 0.9 \\ 0.1 & 0.3 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.15 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -7 & 7 \\ 3 & -6 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 1 & 0.4 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.15 \end{bmatrix},$$
$$\overline{A}_1 = \begin{bmatrix} -6 & 8 \\ 6 & -6 \end{bmatrix}, \qquad \underline{A}_1 = \begin{bmatrix} -8 & 6 \\ 4 & -8 \end{bmatrix},$$
$$\overline{A}_2 = \begin{bmatrix} -6 & 8 \\ 4 & -5 \end{bmatrix}, \qquad \underline{A}_2 = \begin{bmatrix} -8 & 6 \\ 2 & -7 \end{bmatrix}, \tag{44}$$
$$\overline{B}_1 = \begin{bmatrix} 0.3 & 1 \\ 0.2 & 0.4 \end{bmatrix}, \qquad \underline{B}_1 = \begin{bmatrix} 0.1 & 0.8 \\ 0 & 0.2 \end{bmatrix},$$
$$\overline{B}_2 = \begin{bmatrix} 0.3 & 0.2 \\ 1.1 & 0.5 \end{bmatrix}, \qquad \underline{B}_2 = \begin{bmatrix} 0.1 & 0 \\ 0.9 & 0.3 \end{bmatrix},$$
$$h(t) = 0.1t, \qquad \rho = 0.1, \qquad T_f = 10, \qquad d = 1,$$
$$\varphi(t) = \begin{bmatrix} 55 & 55 \end{bmatrix}^T, \qquad \omega^T(t) = \begin{bmatrix} \sin(t) + 1 & 2\cos(t) \end{bmatrix},$$
$$n = 2, \qquad \gamma = 2.$$

Obviously, $C_1 = 110$. Specify $\lambda = 0.1$, $\beta = 1.05$, $C_2 = 200$. Substituting these known parameters into (11) and (12) leads to feasible solutions that

$$v_1 = \begin{bmatrix} 15.475 & 25.888 \end{bmatrix}^T, \qquad v_2 = \begin{bmatrix} 27.692 & 13.415 \end{bmatrix}^T,$$
$$\eta_1 = \begin{bmatrix} 40.366 & 48.327 \end{bmatrix}^T, \qquad \eta_2 = \begin{bmatrix} 46.951 & 41.572 \end{bmatrix}^T,$$
$$\zeta_1 = \begin{bmatrix} 33.802 & 33.802 \end{bmatrix}^T, \qquad \zeta_2 = \begin{bmatrix} 33.802 & 33.802 \end{bmatrix}^T,$$
$$\xi_1 = \begin{bmatrix} -168.9 & -81.818 \end{bmatrix}^T, \qquad \xi_2 = \begin{bmatrix} 0.71057 & -262.3 \end{bmatrix}^T. \tag{45}$$

Then, it follows that

$$K_0 = 0.2781, \qquad K_1 = 27.692,$$
$$K_2 = 48.327, \qquad K_3 = 33.802, \tag{46}$$
$$K_4 = 13.415.$$

Noting that $\xi_i = (F_i H_i)^T v_i$, we obtain

$$H_1 = \begin{bmatrix} -2.1843 & 0 \\ 0 & 2.6114 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 2.4413 & 0 \\ 0 & 2.5197 \end{bmatrix},$$

$$\underline{A}_1 + F_1 H_1 = \begin{bmatrix} -5.8157 & 6 \\ 4 & -6.6943 \end{bmatrix} \in M,$$

$$\underline{A}_2 + F_2 H_2 = \begin{bmatrix} -6.77935 & 6 \\ 2 & -4.4803 \end{bmatrix} \in M. \tag{47}$$

Consequently, H_1 and H_2 meet the requirements of the design. Then the controller $u(t) = H_{\sigma(t)} x(t)$ is obtained.

On the other hand, from (14), we get

$$\tau_a > 0.5469. \tag{48}$$

According to Theorem 10, if we choose a $\sigma(t)$ which satisfies $\tau_a > 0.5469$, then system (43) would be finite-time bounded with respect to $(C_1, C_2, T_f, \gamma, \sigma(t))$ under the designed controller $u(t) = H_{\sigma(t)} x(t)$.

Finally, the simulation of system (43) with the designed controller is performed. The simulation results are shown in Figures 1–3. The switching law of system (43) is shown in Figure 1. Obviously, the switching times of system (43) is 7 over $[0, 10]$. Thus, $\tau_a = (10/7) > 0.5469$ and meets the requirement of (14). In Figure 2, we could find that every element of $x(t)$ falls in the range $[0, 100]$. Therefore, every element of system state is bounded which implies that the system state is bounded over $[0, 10]$. In other words, system (43) is FTB. The same conclusion could be reached via Figure 3. From Figure 3, it is easy to find $\|x\| < C_2 = 200$ over $[0, 10]$. Thus, the state of system (43) remains within the prescribed bound over $[0, 10]$. Therefore, under the designed controller, system (43) is indeed bounded over $[0, 10]$.

In addition, though system (43) is finite-time bounded, its state trajectory is an oscillating curve over $[0, 10]$. Thus, system (43) is not Lyapunov stable over $[0, 10]$. It is illustrated that FTB is different from LAS. There is not a necessary relation between them.

5. Conclusions

This paper focuses on the FTS analysis for a class of uncertain switched positive linear systems. Based on multiply copositive Lyapunov-Krasovskii functions and average dwell-time method, a robust state feedback controller is built such that the uncertain system is finite-time bounded. The innovation points lie in that a concise definition of FTB is proposed for switched positive system. Since the new definition is adopted, the subsequent mathematical derivation is greatly simplified. Finally, a numerical example illustrates the validities of the obtained results and the difference between LAS and FTB.

However, multiply copositive Lyapunov-Krasovskii functions in this paper are traditional. In the literature [32, 33], barrier Lyapunov functions and P-times differentiable unbounded functions are presented, respectively. In future work, we will try to utilize the two new kinds of functions to deal with the FTS problem for switched positive system and
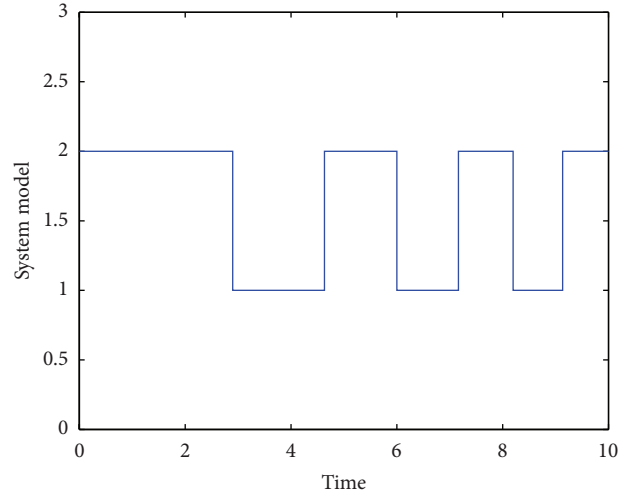

Figure 1: The figure of switching law.

    x_1 : the 1st element of system state
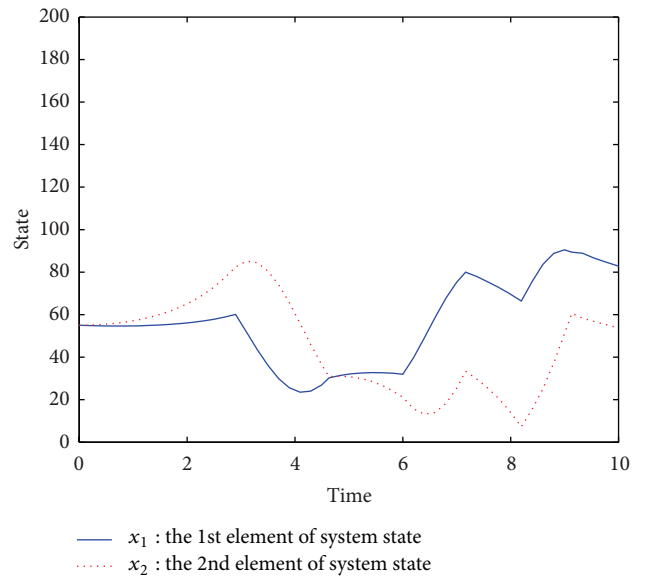 x_2 : the 2nd element of system state

Figure 2: The figure of system state.

make a comparative analysis with the traditional functions. In addition, interval and polytopic uncertainties could not cover all kinds of the uncertainties in practice. Therefore, we will analyze the problem of FTS for switched positive linear system with arbitrary uncertainties. As mentioned in Section 1, the switching behavior is an important factor for the problems of FTB and FTS. For FTB and FTS, it is one of the critical issues to analyze asynchronous switching [34], slow switching, and fast switching impacts on the system state. In this paper, the issue has not been discussed in detail. Therefore we will try to investigate this issue for switched positive system in further study.
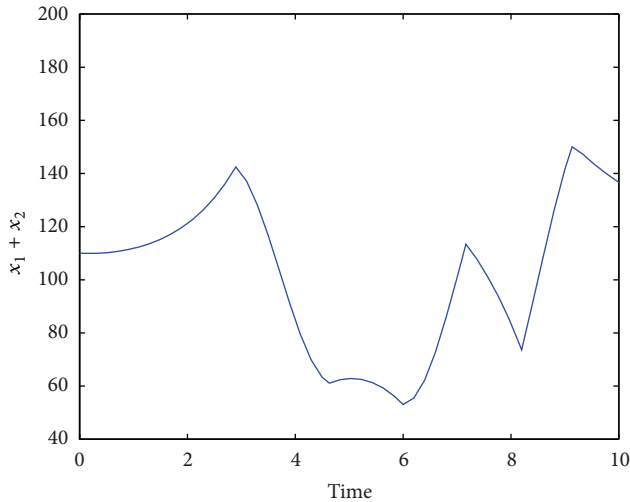
FIGURE 3: The figure of $\|x\|$.

Conflict of Interests

The authors (Tianjian Yu, Yanke Zhong, Tefang Chen, and Chunyang Chen) declare that there is not any conflict of interests regarding the publication of this paper.

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