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Vortex Motion in the Ionosphere and Nonlinear Transport

DAVID J. SOUTHWOOD[1] AND MARGARET G. KIVELSON[2]

Institute of Geophysics and Planetary Physics, University of California, Los Angeles

The relation between vorticity and ionospheric flow patterns is investigated by using a fluid mechanics approach in place of the more customary electromagnetic approach. The focus on the fluid features is justified by the observation that in the incompressible limit appropriate to the ionosphere, vorticity can be regarded as the source of the flow field. We show how vorticity can be introduced into the flow by local ionospheric conditions. However, in the cases of greatest interest, the vorticity is imposed by external sources, which can be in the magnetosphere or in the solar wind. As an important application, we consider traveling ionospheric vortices propagating around the polar cap boundary. We show that such traveling disturbances transport both momentum and magnetic flux in the direction of their phase velocity, typically antisunward. Like other intermittent disturbances of small scale, such as flux transfer events, individual traveling ionospheric vortices transport relatively little flux, but multiple disturbances could conceivably transport an important fraction of the polar cap magnetic flux from the dayside to the tail.

INTRODUCTION

In the past few years there has been increasing interest in the existence of large-scale traveling vortex structures which move tailward in the vicinity of the polar cap boundary. The initial reports [*Todd et al.*, 1986; *Lanzerotti et al.*, 1986,] were intent on identifying the ionospheric footprint of reconnection phenomena at the magnetopause and of flux transfer events (FTEs), which were believed to be examples of impulsive and localized reconnection. In fact, as we indicate below, there now exists a large literature [*Friis Christensen et al.*, 1988; *Bering et al.*, 1988, *Lockwood et al.*, 1990; *Glassmeier et al.*, 1989; *Lanzerotti et al.*, 1990; *Farrugia et al.*, 1989; *Glassmeier and Heppner*, 1992] on the occurrence of large amplitude vortex like motions near the polar cap boundary that resemble the footprints of FTEs predicted in theoretical works like those of *Southwood* [1985, 1987], *Saunders* [1989] and *Lee and Fu* [1985]. However there also is a substantial body of evidence that the events are often unlikely to be associated with reconnection. Theorists have gone on to predict a different geometry for FTEs [*Southwood et al.*, 1988; *Scholer*, 1988] and by implication a different footprint is predicated [*Lockwood et al.*, 1990]. Thus, the generation mechanism may be uncertain, but the existence of traveling vortical flows in the ionosphere is well established. Hence it is worthwhile to consider the ionospheric observations themselves and investigate the flux and momentum transfer properties of vortices like those observed. We show that any traveling vortex in the ionosphere transports both momentum and flux in the direction of travel. Our conclusion is that the flux transferred in such transient disturbances can be significant, a conclusion that is completely independent of the details of the mechanism by which the signatures are excited.

[1]Also at Department of Physics, Imperial College of Science, Technology and Medicine, London, England.

[2]Also at Department of Earth and Space Sciences, University of California, Los Angeles.

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Traveling ionospheric vortices (TIVs) have a horizontal scale of the order of 300-2000 km, and travel away from the noon meridian at speeds ranging from ~1 to 5 km/s (and in one event, as high as 10 km/s). The flow speed within the eddies is typically lower than the phase velocity but the peak flow can exceed 1 km/s. TIVs have been observed by radar backscattering and the associated magnetic perturbations have been tracked by arrays of magnetometers. In addition to flux transfer events (FTEs) or localized reconnection at the magnetopause generating the signatures, they could also be excited by solar wind pressure perturbations traveling along the magnetopause [*Kivelson and Southwood*, 1991, and references therein] or nonlinear Kelvin Helmholtz waves at the magnetopause [*Miura*, 1987; *Pu et al.*, 1990].

The vorticity in TIVs may be actually located in relatively thin sheets that map from the magnetopause or it may be more distributed. The precise distribution is of course a critical feature in determining the source of vorticity but matters little for our argument here. What is important is that the magnetic signature of such patterns over magnetometer arrays in Greenland and in Northern Scandinavia [*Friis-Christensen et al.*, 1988; *McHenry et al.*, 1988; *Glassmeier et al.*, 1989] appear to move away from noon with a longitudinal phase speed which is faster (approximately a few kilometers per second) than the flow speed deduced within the flow pattern but by not more than an order of magnitude. The vortex system has a twin structure. Not only is the speed of fluid motion different from that of the pattern but also the fluid motion in the center of the pattern is directed transverse to the motion of the pattern.

APPROACH AND PURPOSE OF PAPER

In this paper we develop a simple theoretical model for understanding the relationship between vorticity and ionospheric flow patterns using a mechanical approach instead of the usual electromagnetic approach. We then use the formalism to compute the transport associated with TIVs in the high latitude ionosphere, a problem that is rather straightforward in fluid mechanics but would be fairly obscure if tackled as an electromagnetic problem.

Ionospheric vorticity and field-aligned currents are intimately linked as has been described repeatedly [*Hasegawa and Sato*, 1979; *Sato*, 1982; *Vasyliunas*, 1984]. The fundamental role of field aligned currents in the coupling between the ionosphere and the magnetosphere is also well established [*e.g.*, *Vasyliunas*, 1972,]. In an earlier work [*Southwood and Kivelson*, 1991], we elaborated on the ionospheric vorticity specifically as a response to field aligned currents. Here we consider ionospheric flows in a more general sense.

A particularly simple flow regime analyzed in standard fluid mechanics texts is the incompressible limit ($\nabla \cdot \boldsymbol{u} = 0$). The flow induced in the Earth's ionosphere is very similar in that it must be magnetically incompressible; from this it follows that the flow perpendicular to the magnetic field is also incompressible. This is because the earth's field strength is so high that the ionospheric flow cannot significantly compress the Earth's magnetic field. Vorticity plays a very basic role in incompressible hydrodynamic flow. In a two dimensional flow, the magnitude of the Laplacian of the streamfunction is equal to the magnitude of the vorticity. The relationship between vorticity and flow in an incompressible flow is exactly analogous to the relation between electrical current and magnetic field. Thus the vorticity can be seen as the source of the flow field. With these ideas in mind, we derive a vorticity equation for a simple model ionosphere. The equation shows that in the absence of external sources, vorticity is conserved following the motion of the plasma. It also shows how vorticity is put into the flow by the local ionospheric conditions and how it is put in from higher altitude.

As a simple but important application of our approach, we examine the momentum and flux transfer in TIVs. We show that a TIV carries momentum in the direction of travel. What is remarkable is that by determining the speed of travel of TIVs in the ionosphere and the speed of their eddy motion one can quantify their effectiveness for transport without needing to specify what high-altitude magnetospheric process generates them. The observed TIVs typically move in the antisolar direction. Our work implies that diverse boundary perturbations can serve as a momentum source for the ionosphere and at the same time contribute to antisolar flux transfer provided only that they drive vorticity in the ionosphere.

A corollary of our work is that, as both pressure drops and pressure jumps have been predicted to excite antisolar traveling vortices, both transfer tailward momentum to the ionosphere and lead to flux transport to the magnetotail (a point originally made by *Dessler* [1964]). The types of pressure changes in the solar wind that we have in mind are of order $|\delta P|/P \approx 0.3$, capable of displacing the subsolar magnetopause by of order 0.5 R_E and typical changes take place in 5 to 10 minutes.

THE VORTICITY EQUATION

As our prime purpose is to set up a simple model for use in high latitudes where the magnetic field is nearly vertical, we model the ionosphere as flat with a vertical magnetic field. Generalization to a tilted field is straightforward but there is much simplification in presentation if we use the assumption. The high-latitude ionosphere is modeled as a thin collisional layer of thickness h, threaded by a magnetic field $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$. We treat only the horizontal (x,y) variation of physical parameters, but we ignore the variation in field associated with the horizontal variation of the Earth's dipole field. (This assumption is to be consistent with our assumption of a vertical field; on the scale of variation of the dipole field, the direction of the field also changes.) We assume that the ambient magnetic field does not vary significantly with height through the ionosphere and we replace quantities like mass density, ρ and pressure, p, by their height averaged values. The material is a low β plasma with $c_s \ll V_A$, where c_s is the sound speed ($c_s^2 = \gamma p/\rho$ with γ the ratio of specific heats, p the plasma pressure, and ρ the ion density) and V_A is the Alfvén speed ($V_A{}^2 = B^2/\mu_0\rho$). The dominant collision process is between the ions and the pervading neutral atmosphere. We shall parametrize the collisions by a mean collision frequency $\nu(x, y)$ defined by

$$\nu(x, y) = \frac{\int dz\, \rho(x, y, z)\, \nu(x, y, z)}{h\, \rho(x, y)}$$

where h is the height of the atmosphere. We now examine a set of equations describing two dimensional (horizontal) flow. The mass flux equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

or

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

where the differential operator, D/Dt, represents the (Langrangian) derivative taken along the path of a parcel of fluid and we assume no sources of ionization.

To a good approximation the magnetic field is linked to the flow and the electric field by the condition for perfect conductivity (also called Ohm's law)

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} \tag{3}$$

and the field is described as frozen into the flow. Faraday's law can then be written in the ionosphere as

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} + \boldsymbol{B} \nabla \cdot \boldsymbol{u} = \boldsymbol{B} \cdot \nabla \boldsymbol{u}$$

or

$$\frac{D\boldsymbol{B}}{Dt} + \boldsymbol{B} \nabla \cdot \boldsymbol{u} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} \tag{4}$$

We treat the ionosphere as a thin sheet and thus within the ionosphere there is no variation of the flow velocity along the background field. Thus

$$\frac{D\boldsymbol{B}}{Dt} + \boldsymbol{B} \nabla \cdot \boldsymbol{u} = 0 \tag{5}$$

The low Alfvén Mach number of ionospheric flow and the low pressure (low β) imply that only very small changes in magnetic field result from the plasma motion. As well, variations in $\boldsymbol{u}$ along the magnetic field are small with respect to the other terms in a two dimensional approximation. We thus replace equation (5) by

$$\nabla \cdot \boldsymbol{u} = 0 \tag{6}$$

In other words, the field is so strong that only magnetically incompressible flows are possible. However, by virtue of our assumption that the ionospheric plasma is confined to a thin sheet in the vertical direction, the condition also implies that the plasma is not compressed in the flow.

We next derive a momentum equation. In practice the high latitude ionosphere is driven by flow imposed from above, from the magnetosphere or even the solar wind. The momentum is conveyed along the field primarily electromagnetically. The magnetic flux of transverse momentum into the thin layer is $\Delta\boldsymbol{B}\, B/\mu_o$ where $\Delta\boldsymbol{B}$ is a transverse horizontal field perturbation just above the ionospheric layer. Including the vertical magnetic momentum flux, assuming that all the momentum is is absorbed in the layer, gives

$$\frac{D\boldsymbol{u}}{Dt}=\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nu\boldsymbol{u}-\frac{1}{\rho}\left[\nabla\left(\frac{B^2}{2\mu_o}\right)-B\frac{\Delta\boldsymbol{B}}{h\,\mu_o}\right] \tag{7}$$

where $\Delta\boldsymbol{B}$ is a transverse field displacement at the top of the layer and where it should be noted that the horizontal gradient in B^2/μ_o is that due to local ionospheric current flow and not, as remarked earlier, due to variation in the background dipole field. We have ignored the plasma pressure gradient in (7); this can also be neglected relative to magnetic forces in the momentum equation because in the ionosphere β (the ratio of thermal to magnetic pressure) is small.

The curl of equation (7) gives an expression for the time evolution of vorticity, $\boldsymbol{\Omega}=\nabla\times\boldsymbol{u}$

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}+\nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u})=$$

$$-\nu\boldsymbol{\Omega}+\boldsymbol{u}\times\nabla\nu+\frac{\nabla\rho}{\rho^2}\times\left[\nabla\left(\frac{B^2}{2\mu_o}\right)-\frac{B\Delta\boldsymbol{B}}{h\,\mu_o}\right]+\frac{B}{h\,\mu_o\rho}\nabla\times\Delta\boldsymbol{B} \tag{8}$$

where we have allowed for the possibility that ν varies spatially as it would if there were gradients in the neutral density.

As $\boldsymbol{u}\cdot\nabla\boldsymbol{u}=\nabla u^2/2-\boldsymbol{u}\times\boldsymbol{\Omega}$ and $\nabla\cdot\boldsymbol{\Omega}=\nabla\cdot\boldsymbol{u}=0$ and $\boldsymbol{\Omega}\cdot\nabla\boldsymbol{u}=0$ by assumption, it follows that

$$\nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u})=-\nabla\times(\boldsymbol{u}\times\boldsymbol{\Omega})=\boldsymbol{u}\cdot\nabla\boldsymbol{\Omega} \tag{9}$$

$\boldsymbol{\Omega}$ then satisfies

$$\frac{\partial\boldsymbol{\Omega}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{\Omega}=\frac{D\boldsymbol{\Omega}}{Dt}=$$

$$-\nu\boldsymbol{\Omega}+\boldsymbol{u}\times\nabla\nu+\frac{\nabla\rho}{\rho^2}\times\left[\nabla\left(\frac{B^2}{2\mu_o}\right)-B\frac{\Delta\boldsymbol{B}}{h\,\mu_o}\right]+\frac{B}{h\mu_o\rho}(\nabla\times\Delta\boldsymbol{B}) \tag{10}$$

Equation (10) is the vorticity equation for the ionosphere. However it is not in a very useful form because of the close to incompressible nature of the changes in field strength. We can eliminate the field pressure term by using the momentum equation (7) again, thus obtaining

$$\frac{D\boldsymbol{\Omega}}{Dt}+\nu\boldsymbol{\Omega}=\boldsymbol{u}\times\nabla\nu-\frac{\nabla\rho}{\rho}\times\left(\frac{D\boldsymbol{u}}{Dt}+\nu\boldsymbol{u}\right)+\frac{B}{h\,\mu_o\rho}\nabla\times\Delta\boldsymbol{B} \tag{11}$$

On rearrangement and substituting $\mu_o j_{\parallel}$ for $\nabla\times\Delta\boldsymbol{B}$, where $j_{\parallel}$ is the vertical current out of the ionosphere, one has

$$\frac{1}{\nu}\frac{D\boldsymbol{\Omega}}{Dt}+\boldsymbol{\Omega}=\boldsymbol{u}\times\nabla(\ln\nu\rho)+\frac{B}{h\,\rho\nu}j_{\parallel}+\frac{1}{\nu}\frac{D\boldsymbol{u}}{Dt}\times\frac{\nabla\rho}{\rho} \tag{12}$$

Equation (12) describes the creation and dissipation of vorticity in the flow. The time derivative in the equation is the Lagrangian derivative. Thus, when the right-hand side (the source) is zero, the left-hand terms show that the vorticity of any parcel of fluid is brought to an equilibrium value (or zero) on the time scale of the ion-neutral collision time, ν^{-1}, a relatively short time (<< 1 s) compared with high latitude ionospheric flow time scales in the terrestrial ionosphere. On time scales pertinent to the observations that motivate this study, one may take $D/Dt \ll \nu$. Then

$$\boldsymbol{\Omega}=\nabla\times\boldsymbol{u}=\boldsymbol{u}\times\nabla(\ln\nu\rho)+\frac{B}{h\,\rho\nu}j_{\parallel} \tag{13}$$

assuming for now that $j_{\parallel}$ does not vary with time. Note that the equilibrium vorticity and field aligned current are linked but that they are proportional only if $\boldsymbol{u}\times\nabla\nu\rho=0$. In this case $j_{\parallel}=0$ only if

$$\nabla\times(\rho\nu\boldsymbol{u})=0. \tag{14}$$

The meaning of this condition is most easily pictured if the flow is locally rectilinear and in the $\hat{x}$ direction. One can rewrite equation (14) as

$$\frac{\partial}{\partial y}(\rho\nu u)=0 \tag{15}$$

and it is clear that the vorticity imposed on the flow is simply that required to render the momentum deposition through collisions uniform transverse to the flow direction. Figure 1 illustrates the effect by showing how a localized region of enhanced $\rho\nu$ modifies

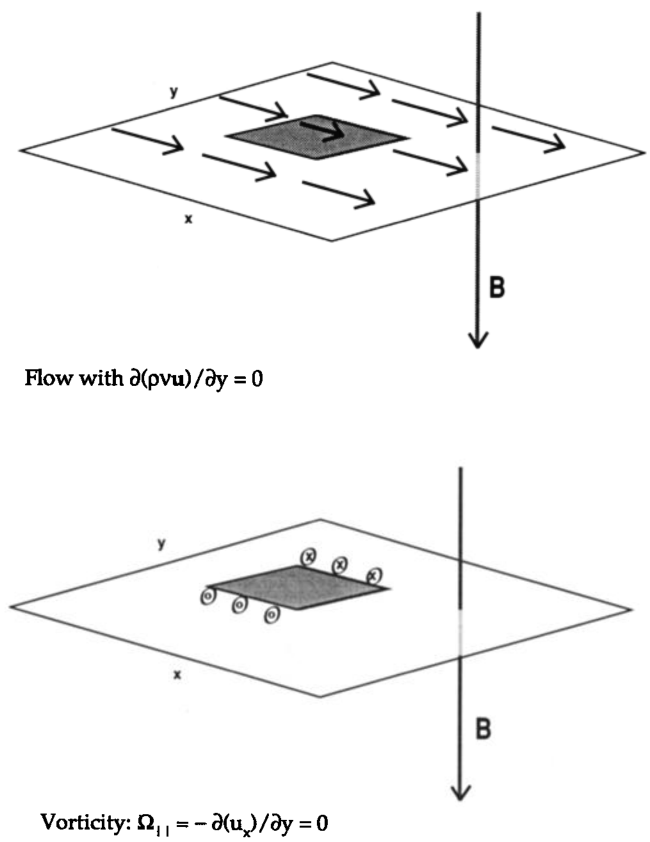

Fig. 1. Schematic of a region of the ionosphere threaded by a vertical magnetic field **B**. Shading identifies a region of enhanced $\rho\nu$. In the upper panel, flow in the x direction is represented by arrows. As the plasma enters the region of larger $\rho\nu$, it slows down. Thus the shaded region is bounded in y by a sheet of vorticity, as illustrated in the lower panel.

the flow in the ionosphere. The localized deceleration sets up ionospheric vorticity, but as no $\boldsymbol{j} \times \boldsymbol{B}$ forces are required to slow the plasma, there is no need to drive field-aligned current. However, the ionospheric vorticity couples into the magnetosphere or the solar wind, and it is the conditions of the external load that determine whether or not field-aligned currents develop in the final steady state.

As the incompressibility of the flow leads one to expect that in steady state there will be no gradients in ρ along the streamlines of the flow (in the absence of ionization sources) and as in such circumstances ν is unlikely to vary much along the streamlines, one sees the equilibrium flow is one in which the momentum deposition is very uniform.

As noted above, the vorticity on a flux tube will not remain constant as the tube moves if the parallel current $j_{\parallel}$ is not maintained steady. Variations of $j_{\parallel}$ depend on conditions in the plasma at higher altitudes on the flux tube in the magnetosphere or even the solar wind. *Southwood and Kivelson* [1991] (henceforth S&K91) have analysed the relationship between vorticity and field aligned current on the magnetospheric part of a flux tube (where the plasma is collision free). S&K91 shows that under fairly general assumptions the amount of field aligned current at a given point in the magnetosphere is proportional to the time integrated rate of variation of the vorticity along the field. The result can be understood by recognizing that the field is sheared or twisted if it carries a field aligned current and, where the field is frozen into the plasma flow, any component of vorticity along the field direction acts to twist (or untwist) the field at each point. Because the ionosphere is collisional and thus dissipative, a continual input of vorticity at high altitude is required to maintain an ionospheric flow. The time scale for dissipating the magnetospheric vorticity is the same as that calculated by S&K91 for the dissipation of field aligned current, namely

$$\tau = \mu_0 \Sigma_p L_{\parallel} \tag{16}$$

where Σ_p is the height integrated conductivity of the ionosphere and $L_{\parallel}$ is the parallel scale length of the flux tube. Using the notation of this paper, one has

$$\tau = \frac{\mu_0 \nu \rho L_{\parallel}}{B} \tag{17}$$

In the Earth's ionosphere, the time scale ranges from tens of seconds to many minutes. This time scale is very much longer than the time $1/\nu$ we identified earlier for dissipating ionospherically generated vorticity. Thus, for example, Alfvén waves carrying vorticity generated by an impulse in the magnetosphere or at its boundary make multiple bounces before damping out.

SOLVING FOR FLOW IN A HORIZONTALLY UNIFORM MODEL IONOSPHERE

If the terms on the left hand side of (13) are zero, that is, the ionosphere is sufficiently uniform and there is no field aligned current input, the equations governing the possible flow systems are

$$\nabla \cdot \boldsymbol{u} = 0 \text{ and } \nabla \times \boldsymbol{u} = 0 \tag{18}$$

In the nearly uniform case, these two equations represent continuity and the momentum equation, respectively, and thus should be capable of producing a unique flow pattern for a given set of boundary conditions. In a flow system governed by equation (18) it is well known that one may describe the flow by either a stream function ψ or a potential, φ. Both, ψ and φ satisfy Laplace's equation

$$\nabla^2 \psi = \nabla^2 \varphi = 0 \tag{19}$$

It is also well known that in an infinite medium or a closed system with no boundaries (such as a thin spherical shell) the only solution of Laplace's equation is the trivial one with ψ, φ constant, leading in either case to the solution, $\boldsymbol{u} = 0$.

For a nontrivial flow solution which closes everywhere or vanishes at large distances, one must have somewhere either the Laplacian of the stream function nonzero or the Laplacian of the velocity potential nonzero. The former condition corresponds to the vorticity being specified somewhere, the latter to there being a source of material somewhere. An alternative possibility is to introduce a boundary in the flow on which inhomogeneous boundary conditions are specified. It turns out that in such a case the boundary conditions are equivalent to there being a source of vorticity or a source of material at the boundary.

MOMENTUM AND FLUX TRANSPORT BY TIVs

We next illustrate the use of the simple model developed so far by examining momentum and flux transport by TIVs.

We continue to regard the ionosphere as an incompressible medium. In such a medium, motion can be instantaneously described by streamlines which all close. In a stationary vortex in an incompressible medium, the streamlines do not vary with time and the closed nature of the streamlines means that an element of material passes repeatedly through the same point. The maximum displacement possible is bounded by the scale size of the streamline which is also the orbit of the element. In such a flow there is no net transport in the mean. The situation is very different when the flow pattern is not time stationary. The streamlines vary continually with time and the trajectories of fluid elements are not the same as the instantaneous streamline pattern. Determining the motion of a fluid element in a time varying incompressible flow becomes more problematic.

When the time variation in the flow field detected at the position of a fluid element is important, the problem becomes nonlinear. The simplest analytic way to tackle the nonlinear problem is by a process of successive approximations. To zeroth order we can expect that the vortex provides no long term displacement. As for the steady vortex case, the displacement is bounded at any time by the spatial amplitude of the vortex streamlines.

An early nonlinear calculation of the analogous problem of a surface wave of sinusoidal form was done by *Stokes* [1848] and reported by *Lamb* [1932]. For a traveling surface wave, it is well known that to first order in the amplitude of the wave the motion of fluid elements is circular in a plane perpendicular to the plane of the boundary. Evidently after an integral number of periods the elements return to their original positions.

However, the flow field itself will vary on the scale of the fluid element orbit. Stokes showed that when the variation of the amplitude of the wave across the orbit of the fluid element in the wave itself is included, the orbit departs from circularity. For a surface wave, the element moves a greater distance in a time δt in the part of its orbit that is closer to the surface of the fluid than in the part of its orbit that is farther from the surface. Because it is moving in the direction of wave propagation at the "top" of its orbit (near the surface), there is a systematic motion imposed in the direction of the wave propagation. Our calculation is similar but is in no way restricted to the simple form of wave chosen by Stokes.

TRANSPORT OF MOMENTUM BY TRAVELING IONOSPHERIC VORTICES

We consider a model in which the (two dimensional) plasma occupies a half space bounded by the line $y=0$ which might be seen as representing the polar cap boundary. The essence of the TIV is its traveling nature and we represent this feature by linking the variation of the flow field in x (longitude) and t (time) to simulate the motion of a pattern along the direction of the boundary.

As vorticity is imposed from above, the curl and the divergence of the flow no longer satisfy equation (18) but instead satisfy

$$\nabla\cdot\boldsymbol{u}=0 \quad \text{and} \quad \nabla\times\boldsymbol{u}=\Omega(x-ct,y) \tag{20}$$

where c is a (phase) velocity of the traveling vortex which propagates in the x direction. The amplitude of the vorticity is regarded as given, that is, it is specified from above by conditions in the magnetosphere/solar wind.

We thus take the flow field to be of the form

$$\boldsymbol{u}(x,y,t)=\boldsymbol{u}(x-ct,y) \tag{21}$$

For the sake of illustration, we show the streamlines of the flow imposed in the ionosphere by a sinusoidal surface wave perturbation in Figure 2*a*. The orbits of fluid elements at different distances from the boundary in the linear (therefore circular) approximation are illustrated in Figure 2*b*.

Let us define $\xi(x_o,y_o,t)$ as the net displacement experienced by the fluid element present at (x_o,y_o) at time t in moving along its orbit from its initial position (x_i,y_i) at $t=-\infty$ prior to the onset of the disturbance. $\xi(x_o,y_o,t)$ is given by

$$\xi(x_o,y_o,t)=\int_{orbit}^{t} dt'\,\boldsymbol{u}(x(t'),y(t'),t') \tag{22}$$

where the integral is carried out over the orbit of the fluid element.

Equation (22) is a non linear integral equation because the path integral is taken along an orbit which is itself determined by the time history of the displacement. In the linear approximation, one calculates the displacement ignoring the variation in the flow field experienced by the fluid element. One can proceed to the second order by repeating the integration using the linear displacement to determine the path of the integral. One can repeat this to arbitrarily high order. In the sinusoidal surface wave example, one can see the effect of going to second order by noting that the radii of the circular orbits of the linear approximation become smaller as one moves away from the boundary (cf. Figure 2*b*). Accordingly in the nonlinear case, the radius of curvature of the orbit varies with y and a cycloidal motion develops as illustrated in Figure 2*c*.

An expansion of the integrand on the right hand side of equation (22) about (x_o,y_o) gives, to first order in displacement,

$$\xi_x(x_o,y_o,t)=$$

$$\int_{orbit}^{t} dt'\,\Big(u_x(x_o,y_o,t')+[x(t')-x_o]\frac{\partial u_x(x_o,y_o,t')}{\partial x}+[y(t')-y_o]\frac{\partial u_x(x_o,y_o,t')}{\partial y}\Big) \tag{23}$$

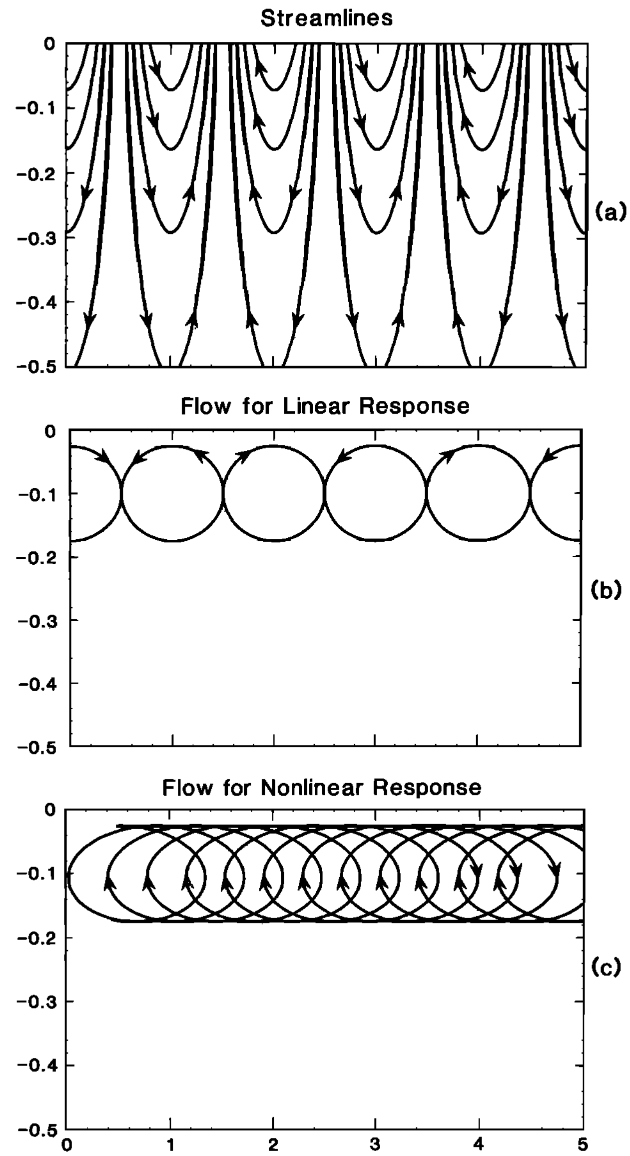

Fig. 2. Streamlines and flow vectors for a traveling surface wave imposed along the boundary of a semi-infinite ionosphere. In (a), streamlines are illustrated. In (b), the motion of selected fluid elements is shown in the linear approximation. In (c), the motion of selected fluid elements is shown in the nonlinear approximation discussed in the text.

The corresponding expression for ξ_y is

$$\xi_y(x_o,y_o,t)=\int_{orbit}^{t} dt'\,u_y(x(t'),y(t'),t')$$

$$=\int_{orbit}^{t} dt'\Big(u_y(x_o,y_o,t')+[x(t')-x_o]\frac{\partial u_y(x_o,y_o,t')}{\partial x}+[y(t')-y_o]\frac{\partial u_y(x_o,y_o,t')}{\partial y}\Big) \tag{24}$$

Evidently the expansion requires that the eddy speed $|\boldsymbol{u}|$ be small compared with the phase speed of the disturbance. Noting that (again with x_i, y_i representing the initial position of the fluid element),

$$x(t')-x_o=[x(t')-x_i]-[x_o-x_i]=\xi_x(t')-\xi_x(t)$$

$$y(t')-y_o=[y(t')-y_i]-[y_o-y_i]=\xi_y(t')-\xi_y(t)$$

and that equation (20) allows us to express $\partial u_x/\partial y$ and $\partial u_y/\partial y$ in terms of the derivatives with respect to x of components of $\boldsymbol{u}$, one finds

$$\xi_x(x_o,y_o,t) = \int_{orbit}^{t} dt'(u_x(x_o,y_o,t') + [\xi_x(t') - \xi_x(t)]\frac{\partial u_x(x_o,y_o,t')}{\partial x} - [\xi_y(t') - \xi_y(t)][\frac{\partial u_y(x_o,y_o,t')}{\partial x} - \Omega]) \quad (25)$$

$$\xi_y(x_o,y_o,t) = \int_{orbit}^{t} dt'(u_y(x_o,y_o,t') + [\xi_x(t') - \xi_x(t)]\frac{\partial u_y(x_o,y_o,t')}{\partial x} - [\xi_y(t') - \xi_y(t)]\frac{\partial u_x(x_o,y_o,t')}{\partial x}) \quad (26)$$

Next we use the assumption that the disturbances depend on time only in the form shown in equation (20), so that

$$\frac{\partial \boldsymbol{u}}{\partial x} = -\frac{1}{c}\frac{\partial \boldsymbol{u}}{\partial t} \quad (27)$$

and also note that the integration path follows the fluid orbit along which Ω is constant, say Ω_o. This gives

$$\xi_x(x_o,y_o,t) = \int_{orbit}^{t} dt'\,(u_x(x_o,y_o,t') - [\xi_x(t') - \xi_x(t)]\frac{\partial u_x(x_o,y_o,t')}{c\,\partial t'} - [\xi_y(t') - \xi_y(t)]\frac{\partial u_y(x_o,y_o,t')}{c\,\partial t'}) - \Omega_o \int_{orbit}^{t} dt'\,[\xi_y(t') - \xi_y(t)] \quad (28)$$

$$\xi_y(x_o,y_o,t) = \int_{orbit}^{t} dt'(u_y(x_o,y_o,t') - [\xi_x(t') - \xi_x(t)]\frac{\partial u_y(x_o,y_o,t')}{c\,\partial t'} + [\xi_y(t') - \xi_y(t)]\frac{\partial u_x(x_o,y_o,t')}{c\,\partial t'}) \quad (29)$$

Partial integrations provide further simplification. As an example, consider the second term on the right hand side of equation (28) which becomes

$$-\int_{-\infty}^{t} dt'[\xi_x(t') - \xi_x(t)]\frac{\partial u_x(x_o,y_o,t')}{c\,\partial t'}$$

$$= -[\xi_x(t') - \xi_x(t)]\,\frac{1}{c}u_x(x_o,y_o,t')\Big|_{-\infty}^{t} + \frac{1}{c}\int_{-\infty}^{t} dt'\,\frac{\partial \xi_x(t')}{\partial t'}u_x(x_o,y_o,t')$$

$$= \int_{-\infty}^{t} dt'\,u_x^2(x_o,y_o,t')/c \quad (30)$$

where the limit terms vanish because the velocity is taken to vanish at times in the distant past and the factor $\xi_x(t') - \xi_x(t) = 0$ at $t' = t$. Treating the other terms in the same fashion, we find

$$\xi_x(x_o,y_o,t) = \int_{orbit}^{t} dt'(u_x(x_o,y_o,t') + [u_x(x_o,y_o,t')^2 + u_y(x_o,y_o,t')^2]/c) - \Omega_o\int_{orbit}^{t} dt'\,[\xi_y(t') - \xi_y(t)] \quad (31)$$

$$\xi_y(x_o,y_o,t) = \int_{orbit}^{t} dt'\,u_y(x_o,y_o,t') \quad (32)$$

In equation (32) the additional terms have cancelled one another. The time derivatives of these equations give the velocity of a fluid element at (x_o,y_o) at time t in the form

$$\frac{\partial \xi_x(x_o,y_o,t)}{\partial t} = u_x(x_o,y_o,t) + [u_x(x_o,y_o,t)^2 + u_y(x_o,y_o,t)^2]/c \quad (33)$$

$$\frac{\partial \xi_y(x_o,y_o,t)}{\partial t} = u_y(x_o,y_o,t) \quad (34)$$

In equation (33), the term proportional to Ω_o has dropped out. Equations (33) and (34), allow us to obtain the momentum and the magnetic flux associated with the perturbed flow system over the full disturbance. The momentum $\boldsymbol{P}$ is the integral over (x,y) of the momentum density

$$\boldsymbol{P} = \iint dx\,dy\,\rho(x,y,t)(\hat{x}\frac{\partial \xi_x(x,y,t)}{\partial t} + \hat{y}\frac{\partial \xi_y(x,y,t)}{\partial t})$$

$$= \iint dx\,dy\,\rho(x,y,t)\,(\hat{x}\,u_x(x,y,t) + \hat{y}\,u_y(x,y,t) + \hat{x}\,[u_x(x,y,t)^2 + u_y(xy,t)^2]/c) \quad (35)$$

where the integrals are taken across the perturbed portion of the ionosphere, whose surface area we take as A.

The linear term (the first on the right hand side) is bounded and cannot exceed $(\rho \boldsymbol{u})_{max}A$. It is likely to be much smaller than this upper limit because the perturbation flows are likely to have different orientations in different parts of the ionosphere. The positive definite quadratic term is likely to dominate. In this case

$$\boldsymbol{P} \approx \hat{x}\frac{1}{c}\iint dx\,dy\,\rho(x,y,t)u^2(x,y,t) \approx A\,\hat{x}\frac{1}{c}\langle\rho u^2\rangle \quad (36)$$

where $\langle\rho\, u^2\rangle$ is the average value over the perturbed area.

Analogous arguments apply to the calculation of the magnetic flux Φ transported by the perturbed flow.

$$\Phi = -\int dt' \int dy \frac{\partial \xi_x(x,y,t)}{\partial t}B(x,y,t) = \frac{1}{c}\int dx \int dy \frac{\partial \xi_x(x,y,t)}{\partial t}B(x,y,t) \quad (37)$$

As $B(x,y,t)$ is roughly constant in the perturbed region of the ionosphere,

$$\Phi = \hat{x}\frac{B}{c^2}\iint dydx\,u^2(x,y,t) \approx \hat{x}\,BA\langle u^2\rangle/c^2 \quad (38)$$

Equation (38) provides the important insight that there is necessarily a net transport of magnetic flux as well as of momentum in the direction of phase motion of a propagating ionospheric disturbance even in the absence of actual net flow. Propagating disturbances, whether quasi-isolated vortical flows or waves, transport magnetic flux and momentum in the direction of phase motion independent of the nature of the boundary disturbance driving the ionosphere. The amount of flux transported increases as the square of the ratio of the mean eddy speed to the phase speed of the disturbance, a quantity that must be small for the approximations that we have made to be valid.

For a typical TIV (described in the introduction) with phase speed of order 3 km/s and rms eddy speed of 500 m/s, the magnetic flux transported is ~3% of the total flux within the area spanned by the eddy. If a typical eddy has a spatial scale of ~1000 km and the polar cap diameter is taken to be of order 5000 km, a single eddy can transport 1% of the polar cap flux. Thus, individual eddies make relatively minor contributions on a global scale, but, as for flux transfer events (FTEs), with frequent repetition their effect can become significant.

Studies of TIVs have sought to characterize the magnetospheric source of the disturbances observed at low altitudes. In considering whether a TIV is the ionospheric signature of a moving FTE, the relative magnitudes of the propagation speed (above referred to as the phase speed) and the rms eddy speed give important clues that have not, to our knowledge, received any attention other than in our own work [*Kivelson and Southwood*, 1991]. For the scales typical of TIVs, the FTE must be of the form originally postulated by *Russell and Elphic* [1979], that is, it must have an equatorial cross section of order 1 R_E or less and be in motion antisunward from the subsolar point. It must transport all the flux linked to it. Thus within the ionospheric footprint, the rms eddy speed and the propagation speed of the signature must be comparable. As noted above, the eddy speed of typical TIVs is 1/6 of their phase speed and they can transport only a relatively small fraction of the magnetic flux that they link. This feature alone makes it unlikely that typical TIVs are FTE footprints in any direct sense. On the other hand, as an FTE moves through the ionosphere, it sets up flows in a larger region that surrounds it. The phase speed of these secondary flows is the FTE transport speed, but their rms eddy speeds are likely to be smaller; equation (38) then applies and shows that the ionospheric perturbations driven by the passage of an FTE themselves transport flux in the antisolar direction. For the nominal reported flow values of FTEs, only if the TIV is large enough to contain both the footprint of an FTE and the surrounding disturbed region does it seem consistent for the observed twin vortex signatures to be linked to transient reconnection.

CONCLUSIONS

We have developed a simple analogy between two-dimensional incompressible flow and high latitude flow systems in the ionosphere. Vorticity acts a source for the stream function for flows and we have developed an equation for time evolution of vorticity and an equation linking vorticity with the flux of field aligned current into/out of the ionosphere.

The results derived are interesting but in many respects are very similar to equations based on an electrodynamic approach. However the usefulness of the approach is illustrated by the calculation that takes up the second half of the paper. Here we have derived a general result that any high altitude (that is, magnetospheric) phenomenon that sets up a moving vortical flow must also transport both momentum and magnetic flux. The net momentum transport is in the direction of the phase motion which is not necessarily the local direction of motion of the plasma. Deriving the result using an electromagnetic formalism would have been possible, of course, but far harder to conceptualize.

An immediate and important observational consequence of the discovery that traveling vortices must transport momentum and flux is that their transport properties can be deduced and monitored on the basis of data obtained from ionospheric flow measurements alone. The flux and momentum transport of a given disturbance do not depend on identifying the mechanism at high altitude whereby the motion is ultimately driven.

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M.G.Kivelson, Department of Earth and Space Sciences, University of California, Los Angeles, CA 90024-1567.

D.J. Southwood, Department of Physics, Imperial College of Science, Technology and Medicine, London SW7 2BZ, England.

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