

R. H. TIENSenior Scientist of Edgar C. Bain
Laboratory for Fundamental Research,
U. S. Steel Corporation, Monroeville, Pa.**G. E. GEIGER**Associate Professor, Department of
Mechanical Engineering, University of
Pittsburgh, Pittsburgh, Pa.

The Unidimensional Solidification of a Binary Eutectic System With a Time-Dependent Surface Temperature¹

The realistic boundary condition of the surface temperature varying with time is used in the heat transfer analysis of the solidification of a binary eutectic alloy system. The system occupies a semi-infinite region and consists of a solid zone, a freezing zone, and a liquid zone. The heat released in the freezing zone is treated as a discontinuous heat generation. The solid-fraction distribution in the freezing zone, given as a function of temperature, is solved simultaneously with the temperature distribution, which is given as a function of distance within the freezing zone. Using the "heat balance integral" technique, the method for the solution of the temperature distribution and the thickness of each region is given. As an example, the results are shown in graphical form for an Al-5 percent Cu alloy.

Introduction

THE change from the liquid to the solid phase during the solidification of an alloy does not take place isothermally, as is the case for a pure metal. The liquid starts to freeze at the liquidus temperature, and freezes partially and gradually until its temperature drops to a eutectic temperature; then the remaining liquid freezes isothermally at that temperature. The physical representation of this process can therefore be constructed by separating the liquid and the solid by a "freezing zone" which is formed by two isothermal surfaces at the liquidus and the solidus temperatures, respectively. The mathematical model consists of the Fourier's conduction equation and the solid-fraction temperature relationship within the freezing zone. A general account of the derivation of the relationship between the temperature and the solid fraction within the freezing zone is given by Brody [1]¹ and Pfann [2].

The one-dimensional solidification of a binary eutectic system with a constant surface temperature, and assumed solid fraction distribution within the freezing zone, was analyzed previously by the authors [3] from the heat transfer point of view. A more realistic model is one in which the cooling temperature at the

surface is time-dependent and the solid fraction distribution and the temperature distribution are solved simultaneously from Fourier's conduction equation and a specific relationship between the temperature and the solid fraction.

The time-dependent boundary condition prohibits an error function form of solution. Applying Goodman's [4] approximate method of the heat balance integral to both the solidified part and the freezing zone reduces the system from a set of two second-order partial differential equations to a set of two first-order ordinary differential equations, with the thickness of the solidified part and the freezing zone as dependent variables. The latter then is solved by the Runge-Kutta method, a simple but reliable numerical technique.

The temperature distribution within the freezing zone depends on the heat generation arising from freezing, and is dependent upon the solid fraction distribution. If this temperature distribution can be expressed only in terms of a dimensionless space variable, \bar{x} , such that the functional relationship is independent of the form of an assumed solid fraction distribution, then the combination of this temperature distribution with a specific solid-fraction temperature relationship is the actual solid-fraction distribution in terms of the same dimensionless space variable.

¹ The material presented in this paper consists of a portion of a PhD dissertation submitted by R. H. Tien to the Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, Pa.

² Numbers in brackets designate References at end of paper.

Contributed by the Heat Transfer Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS and presented at the ASME-AIChE Heat Transfer Conference and Exhibit, Seattle, Wash., August 6-9, 1967. Manuscript received by Heat Transfer Division, February 21, 1967. Paper No. 67-HT-43.

Statement of the Problem

The liquid, at liquidus temperature, initially occupies a semi-infinite region which extends over the positive x -axis. At time zero, the surface ($x = 0$) is at solidus temperature. At later times the temperature is allowed to decrease in an arbitrary

Nomenclature

A = internal heat generation, Btu/ft³-hr
 c = specific heat, Btu/lb deg F
 C_0 = initial concentration of the alloy
 ϵ = position of solidus front, ft
 $\Delta\epsilon$ = thickness of freezing zone, ft
 f_s = solid fraction
 f_{su} = eutectic solid fraction
 k = equilibrium distribution coefficient
 K = thermal conductivity, Btu/ft-hr-deg F

L = latent heat of fusion, Btu/lb
 M_l = magnitude of the slope of the liquidus line, deg F
 α = thermal diffusivity, ft²/hr
 ρ = density, lb/ft³
 R = rate of temperature change at cooling surface, deg F/hr
 t = time, hr
 T = temperature, deg F
 T_0 = equilibrium temperature for zero concentration, deg F

T_s = solidus temperature, deg F
 T_l = liquidus temperature, deg F
 T_c = surface temperature, deg F
 ΔT = temperature difference between liquidus and solidus, deg F
 x = distance, ft
subscript 1 = solidified part
subscript 2 = freezing zone

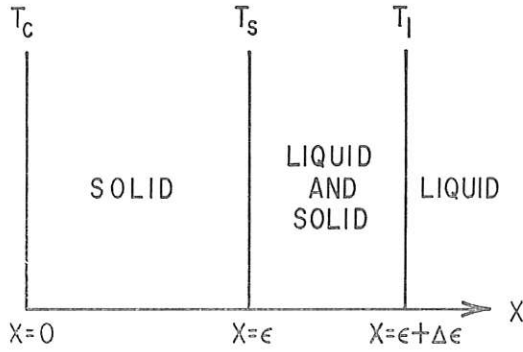


Fig. 1 Schematic representation

manner. A schematic representation of this problem is shown in Fig. 1. The rate of change of the solid fraction with respect to time within the freezing zone provides the latent heat effect. A solid-fraction temperature relationship can be written as,

$$f_s = 1 - \left(\frac{M_l C_0}{T_0 - T} \right)^{\frac{1}{1-k}} \quad (1)$$

For the normal nonequilibrium mode of freezing, the derivation of equation (1) is given by Pfann [2].

Assuming constant physical properties, one-dimensional temperature variation, conduction as the only heat transfer mechanism, and the heat of fusion the only internal heat generation, Fourier's conduction equation for the solidified part and the freezing zone can be written as:

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial T_1}{\partial t} \quad \text{for } 0 \leq x \leq \epsilon \quad (2)$$

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{A}{K_2} = \frac{1}{\alpha_2} \frac{\partial T_2}{\partial t} \quad \text{for } \epsilon \leq x \leq \epsilon + \Delta\epsilon \quad (3)$$

where T_1 and T_2 are relative to the solidus temperature, T_s . The boundary conditions are:

- 1 $T_1(x=0) = -(T_s - T_c)$
- 2 $T_1(x=\epsilon) = 0$
- 3 $T_2(x=\epsilon) = 0$
- 4 $T_2(x=\epsilon + \Delta\epsilon) = \Delta T$
- 5 $\frac{\partial T_2}{\partial x}(x=\epsilon + \Delta\epsilon) = 0$
- 6 $K_1 \frac{\partial T_1}{\partial x}(x=\epsilon) = K_2 \frac{\partial T_2}{\partial x}(x=\epsilon) + \rho L(1 - f_{su}) \frac{d\epsilon}{dt}$

The initial conditions are:

- 7 $\epsilon(t=0) = 0$
- 8 $\Delta\epsilon(t=0) = 0$

The first five boundary conditions specify the temperature or the temperature gradient at the respective positions. The sixth boundary condition represents the conservation of energy at the eutectic front. The second term on the right side of the equation represents the latent heat effect due to freezing of the remaining liquid at the eutectic temperature.

The internal heat generation term, A , associated with the latent heat effect, can be written as

$$A = \rho L \frac{\partial f_s}{\partial t} \quad (4)$$

where the solid fraction, f_s , can be represented generally by a n th degree polynomial.

$$f_s = f_{su} \left[1 - \sum_{i=1}^n C_i \bar{x}^i \right] \quad (5)$$

The coefficients C_i 's are chosen such that

$$f_s(\bar{x}=0) = f_{su} \text{ i.e., } C_i \text{'s are all finite}$$

and

$$f_s(\bar{x}=1) = 0 \quad \text{i.e., } \sum_{i=1}^n C_i = 1$$

Combining equations (4) and (5) provides

$$A = \frac{\rho L}{\Delta\epsilon} \left(\frac{d\epsilon}{dt} + \bar{x} \frac{d\Delta\epsilon}{dt} \right) \left[\sum_{i=1}^n i C_i \bar{x}^{i-1} \right] f_{su} \quad (6)$$

where $\bar{x} = \frac{x - \epsilon}{\Delta\epsilon}$.

Solution

It will be assumed that the "thermal layer" which has been defined by Goodman [4] in the heat-balance-integral method is the entire solidified part of the system in which equation (2) is applicable. Then integrating equation (2) with respect to x from $x=0$ to $x=\epsilon$ results in

$$\frac{\partial T_1}{\partial x}(x=\epsilon, t) - \frac{\partial T_1}{\partial x}(x=0, t) = \frac{1}{\alpha_1} \int_0^\epsilon \frac{\partial T_1}{\partial t} dx \quad (7)$$

By defining

$$\theta_1 = \int_0^\epsilon T_1(x, t) dx$$

the right side of equation (7) can be reduced to

$$\frac{1}{\alpha_1} \left[\frac{d\theta_1}{dt} + T_1(x=\epsilon, t) \frac{d\epsilon}{dt} \right] \quad (8)$$

by using Leibniz's formula.

By combining equations (7), (8) and the boundary conditions (2) and (6), equation (2) becomes

$$\frac{K_2}{K_1} \frac{\partial T_2}{\partial x}(x=\epsilon, t) + \frac{\rho_2 L}{K_1} (1 - f_{su}) \frac{d\epsilon}{dt} - \frac{\partial T_1}{\partial x}(x=0, t) = \frac{1}{\alpha_1} \frac{d\theta_1}{dt} \quad (9)$$

Next the "thermal layer" for equation (3) is defined as the entire freezing zone. Then integrating equation (3) with respect to x from $x=\epsilon$ to $x=\epsilon + \Delta\epsilon$ yields

$$\frac{\partial T_2}{\partial x}(x=\epsilon + \Delta\epsilon, t) - \frac{\partial T_2}{\partial x}(x=\epsilon, t) + \frac{1}{K_2} \int_\epsilon^{\epsilon + \Delta\epsilon} A dx = \frac{1}{\alpha_2} \int_\epsilon^{\epsilon + \Delta\epsilon} \frac{\partial T_2}{\partial t} dx \quad (10)$$

On the left side of equation (10), the first term is eliminated by applying boundary condition (5), the second term is replaced by boundary condition (6), while the last term can be integrated with the aid of equation (6). The right side of equation (10) can be reduced in a similar manner to

$$\frac{1}{\alpha_2} \left[\frac{d\theta_2}{dt} + \Delta T \left(\frac{d\epsilon}{dt} + \frac{d\Delta\epsilon}{dt} \right) \right]$$

where

$$\theta_2 = \int_\epsilon^{\epsilon + \Delta\epsilon} T_2(x, t) dx$$

With the aid of all the foregoing conditions, equation (3) is reduced to the following final form:

$$\left(\frac{L}{c_2} + \Delta T\right) \frac{d\epsilon}{dt} + \left[\frac{L f_{su}}{c_2} \left(\sum_{i=1}^n \frac{i}{i+1} C_i \right) + \Delta T \right] \frac{d\Delta\epsilon}{dt} - \frac{K_1}{\rho c_2} \frac{\partial T_1}{\partial x} (x = \epsilon, t) = \frac{d\theta_2}{dt} \quad (11)$$

Now a quadratic temperature distribution in both the solidified part and freezing zone is assumed,

$$\begin{aligned} T_1(x, t) &= A_0 + A_1(\epsilon - x) + A_2(\epsilon - x)^2 \\ T_2(x, t) &= A_3 + A_4(x - \epsilon) + A_5(x - \epsilon)^2 \end{aligned} \quad (12)$$

where the A_i 's are independent of x , and can be determined by the prescribed boundary conditions.

From the boundary conditions (2) and (3), A_0 and A_3 are zero. By using the boundary conditions (4) and (5), A_4 and A_5 are determined as:

$$\begin{aligned} A_4 &= \frac{2\Delta T}{\Delta\epsilon} \\ A_5 &= -\frac{\Delta T}{\Delta\epsilon^2} \end{aligned} \quad (13)$$

A_1 and A_2 must be solved by the use of boundary condition (1) and the combination of boundary conditions (2) and (6), and equation (2). The boundary condition (6) alone is not suitable for the present case. Since $\frac{d\epsilon}{dt}$ appears in boundary condition (6), this will make equation (9) a second-order differential equation in ϵ which cannot be solved because only one initial condition for ϵ is available.

$$\begin{aligned} A_1 &= -\frac{1}{\epsilon} (\beta_1 + \beta_2) \\ A_2 &= -\frac{1}{\epsilon^2} [(T_s - T_c) - (\beta_1 + \beta_2)] \end{aligned} \quad (14)$$

where

$$\begin{aligned} \beta_1 &= \frac{K_2}{K_1} \Delta T \frac{\epsilon}{\Delta\epsilon} - \frac{L}{c_1} (1 - f_{su}) \\ \beta_2 &= \sqrt{\beta_1^2 + 2 \frac{L}{c_2} (1 - f_{su})(T_s - T_c)} \end{aligned} \quad (15)$$

Hence the temperature distribution in the two regions is:

$$\begin{aligned} T_1(x, t) &= -\frac{1}{\epsilon} (\beta_1 + \beta_2)(\epsilon - x) \\ &\quad - \frac{1}{\epsilon^2} [(T_s - T_c) - (\beta_1 + \beta_2)](\epsilon - x)^2 \end{aligned} \quad (16)$$

$$T_2(x, t) = \frac{2\Delta T}{\Delta\epsilon} (x - \epsilon) - \frac{\Delta T}{\Delta\epsilon^2} (x - \epsilon)^2 \quad (17)$$

By defining $\bar{x} = \frac{x - \epsilon}{\Delta\epsilon}$, equation (17) can also be written as

$$T_2(\bar{x}) = 2\Delta T \bar{x} - \Delta T \bar{x}^2 \quad (18)$$

The functional relationship between T_2 and \bar{x} as shown in equation (18) appears to be independent of the distribution of the solid fraction f_s , however ϵ and $\Delta\epsilon$ which appear in \bar{x} are functions of f_s . Hence, substituting equation (18) into equation (1) will give the true solid fraction distribution in terms of \bar{x} , and is shown as:

$$f_s = 1 - \left[\frac{M_i C_0}{(T_0 - T_s) - \Delta T (2\bar{x} - \bar{x}^2)} \right]^{1-k} \quad (19)$$

Equation (19) can also be represented by an n th degree polynomial using a modified least-squares method as shown in equation (5). However, the coefficients now are not assumed but calculated from equation (19).

With the temperature distributions T_1 and T_2 as known functions of x , ϵ , and $\Delta\epsilon$ as shown in equations (16) and (17), the parameters θ_1 and θ_2 are calculated according to their definitions. Upon substituting these into equations (9) and (11), two first-order ordinary differential equations of ϵ and $\Delta\epsilon$ are obtained with respect to time, t , as independent variables as:

$$\begin{aligned} \frac{d\epsilon}{dt} &= \frac{N_3 M_2 - N_2 M_3}{N_1 M_2 - N_2 M_1} = f_1(\epsilon, \Delta\epsilon, t) \\ \frac{d\Delta\epsilon}{dt} &= \frac{N_1 M_3 - N_3 M_1}{N_1 M_2 - N_2 M_1} = f_2(\epsilon, \Delta\epsilon, t) \end{aligned} \quad (20)$$

where

$$\begin{aligned} N_1 &= \Phi_1 + \frac{L}{c_1} (1 - f_{su}) \\ N_2 &= -\frac{1}{6} \frac{K_2}{K_1} \Delta T \left(\frac{\epsilon}{\Delta\epsilon} \right)^2 \left(1 + \frac{\beta_1}{\beta_2} \right) \\ N_3 &= \frac{1}{3} \epsilon \frac{dT_c}{dt} \Phi_2 - \alpha_1 \left(A_1 + 2A_2\epsilon + \frac{K_2}{K_1} \frac{2\Delta T}{\Delta\epsilon} \right) \\ M_1 &= \frac{L}{c_2} + \Delta T \\ M_2 &= \frac{L}{c_2} f_{su} \left(\sum_{i=1}^n \frac{i}{i+1} C_i \right) + \frac{1}{3} \Delta T \\ M_3 &= -\frac{K_1}{K_2} A_1 \alpha_2 \\ \Phi_1 &= \frac{1}{6} \left[\beta_1 + \beta_2 + 2(T_s - T_c) + \frac{K_2}{K_1} \Delta T \frac{\epsilon}{\Delta\epsilon} \left(1 + \frac{\beta_1}{\beta_2} \right) \right] \\ \Phi_2 &= 1 + \frac{1}{2} \frac{L}{c_1} (1 - f_{su}) \frac{1}{\beta_2} \end{aligned}$$

and A_1 , A_2 , β_1 , and β_2 are given in equations (14) and (15).

To generalize the solution, the following dimensionless variables are defined:

$$\begin{aligned} \bar{\epsilon} &= \frac{\epsilon}{D}; \quad \Delta\bar{\epsilon} = \frac{\Delta\epsilon}{D}; \quad \bar{t} = \frac{t\alpha_1}{D^2}; \quad \bar{T} = \frac{T - T_s}{\Delta T}; \\ \bar{K} &= \frac{K_2}{K_1}; \quad \bar{c} = \frac{c_2}{c_1}; \quad \bar{\alpha} = \frac{\alpha_2}{\alpha_1}; \quad \bar{L} = \frac{L}{c_1 \Delta T}; \end{aligned}$$

where D is a characteristic length. Then

$$\begin{aligned} \bar{A}_1 &= A_1 \left(\frac{D}{\Delta T} \right) = -\frac{1}{\bar{\epsilon}} (\bar{\beta}_1 + \bar{\beta}_2) \\ \bar{A}_2 &= A_2 \left(\frac{D^2}{\Delta T} \right) = \frac{1}{\bar{\epsilon}^2} [\bar{T}_c + (\bar{\beta}_1 + \bar{\beta}_2)] \\ \bar{\beta}_1 &= \frac{\beta_1}{\Delta T} = \bar{K} \frac{\bar{\epsilon}}{\Delta\bar{\epsilon}} - \bar{L}(1 - f_{su}) \\ \bar{\beta}_2 &= \frac{\beta_2}{\Delta T} = \sqrt{\bar{\beta}_1^2 - \frac{2\bar{L}}{\bar{c}} (1 - f_{su}) \bar{T}_c} \\ \bar{\Phi}_1 &= \frac{\Phi_1}{\Delta T} = \frac{1}{6} \left[\bar{\beta}_1 + \bar{\beta}_2 - 2\bar{T}_c + \bar{K} \frac{\bar{\epsilon}}{\Delta\bar{\epsilon}} \left(1 + \frac{\bar{\beta}_1}{\bar{\beta}_2} \right) \right] \\ \bar{\Phi}_2 &= \Phi_2 = 1 + \frac{1}{2} \bar{L}(1 - f_{su}) \frac{1}{\bar{\beta}_2} \end{aligned}$$

$$\begin{aligned}\bar{M}_1 &= \frac{M_1}{\Delta T} = 1 + \frac{\bar{L}}{\bar{c}} \\ \bar{M}_2 &= \frac{M_2}{\Delta T} = \frac{\bar{L}}{\bar{c}} f_{su} \left(\sum_{i=1}^n \frac{i}{i+1} C_i \right) + \frac{1}{3} \\ \bar{M}_3 &= \frac{M_3}{\left(\frac{\Delta T \alpha_1}{D} \right)} = \frac{\bar{\alpha}}{\bar{K}} \frac{1}{\bar{\epsilon}} (\bar{\beta}_1 + \bar{\beta}_2) \\ \bar{N}_1 &= \frac{N_1}{\Delta T} = \bar{\Phi}_1 + \bar{L}(1 - f_{su}) \\ \bar{N}_2 &= \frac{N_2}{\Delta T} = -\frac{1}{6} \bar{K} \left(\frac{\bar{\epsilon}}{\Delta \bar{\epsilon}} \right)^2 \left(1 + \frac{\bar{\beta}_1}{\bar{\beta}_2} \right) \\ \bar{N}_3 &= \frac{N_3}{\left(\frac{\Delta T \alpha_1}{D} \right)} = \frac{1}{3} \bar{\epsilon} \frac{d\bar{T}_c}{d\bar{t}} \bar{\Phi}_2 - \left[\bar{A}_1 + 2\bar{A}_2 \bar{\epsilon} + \frac{2\bar{K}}{\Delta \bar{\epsilon}} \right].\end{aligned}$$

Therefore equation (20) can also be written as:

$$\begin{aligned}\frac{d\bar{\epsilon}}{d\bar{t}} &= \frac{\bar{N}_3 \bar{M}_2 - \bar{N}_2 \bar{M}_3}{\bar{N}_1 \bar{M}_2 - \bar{N}_2 \bar{M}_1} = \bar{f}_1(\bar{\epsilon}, \Delta \bar{\epsilon}, \bar{t}) \\ \frac{d\Delta \bar{\epsilon}}{d\bar{t}} &= \frac{\bar{N}_1 \bar{M}_3 - \bar{N}_3 \bar{M}_1}{\bar{N}_1 \bar{M}_2 - \bar{N}_2 \bar{M}_1} = \bar{f}_2(\bar{\epsilon}, \Delta \bar{\epsilon}, \bar{t})\end{aligned}\quad (21)$$

The Runge-Kutta [5] method is used to solve equation (21) with the initial conditions (7) and (8) for $\bar{\epsilon}$ and $\Delta \bar{\epsilon}$. With a uniform interval $\Delta \bar{t}$, the $(n+1)$ th values of $\bar{\epsilon}$ and $\Delta \bar{\epsilon}$ can then be calculated in terms of n th values of $\bar{\epsilon}$ and $\Delta \bar{\epsilon}$ as:

$$\begin{aligned}\bar{\epsilon}_{n+1} &= \bar{\epsilon}_n + \frac{1}{6} (n_0 + 2n_1 + 2n_2 + n_3) \\ \Delta \bar{\epsilon}_{n+1} &= \Delta \bar{\epsilon}_n + \frac{1}{6} (m_0 + 2m_1 + 2m_2 + m_3)\end{aligned}\quad (22)$$

where

$$\begin{aligned}n_0 &= \Delta \bar{t} \bar{f}_1(\bar{i}_n, \bar{\epsilon}_n, \Delta \bar{\epsilon}_n) \\ m_0 &= \Delta \bar{t} \bar{f}_2(\bar{i}_n, \bar{\epsilon}_n, \Delta \bar{\epsilon}_n) \\ n_1 &= \Delta \bar{t} \bar{f}_1 \left(\bar{i}_n + \frac{1}{2} \Delta \bar{t}, \bar{\epsilon}_n + \frac{1}{2} n_0, \Delta \bar{\epsilon}_n + \frac{1}{2} m_0 \right) \\ m_1 &= \Delta \bar{t} \bar{f}_2 \left(\bar{i}_n + \frac{1}{2} \Delta \bar{t}, \bar{\epsilon}_n + \frac{1}{2} n_0, \Delta \bar{\epsilon}_n + \frac{1}{2} m_0 \right) \\ n_2 &= \Delta \bar{t} \bar{f}_1 \left(\bar{i}_n + \frac{1}{2} \Delta \bar{t}, \bar{\epsilon}_n + \frac{1}{2} n_1, \Delta \bar{\epsilon}_n + \frac{1}{2} m_1 \right) \\ m_2 &= \Delta \bar{t} \bar{f}_2 \left(\bar{i}_n + \frac{1}{2} \Delta \bar{t}, \bar{\epsilon}_n + \frac{1}{2} n_1, \Delta \bar{\epsilon}_n + \frac{1}{2} m_1 \right) \\ n_3 &= \Delta \bar{t} \bar{f}_1(\bar{i}_n + \Delta \bar{t}, \bar{\epsilon}_n + n_2, \Delta \bar{\epsilon}_n + m_2) \\ m_3 &= \Delta \bar{t} \bar{f}_2(\bar{i}_n + \Delta \bar{t}, \bar{\epsilon}_n + n_2, \Delta \bar{\epsilon}_n + m_2)\end{aligned}$$

with $\bar{\epsilon}_0 = \Delta \bar{\epsilon}_0 = 0$ at $\bar{t}_0 = 0$.

Numerical Example and Discussion

The distribution of the solid fraction within the freezing zone in terms of \bar{x} is calculated from equation (19) by use of the following numerical data for 5 percent Cu-aluminum alloy.

$$\begin{aligned}k &= 0.1687 \\ M_l &= 5.78 \\ C_0 &= 5.0 \text{ percent}\end{aligned}$$

$$T_0 = 1220 \text{ deg F}$$

$$T_s = 1018 \text{ deg F}$$

$$\Delta T = 173.1 \text{ deg F}$$

This $f_s \sim \bar{x}$ curve is then fitted by a tenth degree polynomial with the maximum difference less than 7×10^{-5} through a modified least-squares method. Thus,

$$f_s = f_{su} \left[1 - \sum_{i=1}^9 C_i \bar{x}^i \right] \quad (23)$$

where

$$C_1 = 1.9516656 \times 10^{-1}$$

$$C_2 = 2.6572437 \times 10^{-1}$$

$$C_3 = 1.940831 \times 10^{-1}$$

$$C_4 = 3.52208$$

$$C_5 = -2.379364 \times 10^1$$

$$C_6 = 9.227445 \times 10^1$$

$$C_7 = -1.9979674 \times 10^2$$

$$C_8 = 2.4912653 \times 10^2$$

$$C_9 = -1.6538519 \times 10^2$$

$$C_{10} = 4.4397551 \times 10^1$$

With a solid fraction distribution as in equation (23) and a linearly decreasing surface temperature such as

$$T_c = T_s - R \cdot \bar{t}; \quad \text{or} \quad \bar{T}_c = -\bar{R} \bar{t} \quad (24)$$

where $\bar{R} = \left(\frac{D^2}{\alpha_1 \Delta T} \right) R$, the positions of those two isothermal fronts $\bar{\epsilon}$ and $\Delta \bar{\epsilon}$ are calculated by equation (22) with a carefully selected time interval, $\Delta \bar{t}$, to insure convergence. The following physical properties are necessary to proceed with the calculation:

$$K_1 = 114 \text{ (Btu/ft-hr deg F)}$$

$$K_2 = 105$$

$$\bar{K} = 0.921$$

$$c_1 = 0.25 \text{ (Btu/lb deg F)}$$

$$\bar{c} = 1.2$$

$$c_2 = 0.3 \text{ (Btu/lb deg F)}$$

$$\bar{\alpha} = 0.7675$$

$$\rho = 170 \text{ (lb/ft}^3\text{)}$$

$$\text{or} \quad \bar{L} = 3.928$$

$$L = 170 \text{ (Btu/lb)}$$

$$f_{su} = 0.903$$

$$f_{su} = 0.903$$

$$T_l = 1191.1 \text{ (deg F)}$$

$$\bar{R} = 5.75$$

$$T_s = 1018 \text{ (deg F)}$$

The results which include the growth of $\bar{\epsilon}$ and $\Delta \bar{\epsilon}$, and their time derivatives, $\frac{d\bar{\epsilon}}{d\bar{t}}$ and $\frac{d\Delta \bar{\epsilon}}{d\bar{t}}$, against time are shown in Fig. 2.

The corresponding temperature distributions within the freezing zone as well as in the solidified part are calculated by equations (16) and (17) for several values of \bar{t} , and are shown in Fig. 3. The heat flux at the surface, $x = 0$, and the temperature, \bar{T}_c , defined in equation (24) are plotted against time in Fig. 4.

As seen from Fig. 2 the rate of movement of the solidus front, $\frac{d\bar{\epsilon}}{d\bar{t}}$, increases rapidly at first, reaches a maximum, then decreases gradually. On the other hand, $\frac{d\Delta \bar{\epsilon}}{d\bar{t}}$ for the liquidus front decreases with freezing time. If the surface temperature remains constant during freezing, as shown in the previous study [3], both $\frac{d\bar{\epsilon}}{d\bar{t}}$ and $\frac{d\Delta \bar{\epsilon}}{d\bar{t}}$ for solidus and liquidus fronts decrease with freezing time.

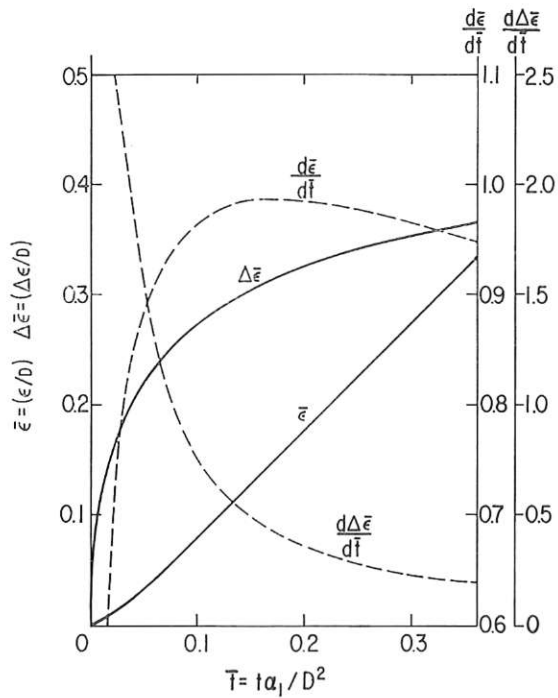


Fig. 2 Growth of solidified and freezing zones and their rates

Variations in the heat flux and temperature at the surface with freezing time are shown in Fig. 4. At the initial stage of solidification the heat flux at the surface decreases rapidly with time, then becomes essentially constant even though the temperature, \bar{T}_c , decreases at a constant rate. This is consistent with the fact that $\frac{d\bar{\epsilon}}{d\bar{t}}$ and $\frac{d\Delta\bar{\epsilon}}{d\bar{t}}$ change rapidly only during the initial stage of solidification.

References

- 1 Brody, H. D., "Solute Redistribution in Dendritic Solidification," ScD thesis, MIT, 1965.
- 2 Pfann, W. G., "Principles of Zone-Melting," *Trans. AIME*, Vol. 135, July 1952, p. 85.
- 3 Tien, R. H., and Geiger, G. E., "A Heat Transfer Analysis of the Solidification of a Binary Eutectic System," ASME Paper No. 66-WA/HT-23. To be published in *Journal of Heat Transfer*.
- 4 Goodman, T. R., "The Heat-Balance Integral and Its Application to Problems Involving a Change of Phase," *TRANS. ASME*, Vol. 30, 1958, p. 335.
- 5 Hildebrand, F. B., *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956, p. 233.

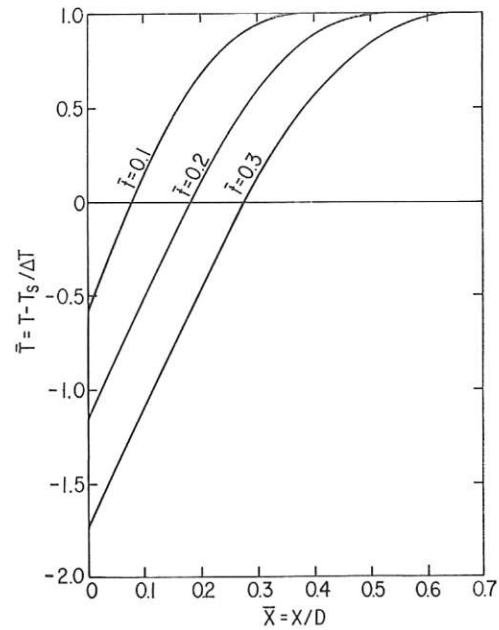


Fig. 3 Temperature distribution

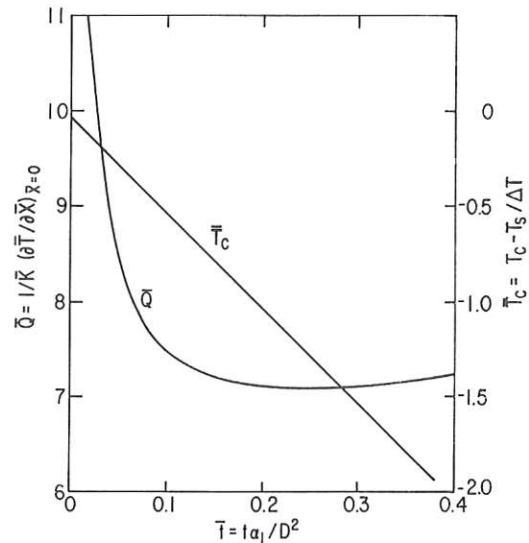


Fig. 4 Heat flux and temperature at surface