Marco Carricato
Department of Industrial Engineering and Interdepartmental Center for Health Sciences and Technologies, University of Bologna, Viale Risorgimento 2, Bologna 40136, Italy
e-mail: marco.carricato@unibo.it

Inverse Geometrico-Static Problem of Underconstrained Cable-Driven Parallel Robots With Three Cables[1]

This paper studies underconstrained cable-driven parallel robots (CDPRs) with three cables. A major challenge in the study of these robots is the intrinsic coupling between kinematics and statics, which must be tackled simultaneously. Effective elimination procedures are presented which provide the complete solution sets of the inverse geometrico-static problems (IGPs) with assigned orientation or position. In the former case, the platform orientation is given, whereas the platform position and the cable lengths and tensions must be computed. In the latter case, the platform position is known, whereas the platform orientation and the cable lengths and tensions are to be calculated. The described problems are proven to admit at the most 1 and 24 real solutions, respectively. [DOI: 10.1115/1.4024291]

1 Introduction

CDPRs employ cables in place of rigid-body extensible legs in order to control the end-effector pose. A CDPR is referred to as *fully constrained* if the end-effector pose is completely determined when actuators are locked and, thus, all cable lengths are assigned. Conversely, a CDPR is *underconstrained* if the end-effector preserves some freedoms once actuators are locked. This occurs either when the end-effector is controlled by a number of cables smaller than the number of degrees of freedom (dofs) that it possesses with respect to the base or when some cable becomes slack in a fully constrained robot [1–3]. The use of CDPRs with a limited number of cables is justified in several applications, in which the task to be performed requires a limited number of controlled freedoms (only n dofs of the end-effector may be governed by n cables) or a limitation of dexterity is acceptable in order to decrease complexity, cost, set-up time, likelihood of cable interference, etc. While a rich literature exists for fully constrained CDPRs, whose features were studied under several viewpoints, including type synthesis [4–10], workspace analysis [11–17], stiffness [18–20], cable interference [21,22], calibration [23,24], failure recovery [25], etc., little research has been conducted on underconstrained robots [26–32].

A major challenge in the kinetostatic analysis of underconstrained CDPRs comes from the fact that, when the actuators are locked and the cable lengths are assigned, the end-effector is still movable, so that the actual configuration is determined by the applied forces. Accordingly, *loop-closure* and *mechanical-equilibrium equations* must be simultaneously solved and displacement analyses become, more properly speaking, *geometrico-static problems* [1,2]. These are significantly more complex than displacement analyses of rigid-link fully constrained parallel manipulators. A meaningful element of comparison is provided by parallel robots equipped with telescoping legs connected to the base and the platform by ball-and-socket joints, whose position problems parallel those of CDPRs (with the obvious difference that, in the latter, configurations involving negative forces in the cables are not admissible). For fully constrained manipulators, such as the Gough–Stewart manipulator, the inverse displacement analysis, aiming at determining the leg lengths when the platform posture is assigned, is a trivial task. Conversely, the direct analysis, which aims at finding the platform pose when the leg lengths are assigned, is a difficult algebraic problem that has attracted the interest of researchers for several years [33]. For underconstrained robots, *both* kinds of analyses gain significant complexity, because of the contribution of equilibrium equations. According to McCarthy [34], the displacement analysis of underconstrained CDPRs is a major pending challenge in current kinematics.

A general framework for the geometrico-static analysis of underconstrained CDPRs, tailored to obtain the *complete* solution sets of the nonlinear equations governing the problems at hand, was proposed in Refs. [1] and [2]. By taking advantage of this framework, this paper studies the IGP of the general 3–3 CDPR. This locution denotes a parallel robot in which a fixed base and a mobile platform are connected to each other by three cables, with cable exit points on the base and anchor points on the platform being distinct. Cables are treated as inextensible and massless, and the platform is acted upon by a constant force, e.g., gravity. The aim of the IGP is to determine the overall configuration of the robot, as well as cable tensions, when three pose coordinates of the platform are assigned, namely, its *orientation* or the *position of a reference point* on it. The solution of the IGP was briefly sketched in Ref. [1], whereas full details and algorithms are given here. In particular, a novel and improved elimination procedure solving the IGP with assigned platform position is provided and positive-dimensional solution sets are analyzed. The solution of the *direct* geometrico-static problem (DGP) of the 3–3 CDPR is presented in Refs. [35–37], whereas the IGP and the DGP of the 4–4 CDPR are presented in Refs. [38] and [39], respectively.

In all numerical examples presented in the text, measurements are expressed in SI units.

2 Geometrico-Static Model

A mobile platform is connected to a fixed base by three cables (Fig. 1). The ith cable ($i = 1, \ldots, 3$) exits from the fixed base at point A_i, and it is connected to the mobile platform at point B_i. The cable length is ρ_i. $Oxyz$ is a Cartesian coordinate frame that is fixed to the base, with **i**, **j**, and **k** being unit vectors along the

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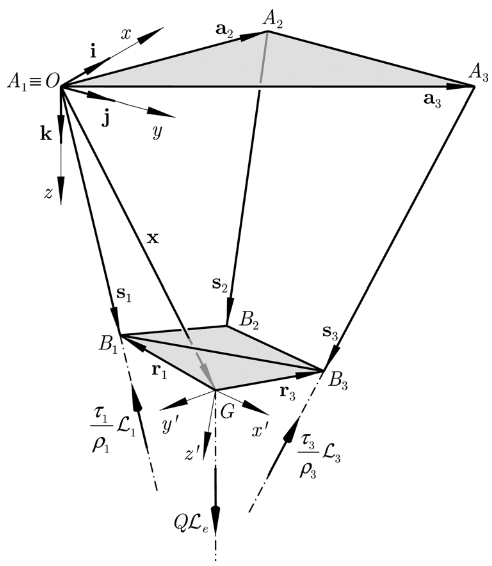

Fig. 1 Model of a cable-driven parallel robot with three cables

coordinate axes. $Gx'y'z'$ is a Cartesian frame that is attached to the end-effector. Without loss of generality, O is chosen to coincide with A_1. The platform pose is described by $\mathbf{X} = [\mathbf{x}^T; \Phi^T]^T$, where $\mathbf{x} = [x, y, z]^T$ is the position vector of G in the fixed frame and Φ is the array grouping the variables parameterizing the platform orientation with respect to $Oxyz$. For the sake of brevity, the following symbols are also introduced

$$\mathbf{a}_i = A_i - O, \quad \mathbf{r}_i = B_i - G$$
$$\mathbf{s}_i = B_i - A_i = \mathbf{x} + \mathbf{r}_i - \mathbf{a}_i$$

Moreover, if $\mathbf{b}_i$ is the position vector of B_i in $Gx'y'z'$ and $\mathbf{R}(\Phi)$ is the rotation matrix between the mobile and the fixed frame, then $\mathbf{r}_i = \mathbf{R}(\Phi)\mathbf{b}_i$. Vector components along the coordinate axes are denoted by right subscripts reporting the axis names.

The platform is acted upon by a constant force, e.g., gravity, which is assumed to be oriented as $\mathbf{k}$ and applied at G, without loss of generality. This force may be described by a 0-pitch wrench $Q\mathcal{L}_e$, where Q is the intensity of the force and $\mathcal{L}_e$ is the normalized Plücker vector of the force line of action. The normalized Plücker vector of the line associated with the ith cable is $\mathcal{L}_i/\rho_i$, where, in axis coordinates, $\mathcal{L}_i = -[\mathbf{s}_i; \mathbf{p}_i \times \mathbf{s}_i]$, and $\mathbf{p}_i$ is any vector from an arbitrarily chosen reference point (called for brevity *moment pole*) to the cable line. So, the wrench exerted by the ith cable on the platform is $(\tau_i/\rho_i)\mathcal{L}_i$, with τ_i being a positive scalar representing the intensity of the cable tensile force.

For practical reasons, the following is finally assumed:

(1) $\rho_i > 0$ and, thus, $\mathbf{s}_i \neq \mathbf{0}, i = 1, \ldots, 3$ (Assumption I)
(2) $0 < ||B_j - B_i|| < ||(A_j - A_i)_{xy}||, i \neq j$ (Assumption II).

The latter assumption, according to which the segment B_iB_j is strictly smaller than the projection of the segment A_iA_j on the xy-plane, is not conceptually necessary, but it rules out some special configurations, which could be handled with no difficulty, but whose analysis would burden the presentation. In particular, the possibility that any two cables may be simultaneously parallel to $\mathbf{k}$ is discarded.

When all cables of the robot are in tension, the set $\mathcal{C}$ of geometrical constraints imposed on the platform comprises the three relations

$$||\mathbf{s}_i|| = \rho_i, \quad i = 1, \ldots, 3 \tag{1}$$

Since only three geometrical restraints are enforced, the platform preserves 3 dofs, with its pose being determined by equilibrium laws, namely [1]

$$\underbrace{\begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_e \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} (\tau_1/\rho_1) \\ (\tau_2/\rho_2) \\ (\tau_3/\rho_3) \\ Q \end{bmatrix} = \mathbf{0} \tag{2}$$

with $\tau_i \geq 0, i = 1, \ldots, 3$.

Equations (1) and (2) amount to nine scalar relations in 12 variables, namely, $\mathbf{x}$, Φ, ρ_i and $\tau_i, i = 1, \ldots, 3$. A finite set of system configurations may be generally determined if any three of these variables are known. In particular, when three variables concerning the platform pose are assigned, an IGP must be solved. Typically, one may wish to assign either the orientation Φ of the platform or the position $\mathbf{x}$ of G. In the former case, the physical meaning of the IGP is the following: which cable lengths must be imposed so that the platform may rest in equilibrium with the assigned orientation? What is the resulting value of $\mathbf{x}$? Which are the cable tensions? In the latter case, the physical meaning of the IGP is, conversely: which cable lengths must be imposed so that the platform may rest in equilibrium with G located in $\mathbf{x}$? What is the resulting orientation of the platform? Which are the cable tensions?

In both cases, the IGP may be simplified by eliminating cable tensions from Eq. (2). By following Refs. [1] and [2], a convenient elimination strategy emerges by observing that Eq. (2) holds only if

$$\text{rank}(\mathbf{M}) \leq 3 \tag{3}$$

namely if $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_e$ are linearly dependent. This is a purely geometrical condition, since $\mathbf{M}$ is a 6×4 matrix only depending on the platform pose. By setting all 4×4 minors of $\mathbf{M}$ equal to zero, a set of suitable independent scalar relations that do not contain cable tensions may be obtained. Moreover, such relations do *not* comprise the cable lengths, so that they lead to a partial decoupling of the system equations, with cable lengths only appearing in Eq. (1).

The IGP takes particular advantage of the mentioned decoupling, since the platform configuration may be directly computed by way of three relations emerging from Eq. (3). In particular, letting all 4×4 minors of $\mathbf{M}$ vanish leads to 15 polynomial equations in $\mathbf{x}$ and Φ of the form $p_j = 0$. The solution of the problem coincides with the variety V of the ideal generated by such equations. When three configuration variables are known (typically, either $\mathbf{x}$ or Φ), any three p_j, say, p_l, p_h, p_k, may be chosen and a corresponding (generally zero-dimensional) variety V_{lhk} is obtained. V is the intersection of the five varieties that may be generated this way. A primary objective of the problem-solving algorithm consists in limiting the number of varieties to be computed to the lowest possible value, possibly to just one.

Once $\mathbf{X}$ is known, cable lengths may be directly calculated by Eq. (1). Cable tensions may then be computed by any three suitable relations chosen within Eq. (2), e.g., if $\mathbf{s}_1, \mathbf{s}_2$, and $\mathbf{s}_3$ are linearly independent, as

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = -\begin{bmatrix} \frac{\mathbf{s}_1}{\rho_1} & \frac{\mathbf{s}_2}{\rho_2} & \frac{\mathbf{s}_3}{\rho_3} \end{bmatrix}^{-1} Q\mathbf{k} \tag{4}$$

The robot configuration is *feasible* if all cable tensions are nonnegative and equilibrium is stable. Stability may be evaluated by assessing the definiteness of the reduced Hessian matrix $\mathbf{H}_r$, as defined in Ref. [2]. The algorithm presented in Ref. [2] is based on a purely algebraic formulation, which allows the equilibrium stability of a robot of arbitrary geometry to be evaluated in a very efficient way. In the examples reported in this paper, the symbols $>$, $\geq$, $<$, $\leq$, and $<>$ denote, respectively, a positive-definite, a positive-semidefinite, a negative-definite, a negative-semidefinite, and an indefinite matrix.

3 IGP With Assigned Orientation

When Φ is assigned, all vectors $\mathbf{r}_i, i = 1, \ldots, 3$, are known. In this case, the position vector $\mathbf{x}$ must be determined.

If O is chosen as the moment pole, $\mathcal{L}_i$ and $\mathcal{L}_e$ may be, respectively, expressed, in axis coordinates, as $-[\mathbf{s}_i; \mathbf{a}_i \times \mathbf{s}_i]$ and $[\mathbf{k}; \mathbf{x} \times \mathbf{k}]$. Accordingly, $\mathbf{M}$ becomes

$$\mathbf{M}(O) = \begin{bmatrix} -\mathbf{s}_1 & -\mathbf{s}_2 & -\mathbf{s}_3 & \mathbf{k} \\ \mathbf{0} & -\mathbf{a}_2 \times \mathbf{s}_2 & -\mathbf{a}_3 \times \mathbf{s}_3 & \mathbf{x} \times \mathbf{k} \end{bmatrix} \tag{5}$$

or equivalently, by expanding $\mathbf{s}_i, i = 1, \ldots, 3$, and by performing elementary column transformations

$$\mathbf{M}'(O) = \begin{bmatrix} \mathbf{x} + \mathbf{r}_1 & \mathbf{r}_{21} - \mathbf{a}_2 & \mathbf{r}_{31} - \mathbf{a}_3 & \mathbf{k} \\ \mathbf{0} & \mathbf{a}_2 \times (\mathbf{x} + \mathbf{r}_2) & \mathbf{a}_3 \times (\mathbf{x} + \mathbf{r}_3) & \mathbf{x} \times \mathbf{k} \end{bmatrix} \tag{6}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, i \neq j$.

Since the cables may not all be parallel to $\mathbf{k}$ (cf. Assumption II in Sec. 2), the ranks of the block matrices[2] $\mathbf{M}_{123}(O)$ and $\mathbf{M}'_{123}(O)$ must be equal to either 2 or 3. By letting

$$\Delta := \det \mathbf{M}'_{123,234}(O) = (\mathbf{r}_{21} - \mathbf{a}_2)_x (\mathbf{r}_{31} - \mathbf{a}_3)_y - (\mathbf{r}_{21} - \mathbf{a}_2)_y (\mathbf{r}_{31} - \mathbf{a}_3)_x \tag{7}$$

two cases may be distinguished, depending on Δ being zero or not (notice that Δ only depends on the input parameters, when Φ is assigned). In general, Eq. (3) is satisfied if $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_e$ belong to the same three-dimensional subspace of lines,[3] namely [40]:

(1) a regulus on a one-sheeted hyperboloid;
(2) the union of two planar pencils sharing a line, none of which has its center at infinity;
(3) a bundle through a point;
(4) a regulus on a hyperbolic paraboloid;
(5) the union of two planar pencils sharing a line, with one of them having its center at infinity;
(6) a bundle of parallel lines; and
(7) a planar field.

Cases 1–3 are the most general ones and they are characterized by $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$, and $\mathbf{k}$ not being perpendicular to a common direction. In this case, $\operatorname{rank}\mathbf{M}'_{123}(O) = 3$ and $\Delta \neq 0$.

Cases 4–7, instead, have $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_e$ lying perpendicularly to a common direction. In this case, $\operatorname{rank}\mathbf{M}'_{123}(O) = 2$ and $\Delta = 0$. Assumption II in Sec. 2 rules out the possibility that case 6 may actually occur. Furthermore, case 7 must be taken into consideration only when A_1, A_2, and A_3 lie on a plane Σ parallel to $\mathbf{k}$, and G, B_1, B_2, and B_3 are coplanar, since, in this case, it may happen that the two planes coincide. In this circumstance, Eq. (3) is identically satisfied and an ad hoc procedure is in order. In particular, since the theoretical constraints imposed by cables are sufficient to determine the platform pose on Σ, the "flattened" 3–3 robot is geometrically equivalent to a planar parallel manipulator with telescoping legs connected to the base and the platform by revolute joints, so that it may take full advantage of the algorithms available for its geometric and static analyses [33].[4]

It is important to observe that letting $B_i \equiv A_i$, with $i = 1, \ldots, 3$, causes the ith column of $\mathbf{M}$ to vanish (since $\mathbf{s}_i = \mathbf{a}_i \times \mathbf{s}_i = \mathbf{0}$) and, thus, it causes all 4×4 minors of $\mathbf{M}$ (and of $\mathbf{M}'$) to be zero. It follows that, when a configuration for which $B_i \equiv A_i$ is compatible with the assigned constraints, it necessarily belongs to V: we call it a *trivial* solution and we need to discard it (cf. Assumption I in Sec. 2). This observation is particularly important for the IGP with assigned orientation. In this case, in fact, it is always possible to displace the platform (with a given orientation) so as to superimpose B_i onto A_i. Consequently, all varieties V_{lhk} necessarily contain the trivial solutions corresponding to $B_1 \equiv A_1, B_2 \equiv A_2$ and $B_3 \equiv A_3$, namely

$$\bar{\mathbf{x}}_i := [\bar{x}_i, \bar{y}_i, \bar{z}_i]^T = \mathbf{a}_i - \mathbf{r}_i, \quad i = 1, \ldots, 3 \tag{8}$$

3.1 Case $\Delta \neq 0$.

The equations

$$p_1 := \det \mathbf{M}'_{1236}(O) = A_{20}x^2 - (B_{011} + C_{101})xy + A_{02}y^2 + A_{10}x + A_{01}y + A_{00} = 0 \tag{9a}$$

$$p_2 := \det \mathbf{M}'_{1235}(O) = B_{110}xy - A_{20}xz + B_{020}y^2 + B_{011}yz + B_{100}x + B_{010}y + B_{001}z + B_{000} = 0 \tag{9b}$$

$$p_3 := \det \mathbf{M}'_{1234}(O) = B_{110}x^2 + B_{020}xy - C_{101}xz + A_{02}yz - C_{100}x - C_{010}y - C_{001}z - C_{000} = 0 \tag{9c}$$

comprise the lowest-degree polynomials among all minors of $\mathbf{M}'(O)$[5]. Since $\operatorname{rank}\mathbf{M}'_{123}(O) = 3$, satisfying Eqs. (9a)–(9c) ensures that $\operatorname{rank}\mathbf{M}'(O) = \operatorname{rank}\mathbf{M}'_{123}(O)$ and thus $V \equiv V_{123}$.

p_1 is quadratic in x and y, and it does not contain the variable z. p_2 and p_3 are quadratic in x, y, and z, but they do not contain the monomial z^2. Eliminating z from p_2 and p_3 gives a cubic polynomial p_{23} in x and y, whereas eliminating y from p_1 and p_{23} yields a fourth-degree equation in x, namely

$$p_{123} := \sum_{h=0}^{4} E_h x^h = 0 \tag{10}$$

whose coefficients E_h, $h = 0, \ldots, 4$, are known functions of the geometric and orientation parameters only.

It emerges from Eqs. (9a)–(9c) that the coefficients of the leading monomials y^2 and y^3 in p_1 and p_{23} are, respectively, A_{02} and $A_{02}B_{020}$ (with, in particular, $A_{02} = a_{3x}r_{21x} - a_{2x}r_{31x}$). Since A_{02} appears in both coefficients, it factors all terms of the resultant of p_1 and p_{23} with respect to y. p_{123} is the polynomial obtained by clearing A_{02} from such a resultant. This simplification is obvious when $A_{02} \neq 0$, but it is still valid when $A_{02} = 0$, with a caveat. In fact, if the resultant of p_1 and p_{23} is calculated after setting $A_{02} = 0$, a fourth-degree polynomial in x is again obtained and it proves to be equal to p_{123} times the constant

$$\Gamma_0 := B_{020}(C_{001} + A_{01}) - C_{010}B_{011} \tag{11}$$

Hence, as long as $\Gamma_0 \neq 0$, p_{123} is still a legitimate elimination ideal for p_1 and p_{23}. The case $\Gamma_0 = 0$ is discussed in Appendix A, in the section concerning empty solution sets.

Since three roots of p_{123} must necessarily coincide with the trivial solutions $\bar{x}_1, \bar{x}_2$, and $\bar{x}_3$, and these are known real numbers, p_{123} may be factorized by using Vieta's formulas. Accordingly, the fourth root $\bar{x}_4$ must be *real*, and it may be expressed as

$$\bar{x}_4 = \frac{E_0}{\bar{x}_1 \bar{x}_2 \bar{x}_3 E_4} = \frac{\hat{E}_0}{\hat{E}_4} \tag{12}$$

[2]The notation $\mathbf{M}_{hij,klm}$ denotes the block matrix obtained from rows h, i, and j, and columns k, l, and m, of $\mathbf{M}$. When all columns are used, the corresponding subscripts are omitted.

[3]It may happen that Eq. (3) is fulfilled because $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ become linearly dependent. In these situations, equilibrium is possible only if $\operatorname{rank}(\mathbf{M}) \leq 2$, since the external wrench must, however, belong to the screw subspace generated by the cable lines. Cases like these are very special and they must be studied separately.

[4]It is worth emphasizing that having A_1, A_2, and A_3 on a plane parallel to $\mathbf{k}$, and G, B_1, B_2, and B_3 coplanar, does *not* necessarily cause the 3–3 robot to behave like a planar mechanism: this is a possibility, to be considered together with the general *spatial* study of the manipulator. For example, when the DGP is considered, a number of feasible stable configurations may be found, for which the base and the platform are skew.

[5]Equations (9a)–(9c) coincide with those that would emerge by computing cable tensions by the first three relations in Eq. (2) and by substituting them back into the remaining ones.

Table 1 IGP of a 3–3 robot with assigned orientation and $\Delta \neq 0$

Geometric dimensions and load: $\mathbf{a}_2=[10;0;0], \mathbf{a}_3=[0;12;0], \mathbf{r}_1=[0.6;0.10;-0.8], \mathbf{r}_2=[-0.4;-0.9;-0.3], \mathbf{r}_3=[-0.7;0.5;-0.5], Q=10$ Trivial solutions: $\bar{\mathbf{x}}_1=[-0.6;-0.1;0.8], \bar{\mathbf{x}}_2=[10.4;0.9;0.3], \bar{\mathbf{x}}_3=[0.7;11.5;0.5]; \Delta=126.30$			
Univariate polynomial in x: $p_{123}=43069563/1562500\cdot(5x+3)(5x-52)(10x-7)(1805x-5196)$			
Nontrivial solution	(ρ_1,ρ_2,ρ_3)	(τ_1,τ_2,τ_3)	$\mathbf{H}_r$
$\bar{\mathbf{x}}_4=[5196/1805;2973/722;139299/20938]=[2.88;4.12;6.65]$	(8.01 10.36 9.85)	$(+5.05+4.69+5.49)$	$>$

where the expression at the right-hand side of Eq. (12) takes advantage of the fact that it is possible to factorize E_0 and E_4 as $\bar{x}_1\bar{x}_2\bar{x}_3\Delta^2\hat{E}_0$ and $\Delta^2\hat{E}_4$, respectively.

By eliminating x (instead of y) from p_1 and p_{23}, one may obtain a univariate fourth-degree equation in y (instead of x), i.e.

$$p'_{123} := \sum_{h=0}^{4} F_h y^h = 0 \tag{13}$$

from which an expression for the direct computation of $\bar{y}_4$ immediately results, namely

$$\bar{y}_4 = \frac{F_0}{\bar{y}_1\bar{y}_2\bar{y}_3F_4} = \frac{\hat{F}_0}{\hat{F}_4} \tag{14}$$

where $F_0=\bar{y}_1\bar{y}_2\bar{y}_3\Delta^2\hat{F}_0$ and $F_4=\Delta^2\hat{F}_4$.

Since $\hat{E}_0, \hat{E}_4, \hat{F}_0$, and $\hat{F}_4$ are known functions of the geometric and orientation parameters, and they are available in *symbolic* form, Eqs. (12) and (14) provide a *closed-form* solution of the problem at hand. Indeed, once $\bar{x}_4$ and $\bar{y}_4$ are known, $\bar{z}_4$ may be directly calculated by either Eq. (9*b*) or Eq. (9*c*), which are linear in z. Cable lengths and tensions follow then from Eqs. (1) and (4), respectively, whereas equilibrium stability may be assessed by solving a linear eigenproblem of order 3 [2]. Clearly, $\bar{\mathbf{x}}_4=(\bar{x}_4,\bar{y}_4,\bar{z}_4)$ is feasible if cable tensions are non-negative and equilibrium is stable. Since a closed-form expression of $\bar{\mathbf{x}}_4$ is available, the entire computation (including stability analysis) may be completed in a very short time, usually in less than 1 ms. A numeric example is reported in Table 1. In this case, the axes of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_e$ form a regulus on a one-sheeted hyperboloid.

The above study proves that, as long as $\hat{E}_4 \neq 0 \neq \hat{F}_4$, the IGP with assigned orientation (and $\Delta \neq 0$) has a *single real* solution, i.e., $\bar{\mathbf{x}}_4$. However, there are special combinations of the geometric parameters and input orientations for which Eqs. (9*a*)–(9*c*) admit a positive-dimensional solution set. In these cases, G may follow quasi-static paths along assigned curves in space, with the platform orientation remaining constant. These cases may have strong practical relevance and, for this reason, they are discussed in detail in Appendix A.

3.2 Case $\Delta = 0$.

By expanding the coefficients of p_{123}, it is possible to verify that all of them comprise the factor Δ^2. Consequently, when $\Delta=0$, p_{123} degenerates and the procedure described in Sec. 3.1 is not adequate to solve the problem.

Since Assumption II at the end of Sec. 2 guarantees that $(\mathbf{r}_{21}-\mathbf{a}_2)_{xy}$ and $(\mathbf{r}_{31}-\mathbf{a}_3)_{xy}$ do not vanish, when $\Delta=0$ these two vectors must be parallel (cf. Eq. (7)), i.e.

$$(\mathbf{r}_{31}-\mathbf{a}_3)_{xy} = \alpha(\mathbf{r}_{21}-\mathbf{a}_2)_{xy} \tag{15}$$

where $\alpha \in R - \{0\}$.

By enforcing both Eq. (15) and $\Delta=0$, the polynomial relations in Eq. (9) may be factored as

$$p_1 := f_{10}g = 0 \tag{16a}$$

$$p_2 := f_{20}g = 0 \tag{16b}$$

$$p_3 := f_{30}g = 0 \tag{16c}$$

where f_{10} and g are linear polynomials in x and y, and f_{20} and f_{30} are linear polynomials in x, y, and z. In particular

$$g := \det \mathbf{M}'_{123,124}(O) = (\bar{y}_1-\bar{y}_2)x - (\bar{x}_1-\bar{x}_2)y + \bar{x}_1\bar{y}_2 - \bar{y}_1\bar{x}_2 \tag{17}$$

where the coefficients multiplying x and y cannot simultaneously vanish, since this would infer $(\bar{\mathbf{x}}_1-\bar{\mathbf{x}}_2)_{xy} = (\mathbf{r}_{21}-\mathbf{a}_2)_{xy} = \mathbf{0}$.

Equation (16) holds if either $f_{10}=f_{20}=f_{30}=0$ or $g=0$. Since the former requirement amounts to a linear nonhomogeneous system in x, y, and z whose coefficient matrix is singular (and, hence, it generally admits no solution), only the latter condition applies. Together with the requisite $\Delta=0$, this brings about that the first two rows of $\mathbf{M}'(O)$ are linearly dependent and that $\operatorname{rank}\mathbf{M}'_{123}(O)=2$. The problem may, thus, be solved by considering the system made up by the relation

$$p_4 := g = 0 \tag{18}$$

and any two between the equations

$$p_5 := \det \mathbf{M}'_{2345}(O) = 0 \tag{19a}$$

$$p_6 := \det \mathbf{M}'_{2346}(O) = 0 \tag{19b}$$

$$p_7 := \det \mathbf{M}'_{2356}(O) = 0 \tag{19c}$$

where p_5, p_6, and p_7 are cubic polynomials in x, y, and z (with p_6 and p_7 being linear in z and p_5 being quadratic in the same variable). If, for instance, p_6 and p_7 are considered, eliminating z from them gives a fourth-degree polynomial p_{67} in x and y, and further eliminating y from p_4 and p_{67} yields a quartic equation in x, i.e., $p_{467}=0$, which admits a single solution besides the trivial ones. This solution formally coincides with the quotient $\hat{E}_0/\hat{E}_4$ at the right-hand side of Eq. (12). Such an expression provides, thus, the *general* single solution of the problem at hand for all cases 1–5 listed in the first part of this section. If $\mathbf{M}'_{236}(O)$ has full rank, then $\operatorname{rank}\mathbf{M}'(O)=\operatorname{rank}\mathbf{M}'_{236}(O)=3$ and $V=V_{467}$. A numeric example is reported in Table 2. In this case, $\Delta=0$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_e$ form a regulus on a hyperbolic paraboloid.

It may happen that, in very special cases, p_{467} degenerates and the procedure described above becomes deficient. Typically, this may occur when $\operatorname{rank}\mathbf{M}'_{236}=2$. A different choice of the minors to be considered normally allows one to conclude the analysis.[6]

4 IGP With Assigned Position

In this case, $\mathbf{x}$ is assigned, whereas Φ and, thus, vectors $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$ must be ascertained (only the position vector $\mathbf{b}_i$ of B_i in $Gx'y'z'$ is known).

If the platform orientation is described by means of Euler parameters, i.e., $\Phi=[e_o;e_1;e_2;e_3]$, the rotation matrix has the form

[6]A dialytic algorithm, similar to the one described in the subsequent Sec. 4, may be used as an alternative procedure with respect to the one presented in this section to solve the IGP with assigned orientation. In this way, a univariate resultant of degree 1 in x (or y) may be computed as the determinant of a 5×5 eliminant matrix.

Table 2 IGP of a 3–3 robot with assigned orientation and $\Delta = 0$

Geometric dimensions and load: $\mathbf{a}_2 = [10; 0; 0], \mathbf{a}_3 = [0; 12; 0], \mathbf{r}_1 = [1; 1; 2], \mathbf{r}_2 = [5; -3; 3], \mathbf{r}_3 = [-3.2; 10.2; -0.5], Q = 10$ Trivial solutions: $\bar{\mathbf{x}}_1 = [-1; -1; -2], \bar{\mathbf{x}}_2 = [5; 3; -3], \bar{\mathbf{x}}_3 = [3.2; 1.8; 0.5]; \Delta = 0$			
Univariate polynomial in x: $p_{467} = 21504/5 \cdot (x+1)(x-5)(5x-16)(137x+989)$			
Nontrivial solution	(ρ_1, ρ_2, ρ_3)	(τ_1, τ_2, τ_3)	$\mathbf{H}_r$
$\bar{\mathbf{x}}_4 = [-989/137; -705/137; 396757/10823] = [-7.22; -5.15; 36.66]$	(39.37 42.29 38.27)	(+21.91,−7.70,−4.54)	<>

$$\mathbf{R} = \mathbf{I}_3 + 2\frac{e_0\widetilde{\Phi}_{123} + \widetilde{\Phi}_{123}\widetilde{\Phi}_{123}}{e_0^2 + e_1^2 + e_2^2 + e_3^2} \quad (20)$$

where $\Phi_{123} = [e_1; e_2; e_3]$ and $\widetilde{\Phi}_{123}$ denotes the skew-symmetric matrix expressing the operator $\Phi_{123}\times$.

If Euler parameters are normalized in the Rodrigues form [41] by setting $e_0 = 1$, the relations in Eq. (9) assume a particularly favorable structure. Indeed, by letting $\mathbf{r}_i = \mathbf{R}\mathbf{b}_i, i = 1, \ldots, 3$, and by clearing the nonzero denominator $1 + e_1^2 + e_2^2 + e_3^2$, p_1, p_2, and p_3 become quartic polynomials in the Rodrigues parameters, i.e.

$$p_h := \sum_{\substack{k=0,\ldots,4 \\ m=0,\ldots,4-k \\ n=0,\ldots,4-k-m}} D_{h,kmn} e_1^k e_2^m e_3^n = 0, \quad h = 1, \ldots, 3 \quad (21)$$

Quartic polynomials in e_1, e_2, and e_3 also emerge from the minors $\det \mathbf{M}'_{1245}(O), \det \mathbf{M}'_{1246}(O)$ and $\det \mathbf{M}'_{1256}(O)$, but they linearly depend on p_1, p_2, and p_3, so that they may be discarded. A further quartic emerges by setting $\det \mathbf{M}'_{j456}(O) = 0$ for $j = 1, \ldots, 3$, so that

$$(\mathbf{x} + \mathbf{r}_1)\left[\det \mathbf{M}'_{456,234}(O)\right] = \mathbf{0} \quad (22)$$

The variety defined by Eq. (22) comprises, besides the trivial solution $\mathbf{x} = -\mathbf{r}_1$, also the set of configurations for which

$$p_8 := \det \mathbf{M}'_{456,234}(O) = 0 \quad (23)$$

Equation (23) has, indeed, degree 4 in e_1, e_2, and e_3. All minors of $\mathbf{M}'(O)$ not considered so far, namely, $\det \mathbf{M}'_{1345}(O), \det \mathbf{M}'_{1346}(O)$, $\det \mathbf{M}'_{1356}(O)$, $\det \mathbf{M}'_{2345}(O), \det \mathbf{M}'_{2346}(O)$, and $\det \mathbf{M}'_{2356}(O)$, yield, instead, sextic equations in the Rodrigues parameters.

It is known that three polynomial equations of the same total degree always admit a Sylvester-type resultant free from extraneous polynomial factors [42]. For the case of three quartics, such a resultant is a 64th-degree polynomial in one of the unknowns. This polynomial may be obtained, in the present case, from p_8 and any two between p_1, p_2, and p_3 (the resultant of p_1, p_2, and p_3 is identically nought). This observation led the author to conclude, in Ref. [1], that the IGP with assigned position of the 3–3 robot admits at the most 64 solutions. Such an estimation, however, is too broad, since it is possible to verify, by numerical experimentation, that the varieties V_{128}, V_{138}, and V_{238} only have 48 solutions in common and only 24 of them actually satisfy the aforementioned sextics, thus belonging to V.

The necessity to fulfill multiple relations, mutually dependent in a nonliner way, suggests using Sylvester's dialytic method [43] in order to obtain a univariate polynomial of least degree in one of the unknowns, e.g., e_3. The method consists in rewriting the original relations as linear equations in all monomials involving the original unknowns except one, which is "hidden" in the equation coefficients. Such monomials are then treated as linear homogeneous unknowns. By augmenting the system with new auxiliary equations (an operation that usually introduces new monomials), one may hope to arrive at a square homogeneous linear system. If this is generically nonsingular, the determinant of the coefficient matrix provides a resultant in the hidden variable. However, the resultant may potentially contain an extraneous polynomial factor, a situation that one wishes to avoid. Finding a formulation free of spurious solutions is a crucial issue in the application of the method. The strategy presented here is based on deriving a large set of linearly independent quartics.

Let $\mathbf{M}$ be written by choosing a generic point P as the moment pole, namely

$$\mathbf{M}(P) = \begin{bmatrix} \cdots & -\mathbf{s}_i & \cdots & \mathbf{k} \\ \cdots & -(B_i - P) \times \mathbf{s}_i & \cdots & (G - P) \times \mathbf{k} \end{bmatrix} \quad (24)$$

When $P \equiv B_i$ and $P \equiv A_i, i = 1, \ldots, 3$, the moment vector in the ith column vanishes, so that setting $\det \mathbf{M}_{j456}(B_i) = 0$ and $\det \mathbf{M}_{j456}(A_i) = 0$ for $j = 1, \ldots, 3$ yields, respectively

$$\mathbf{s}_i\left[\det \mathbf{M}_{456,km4}(B_i)\right] = \mathbf{0} \quad (25)$$

and

$$\mathbf{s}_i\left[\det \mathbf{M}_{456,km4}(A_i)\right] = \mathbf{0} \quad (26)$$

with $k, m \in \{1, 2, 3\} - \{i\}$. This way, the equations

$$p_9 := \det \mathbf{M}_{456,234}(B_1) = 0 \quad (27a)$$

$$p_{10} := \det \mathbf{M}_{456,134}(B_2) = 0 \quad (27b)$$

$$p_{11} := \det \mathbf{M}_{456,134}(A_2) = 0 \quad (27c)$$

$$p_{12} := \det \mathbf{M}_{456,124}(B_3) = 0 \quad (27d)$$

$$p_{13} := \det \mathbf{M}_{456,124}(A_3) = 0 \quad (27e)$$

may be obtained. Analogously, by defining a convenient additional point G_0 such that $G - G_0 = \mathbf{k}$, and by setting $P \equiv G$ and $P \equiv G_0$, one may also obtain

$$p_{14} := \det \mathbf{M}_{456,123}(G) = 0 \quad (28a)$$

$$p_{15} := \det \mathbf{M}_{456,123}(G_0) = 0 \quad (28b)$$

All polynomials p_j, $j = 9, \ldots, 15$, have degree 4 in the Rodrigues parameters. No other quartic linearly independent from p_1, p_2, and p_3 may be obtained from the minors of $\mathbf{M}$ by varying the moment pole. The relations in Eqs. (21), (23), (27), and (28) form a system of 11 quartics, comprising 15 monomials in e_1 and e_2. By multiplying such quartics by e_2, 11 additional relations may be introduced, which comprise five novel monomials in e_1 and e_2. Among the new equations, nine may be chosen so as to form, together with the original ones, a linear system of the form

$$\mathbf{S}(e_3)\mathbf{E}_{23} = \mathbf{0} \quad (29)$$

where $\mathbf{S}(e_3)$ is a 20×20 matrix whose entries are known polynomial functions (available in *symbolic* form) of e_3 and $\mathbf{E}_{23}$ is a column vector comprising all monomials of e_1 and e_2 with a degree less than or equal to 5, except e_1^5. The last three entries of $\mathbf{E}_{23}$ are, in particular, e_1, e_2, and 1. The determinant of $\mathbf{S}(e_3)$ provides the desired 24th-degree resultant devoid of spurious roots, namely

$$\det \mathbf{S}(e_3) = \sum_{h=0}^{24} L_h e_3^h = 0 \tag{30}$$

where coefficients L_h, $h = 0, \ldots, 24$, only depend on the position of G and the geometric parameters.

An alternative algorithm with respect to the one described above to obtain a univariate resultant consists in conveniently choosing, among the 22 novel equations derived by multiplying the original ones by e_1 and by e_2, 10 linearly independent relations that contain all six monomials of degree 5 in e_1 and e_2. In this way, the resultant in e_3 may be computed as the determinant of a 21×21 matrix. Numerical tests show that this formulation is more robust when dealing with special cases occurring for particular geometric configurations (for which the 20×20 determinant vanishes).

Since the expansion of the determinant of a 20×20 or 21×21 matrix in completely symbolic form is extremely onerous for a computer algebra system (such as `Maple` or `Mathematica`), this operation must normally be accomplished by assigning numeric values to known geometric quantities. Once this is accomplished, $\mathbf{S}(e_3)$ contains only one symbolic variable, i.e., e_3, and the expansion of its determinant is undemanding. On a PC with a 2.67 GHz Intel Xeon processor and 4GB of RAM, the expansion of $\det \mathbf{S}(e_3)$ for the example reported in Table 3 may be performed by `Maple` in less than 0.5 s.

For each root of Eq. (30), a single value for e_1 and e_2 may be obtained by solving the linear system in Eq. (29). Cable tensions may then be computed by Eq. (4). The set of *feasible* configurations emerges by selecting the real roots of $\det \mathbf{S}(e_3)$ for which cable tensions are non-negative and equilibrium is stable. The overall computation time is essentially given by the time required to expand $\det \mathbf{S}(e_3)$, since all subsequent steps (including finding the real roots of the determinant) are at least one order-of-magnitude faster.

The expansion of $\det \mathbf{S}(e_3)$ may be avoided, and computation quickened, by setting up Eq. (29) as a polynomial eigenvalue problem, in the form

$$\mathbf{S}(e_3)\mathbf{E}_{23} = \left(\sum_{k=0}^{4} e_3^k \mathbf{A}_k\right)\mathbf{E}_{23} = \mathbf{0} \tag{31}$$

where $\mathbf{S}(e_3)$ has been written as a matrix polynomial of order 4, whose coefficients $\mathbf{A}_k, k = 0, \ldots, 4$, are constant matrices only depending on the position of G and the geometric parameters. The roots of $\det \mathbf{S}(e_3)$ and the vectors $\mathbf{E}_{23}$ solving Eq. (29) are, by definition, the eigenvalues and eigenvectors of $\mathbf{S}(e_3)$. If $\dim \mathbf{S}(e_3) = N$, these eigenpairs may be numerically computed by converting Eq. (31) into a linear eigenvalue problem of order $4N$ with the same finite eigenvalues, so that classic methods for linear eigenproblems may be pressed into service [44]. This way, all

Table 3 A 3–3 robot whose IGP with assigned position admits 24 real potential solutions

Geometric dimensions and load: $\mathbf{a}_2 = [9; 10; 0], \mathbf{a}_3 = [6; 8; 7], \mathbf{b}_1 = [-7; 3; 2], \mathbf{b}_2 = [-5; 1; -1], \mathbf{b}_3 = [-5; -2; 0], \mathbf{x} = [7; 9; 1], Q = 10$

Univariate polynomial in e_3: $\det \mathbf{S}(e_3) = +820677682620056914502526001540236149664458265234 e_3^{24}$

$$\begin{aligned}
&-1061765379995086980503032620927413883140461664 0847 e_3^{23} - 805201120762915050953742545889256799366411400306 12 e_3^{22}\\
&+62072319163059674570862791390270285891267211834 8292 e_3^{21} + 22444573815923787449510720035728864665054468047315 68 e_3^{20}\\
&-10042557842888364893128923167086735878025704668943759 e_3^{19} - 19029295771254380694758354014911317918738663989654152 e_3^{18}\\
&+72578960982294287507086571892197357912446615077237786 e_3^{17} + 73830200439589669837526253624025758230943986804931990 e_3^{16}\\
&-282390367979251759472889802317563412868139973956341870 e_3^{15} - 147054215376404227324409755922820121888570550749730984 e_3^{14}\\
&+642054361021857602608966458820847086905733936256221504 e_3^{13} + 148405195196987585178762580495747225216253541090816592 e_3^{12}\\
&-885338166160478450118928113693274810166512764272208294 e_3^{11} - 603027629427570356443375591433711293637945769801570 08 e_3^{10}\\
&+751323495984155705731825209490642557900604032459994716 e_3^{9} - 115953796696205014954028842979183472611087300203371 86 e_3^{8}\\
&-388351844813509515413972040723375108190383669878826915 e_3^{7} + 186579398486447271629479540328528783636658787515718 68 e_3^{6}\\
&+116581403236475679656512614572842467737777178404043276 e_3^{5} - 547949521758322508814130433753018363255050285319881 6 e_3^{4}\\
&-180881523933723041575685978361033481977513071654316 59 e_3^{3} + 367298317401261058941260215786258009876669842615928 e_3^{2}\\
&+106945423569192819933217950470346274965399412154988 2 e_3 + 466114829983096414959446521715693507432600971624 58
\end{aligned}$$

Configuration	(e_1, e_2, e_3)	(ρ_1, ρ_2, ρ_3)	(τ_1, τ_2, τ_3)	$\mathbf{H}_r$
1	$(+0.7634246351375643899, +0.2144041953164012399, +0.4807389069063061451)$	$(5.08, 7.47, 9.28)$	$(-8.54, -0.69, -16.68)$	$<>$
2	$(-0.3118079777548289964, +1.3138841327707174979, +0.5017867472282055003)$	$(16.98, 6.58, 4.07)$	$(+2.88, +9.87, +1.56)$	$<>$
3	$(+0.2001608318444858809, -0.2323624476334943307, +0.7979367415323793600)$	$(3.68, 6.46, 9.32)$	$(-29.69, -31.99, +3.74)$	$<$
4	$(-1.1980739835427372418, +0.3120413901385634775, +0.8815736227134803121)$	$(9.28, 7.50, 0.78)$	$(+0.24, -4.20, -12.87)$	$<$
5	$(+1.4317151114911808362, -0.4349823013446592256, +1.1664189736927190095)$	$(6.85, 6.17, 11.44)$	$(+0.94, -0.46, -10.39)$	$<>$
6	$(-1.2294176759025500749, +1.5323521263619918432, +1.2814168119956330214)$	$(15.76, 6.54, 4.79)$	$(+0.55, +11.57, +3.56)$	$<>$
7	$(+0.5876685126742958725, -1.5304987591598525061, +1.3389381429787698737)$	$(11.96, 3.83, 10.53)$	$(-3.36, -12.10, +4.32)$	$<>$
8	$(-2.0793173042033534266, -0.5423326238903700990, +1.4438672936010393110)$	$(4.70, 7.16, 1.82)$	$(-37.83, +0.87, -44.77)$	$<$
9	$(-7.1168007105998403878, -2.2727912287219393891, +1.4867153462984873810)$	$(3.59, 7.48, 5.41)$	$(+17.37, +20.09, -2.10)$	$<>$
10	$(-2.3167391480195777785, -0.4212144251684132813, +1.7790336236054227234)$	$(5.90, 7.12, 1.01)$	$(-8.19, +0.85, -16.57)$	$<$
11	$(+5.7978576157189826538, +0.5703866722573983407, +5.9874716213902005065)$	$(10.98, 5.66, 11.53)$	$(-0.36, -2.33, -8.49)$	$<>$
12	$(+6.5213887323660842474, -9.5688652126571470658, +15.6259039246963154437)$	$(15.07, 3.18, 9.90)$	$(-0.48, -18.96, +10.58)$	$<>$
13	$(+0.2579484191462487771, +0.7260739527980151820, -0.0458987036139806216)$	$(15.06, 7.12, 2.88)$	$(-72.14, -0.73, -78.57)$	$<>$
14	$(+0.0456408825214523273, +0.9618070707480913020, -0.2934072474535058737)$	$(17.75, 6.28, 0.88)$	$(+6.72, -0.60, -9.95)$	$<$
15	$(-0.3231844159276431892, -1.0590220806007764338, -0.6096213673883211171)$	$(17.18, 3.65, 11.36)$	$(+0.56, -2.57, -7.61)$	$<>$
16	$(+0.8805611206146294749, +1.1250708193247081074, -0.6795024237035720852)$	$(14.60, 6.35, 1.63)$	$(-0.37, +7.92, -3.39)$	$<$
17	$(+0.7431375057822628939, +1.2067418279500934783, -0.7808443877651603867)$	$(15.68, 5.98, 0.85)$	$(-0.39, +5.07, -5.23)$	$<$
18	$(-0.0350704155412834709, -1.4151080814907166256, -0.8433915522557506921)$	$(18.47, 2.88, 11.54)$	$(+0.67, +4.73, -14.18)$	$>$
19	$(-0.4329502420956995131, -1.3468211640704336871, -1.0018691108946436935)$	$(18.00, 3.28, 11.52)$	$(+1.76, +0.80, -11.26)$	$<>$
20	$(-1.4218370234199270169, -0.9117398153636558460, -1.6469179198997482355)$	$(16.10, 4.59, 11.27)$	$(-2.36, -5.99, -3.97)$	$<>$
21	$(-1.6901971364929174389, +7.9091089569234085282, -1.6909672945676966476)$	$(18.93, 3.67, 7.68)$	$(-16.63, +18.16, +3.01)$	$<>$
22	$(-0.0068199958458342934, -0.7154426580635924040, -1.7244167908598977683)$	$(19.28, 2.82, 10.54)$	$(+17.96, -16.55, -3.94)$	$<$
23	$(+4.1084117389317057621, -0.4212589725746318291, -4.1764537703635687677)$	$(13.13, 6.08, 4.98)$	$(+3.29, +7.09, -2.10)$	$<>$
24	$(+3.6592836868101469244, -2.8882478193490052005, -6.3408451287042458310)$	$(16.68, 4.85, 7.69)$	$(+8.64, +0.80, -9.19)$	$<>$

feasible solutions of the IGP with assigned position may be ordinarily calculated, for a robot of arbitrary geometry, within a few milliseconds.

Table 3 reports the solution set for a numeric example, whose geometric dimensions are not particularly meaningful under a practical point of view, but which proves that all solutions of Eq. (30) may be *real*.

Since the presented algorithm makes use of all minors of $\mathbf{M}$, it allows the problem at hand to be solved for all cases 1–5 listed in the first part of Sec. 3. The degree of the equations involved in the problem-solving procedure is rather high and all coefficients are generally nonzero, so that positive-dimensional solution sets are unlikely to be found (except, obviously, when $(x, y) = (\bar{x}_i, \bar{y}_i)$ and $z \neq \bar{z}_i$, $i = 1, \ldots, 3$, in which case the robot is allowed to operate like a 1-dof crane, with the ith cable holding the entire charge).

Since Rodrigues parameters are unable to describe orientations requiring $e_0 = 0$, this case must be considered separately. For example, by setting $e_0 = 0$ and $e_j = 1$, for some j, the relations in Eqs. (21), (23), (27), and (28) become quartics in e_h and e_k, $h \neq j \neq k$. Since these equations contain only five monomials in e_k, they are "more" than sufficient to eliminate e_k and, thus, obtain a univariate resultant in e_h. If special geometric conditions are satisfied, common solutions may exist. An example is reported in Appendix B.

5 Conclusions

This paper studied the kinematics and statics of underconstrained cable-driven parallel robots with three cables, in crane configuration. For such robots, kinematics, and statics are coupled and they must be dealt with simultaneously. This poses major challenges, since displacement analysis problems gain remarkable complexity with respect to those of analogous 6-dof rigid-link robots.

An original geometrico-static model was presented, which allowed the IGP to be effectively worked out by elimination algorithms and the complete solution sets of the problem to be determined. It was shown that

- the IGP with assigned orientation, which aims at computing the platform position, the cable lengths and the cable tensions once the platform orientation is given, admits at the most a single solution;
- the IGP with assigned position, which aims at calculating the platform orientation, the cable lengths and the cable tensions once the platform position is known, admits at the most 24 solutions.

The former problem was solved by successive computation of resultants. When the solution exists, it is real. Relevant cases exist that cause the ordinarily single solution to degenerate into a one-dimensional set, thus forming, in the space of the platform position, a one-dimensional curve.

The latter problem was solved by a specialized Sylvester's dialytic procedure. A numerical example was given that proves that all 24 points in the variety of the problem may be real.

It must be observed that all solution counts reported above do not take into account the constraints imposed by the sign of cable tensions and the stability of equilibrium. Once such constraints are imposed and solutions are sifted, the number of feasible configurations drastically decreases. The author is not aware of effective mathematical techniques that would allow the constraint on the cable-tension sign to be incorporated into the equations to be solved, when elimination techniques are used. This issue is open to further research.

When the cable lengths emerging from the solution of the IGP are feeded-in to the actuators of the actual robot, the end-effector may evolve, a priori, in any one of the feasible solutions emerging from the corresponding direct problem and nothing guarantees that the desired configuration is actually reached. In fact, when multiple feasible configurations exist, the end-effector may switch across them, due to inertia forces or external disturbances. Accordingly, the computation of the entire set of equilibrium configurations is essential for robust trajectory planning. This motivates the relevance of the presented algorithms, which allow the complete set of feasible solutions of the IGP to be computed in a very efficient way (the order-of-magnitude of the computation is a few milliseconds, on a normal PC).

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Appendix A: Special Solution Sets of the IGP With Assigned Orientation

There are special combinations of geometric parameters and input orientations for which the IGP with assigned orientation of the 3–3 CDPR admits infinite solutions.[7] In these cases, the platform may follow quasi-static paths, within its feasible workspace, while preserving its orientation unaltered. This property may be very useful in applications in which the payload must be limitedly tilted during handling. Three relevant cases, holding for $\Delta \neq 0$, are studied here. For readers who are not interested in the details of the derivation, the main results are summarized hereafter.

- When planes $A_1A_2A_3$ and $B_1B_2B_3$ are perpendicular to $\mathbf{k}$ and
  - — either $\mathbf{r}_{i,xy}$ and $\mathbf{r}_{j,xy}$ are parallel to $(A_i - A_h)_{xy}$ and $(A_j - A_h)_{xy}$, respectively, for $h, i, j = 1, \ldots, 3$ and $h \neq i \neq j \neq h$ (Fig. 2(*a*)),
  - — or the sides of the projections of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ on the xy-plane are mutually parallel (Fig. 2(*b*)),

  V is a line parallel to $\mathbf{k}$.
- When the sides of the projections of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ on the xy-plane are mutually parallel, V is a curve lying on a cylinder whose axis is parallel to $\mathbf{k}$ and whose cross-section is a conic.
- When the platform orientation is such that $\mathbf{r}_i \times \mathbf{r}_j$ is parallel to $(\mathbf{a}_i - \mathbf{a}_j) \times \mathbf{k}$, for $i, j = 1,2,3$ and $i \neq j$, V is a conic lying on the plane Π_{ij} that is parallel to $\mathbf{k}$ and passes through A_i and A_j, and the robot behaves like a 2–2 CDPR [2], with the plane GB_iB_j coinciding with Π_{ij}, the ith and the jth cable holding the entire load and the hth cable, with $i \neq h \neq j$, being uncharged.
- When the platform orientation is such to require $\mathbf{r}_i$ to be parallel to $\mathbf{k}$, $i = 1,2,3$, the robot behaves like a 1-dof crane, with A_i, B_i and G being aligned and the ith cable holding the entire charge.

For the sake of completeness, the (not-so-rare) case in which the IGP does not admit a solution is also discussed at the end of the Appendix.

One-Dimensional Solution Set: A Line Parallel to k. When $\Delta \neq 0$, the IGP with assigned orientation is governed by Eqs. (9*a*)–(9*c*). The problem was solved in Sec. 3.1 by eliminating, as a first step, z from p_2 and p_3, thus obtaining a polynomial p_{23} in x and y, and by further eliminating y from p_1 and p_{23}, in order to attain a univariate polynomial p_{123} in x. As p_{23} is the resultant of p_2 and p_3 with respect to z, the variety defined by p_{23} necessarily

[7]This is strictly related to the particularly simple form of the equations that govern this problem, especially when $\Delta \neq 0$ (cf. Eq. (9)).

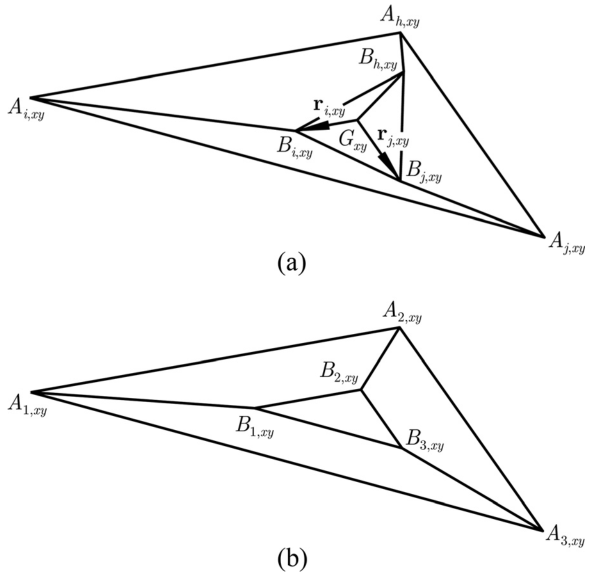

Fig. 2 Conditions for positive-dimensional solution sets of the IGP with assigned orientation: (*a*) $\mathbf{r}_{i,xy} \parallel (A_i - A_h)_{xy}$ and $\mathbf{r}_{j,xy} \parallel (A_j - A_h)_{xy}$; (*b*) the sides of the projections of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ on the *xy*-plane are mutually parallel

comprises the set of pairs (x,y) for which the leading coefficients of z in both p_2 and p_3 vanish [45]. Since p_2 and p_3 have degree one in z and, in both of them, the coefficient of the monomial z is linear in x and y, such a set ordinarily consists of a single element, namely

$$(\hat{x},\hat{y}) = \frac{1}{\delta}\left(a_{3x}n_{12} + a_{2x}n_{31}, a_{3y}n_{12} + a_{2y}n_{31}\right) \tag{A1}$$

where $\delta := n_{12} + n_{23} + n_{31}$ and $n_{ij} := \mathbf{r}_i \times \mathbf{r}_j \cdot \mathbf{k}, i,j = 1,\ldots,3$.

Normally, $(\hat{x},\hat{y})$ is not a solution of the problem, unless $p_1(\hat{x},\hat{y}), p_2(\hat{x},\hat{y})$ and $p_3(\hat{x},\hat{y})$ are simultaneously zero. If the relationship $p_1(\hat{x},\hat{y}) = 0$ is enforced, the following condition in the geometric and orientation parameters is obtained:

$$\Delta \cdot (n_{31}\mathbf{a}_2 \times \mathbf{r}_2 \cdot \mathbf{k} + n_{12}\mathbf{a}_3 \times \mathbf{r}_3 \cdot \mathbf{k}) = 0 \tag{A2}$$

and thus, by considering that $\Delta \neq 0$

$$\Gamma_1 := n_{31}\mathbf{a}_2 \times \mathbf{r}_2 \cdot \mathbf{k} + n_{12}\mathbf{a}_3 \times \mathbf{r}_3 \cdot \mathbf{k} = 0 \tag{A3}$$

If Eq. (A3) is satisfied, both p_1 and p_{23} vanish in $(\hat{x},\hat{y})$, so that $\hat{x}$ belongs to the variety of p_{123} and, accordingly, $(\hat{x},\hat{y}) = (\bar{x}_4,\bar{y}_4)$. $(\hat{x},\hat{y})$ extends to a complete solution if the terms of zero total degree in both p_2 and p_3 also vanish.[8] Setting such coefficients to zero amounts to two further conditions in the geometric and orientation parameters, i.e.

$$\Gamma_2 := \sum_{(h,i,j)\in\sigma_{123}} \left(a_{iy} - a_{jy}\right) n_{ih} n_{hj} \mathbf{r}_{ij} \cdot \mathbf{k} - \left(a_{2z} r_{2y} n_{31} + a_{3z} r_{3y} n_{12}\right)\delta = 0 \tag{A4a}$$

$$\Gamma_3 := \sum_{(h,i,j)\in\sigma_{123}} \left(a_{ix} - a_{jx}\right) n_{ih} n_{hj} \mathbf{r}_{ij} \cdot \mathbf{k} - \left(a_{2z} r_{2x} n_{31} + a_{3z} r_{3x} n_{12}\right)\delta = 0 \tag{A4b}$$

where σ_{123} is the set of cyclic permutations of the triplet (1,2,3).

[8]If p_2 and p_3 do not vanish in $(\hat{x},\hat{y})$, $(\hat{x},\hat{y})$ is a solution of the problem only for $z \to \infty$.

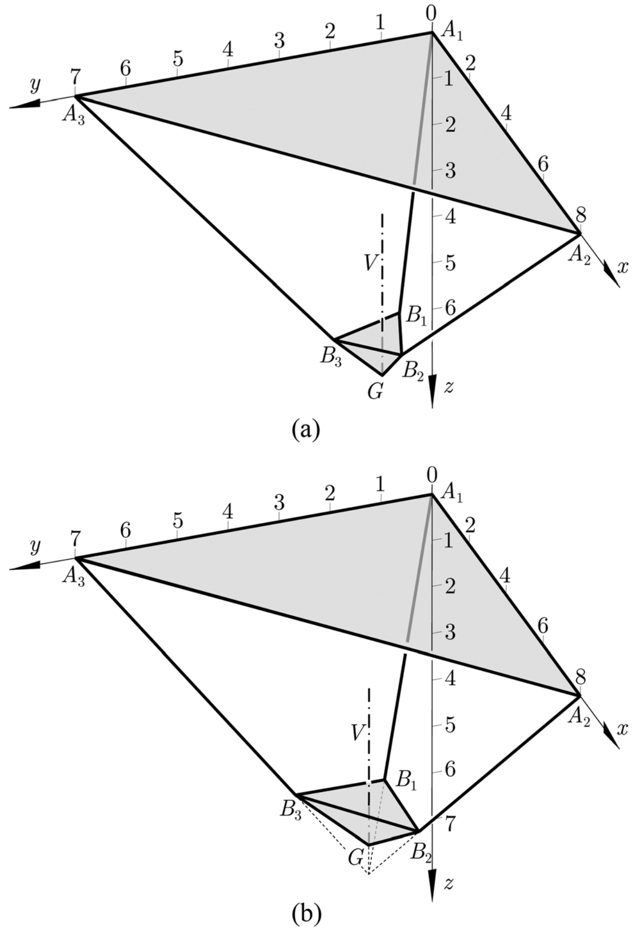

Fig. 3 Examples of one-dimensional solution sets of the IGP with assigned orientation: (*a*) $\mathbf{a}_2 = [8;0;0], \mathbf{a}_3 = [0;7;0], \mathbf{r}_1 = [-0.5;-0.5;-1], \mathbf{r}_2 = [1;0;-1], \mathbf{r}_3 = [0;1;-1], [x,y] = [2,1.75]$ and (*b*) $\mathbf{a}_2 = [8;0;0], \mathbf{a}_3 = [0;7;0], \mathbf{r}_1 = [-0.5;-0.5;-1], \mathbf{r}_2 = [1.5;-0.5;-1], \mathbf{r}_3 = [-0.5;1.25;-1], [x,y] = [2,2]$. In both cases, *V* is a vertical line and equilibrium is feasible along the entire path.

If $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0, (\hat{x},\hat{y})$ is a solution of the IGP for any value of z, and, provided that cable tensions are positive and $\mathbf{H}_r$ is positive definite, the platform may follow, with constant orientation, a quasi-static linear path parallel to $\mathbf{k}$. Simple conditions sufficient to let $\Gamma_i = 0, i = 1,\ldots,3$, are the following:

- $\Gamma_1 = 0$ whenever $\mathbf{r}_{i,xy}$ and $\mathbf{r}_{j,xy}$ are, respectively, parallel to $(A_i - A_h)_{xy}$ and $(A_j - A_h)_{xy}$, for $h,i,j = 1,\ldots,3$ and $h \neq i \neq j \neq h$ (Fig. 2(*a*));
- $\Gamma_1 = 0$ whenever $\mathbf{r}_{j,xy} - \mathbf{r}_{1,xy} = \beta\mathbf{a}_{j,xy}$, for $\beta \in (-1,1)$ and $j = 2,3$, namely, whenever the sides of the projections of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ on the xy-plane are mutually parallel (Fig. 2(*b*));
- $\Gamma_2 = \Gamma_3 = 0$ whenever the planes $A_1A_2A_3$ and $B_1B_2B_3$ are perpendicular to $\mathbf{k}$ (in this case, in fact, $a_{2z} = a_{3z} = \mathbf{r}_{ij} \cdot \mathbf{k} = 0, \forall i, \forall j$);
- $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$ whenever $\mathbf{r}_i \parallel \mathbf{k}$ for some i (this condition infers, in fact, $n_{ij} = n_{jh} = 0, i \neq j \neq h \neq i$; provided that G, B_1, B_2, and B_3 are not coplanar, the inverse implication is also true).

Figures 3(*a*) and 3(*b*) show two examples in which the first and the second condition are, respectively, satisfied, together with the third one. In the latter case, $(\hat{x},\hat{y}) = -\left(r_{1x}, r_{1y}\right)/\beta$ and the wrenches acting on the platform always intersect in a common point, thus forming a bundle of lines (case 3 of those listed at the beginning of Sec. 3). When the fourth condition is met,

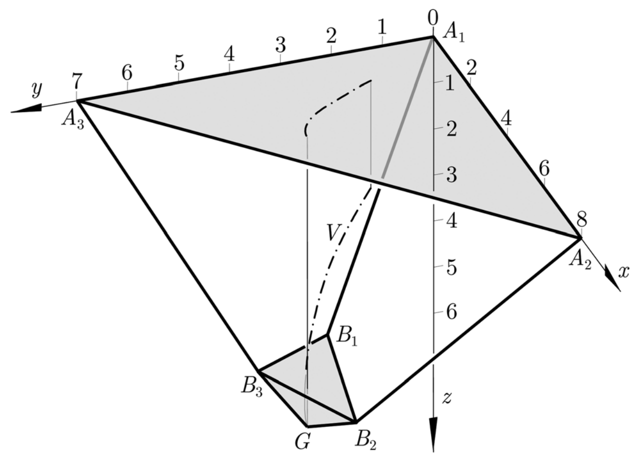

Fig. 4 Example of one-dimensional solution set of the IGP with assigned orientation: $\mathbf{a_2} = [\mathbf{8}; \mathbf{0}; \mathbf{0}], \mathbf{a_3} = [\mathbf{0}; \mathbf{7}; \mathbf{0}], \mathbf{r_1} = [-\mathbf{2/3}; -\mathbf{2/3}; -\mathbf{3/2}], \mathbf{r_2} = [\mathbf{14/15}; -\mathbf{2/3}; -\mathbf{1/2}], \mathbf{r_3} = [-\mathbf{2/3}; \mathbf{11/15}; -\mathbf{1}]$, and $p_{23} := 735x^2 + 567xy - 420y^2 - 6062x + 2534y + 1960 = 0$. V is a curve on a cylindrical surface and equilibrium configurations are feasible along the entire path.

$(\hat{x}, \hat{y}) = (\bar{x}_i, \bar{y}_i)$ and V is a line parallel to $\mathbf{k}$ passing through A_i, namely, the robot behaves like a 1-dof crane, with the ith cable holding the entire charge.

One-Dimensional Solution Set: A Curve on a Cylinder. p_1 has a noteworthy geometric interpretation. Indeed, by writing it in the form

$$p_1 := \det \mathbf{M}'_{1236} = \det \mathbf{M}_{1236} = \det \mathbf{M}_{126,123}$$
$$= -\begin{vmatrix} \mathbf{s}_{1,xy} & \mathbf{s}_{2,xy} & \mathbf{s}_{3,xy} \\ 0 & \mathbf{a}_{2,xy} \times \mathbf{s}_{2,xy} \cdot \mathbf{k} & \mathbf{a}_{3,xy} \times \mathbf{s}_{3,xy} \cdot \mathbf{k} \end{vmatrix} \tag{A5}$$

and by discarding trivial roots, it emerges that p_1 vanishes whenever either $\mathbf{s}_{i,xy} = \mathbf{0}$ for some i (namely when a cable is parallel to $\mathbf{k}$, in which case $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$ and the robot operates as a 1-dof crane), or the projections of the cable lines on the xy-plane are linearly dependent (namely, they meet in a common point, possibly at infinity). It happens that, when the sides of the projections of the triangles $A_1A_2A_3$ and $B_1B_2B_3$ on the xy-plane are mutually parallel (i.e., when the second one of the conditions listed in the previous subsection holds), the latter condition is always fulfilled. In this case, $\mathbf{r}_{j,xy} = \beta \mathbf{a}_{j,xy} + \mathbf{r}_{1,xy}$, for $j = 2,3$, $\beta \in (-1, 1)$, and $\Delta = (1-\beta)^2 \mathbf{a}_2 \times \mathbf{a}_3 \cdot \mathbf{k}$. Accordingly, if $\Delta \neq 0$, the solutions of the IGP are simply given by Eqs. (9*b*) and (9*c*), and the resulting variety is one-dimensional. In particular, since p_{23} represents, in this case, a conic on the xy-plane, V is a curve lying on the cylinder having this conic as a cross-section (Fig. 4). The pair $(\hat{x}, \hat{y})$ (in this case equal to $-(r_{1x}, r_{1y})/\beta$) must ordinarily be subtracted from the conic, since it does not extend to a finite solution of the problem, unless $\Gamma_2 = \Gamma_3 = 0$.

One-Dimensional Solution Set: A Planar Curve. Let Π_{ij}, $i \neq j$, be the plane parallel to $\mathbf{k}$ passing through A_i and A_j, and let the plane GB_iB_j be parallel to Π_{ij}. Ordinarily, p_1 and p_{23} represent a conic and a cubic curve on the xy-plane, respectively. However, in the considered case, p_1 degenerates into the union of two lines, l_1 and l_2, whereas p_{23} degenerates into the union of a line, coinciding with l_1, and a conic, c_1. l_1 is the line passing through $A_{i,xy}$ and $A_{j,xy}$. l_2 and c_1 intersect into a single point, coinciding with $(\bar{x}_h, \bar{y}_h)$, namely, with the projection on the xy-plane of the h-th trivial solution, with $i \neq h \neq j$. When $G_{xy} \in \{l_2 \cap c_1\}$ (so that $p_1 = p_{23} = 0$), letting $p_2 = p_3 = 0$ infers the trivial solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}_h$, which has to be discarded. As a consequence, solutions of the IGP may only exist so that G_{xy} lies on l_1. However, when $G_{xy} \in l_1$, letting $p_2 = p_3 = 0$ requires G to belong to a conic c_2 lying on Π_{ij} (geometrically, c_2 is the common intersection of the quadrics p_2 and p_3 with the plane Π_{ij}). In this case, GB_iB_j lies on Π_{ij} and the robot behaves like a 2–2 CDPR [2], with the i-th and the j-th cable holding the entire load ($\mathcal{L}_i$, $\mathcal{L}_j$, and $\mathcal{L}_e$ intersect in a common point) and the h-th cable being unloaded.

Empty Solution Sets. The ideals $\langle p_1, p_{23} \rangle$ and $\langle p_{123} \rangle$ represent, respectively, the first and the second elimination ideal of Eqs. (9*a*)–(9*c*).

A solution $\bar{x}_4$ of p_{123} may not extend to a partial solution $(\bar{x}_4, \bar{y}_4)$ in the variety defined by p_1 and p_{23}, if the leading coefficients of y in p_1 and p_{23} vanish for $x = \bar{x}_4$ [45]. If $A_{20} \neq 0$, the

Table 4 A 3–3 robot whose IGP with assigned position admits 4 potential solutions for which $e_0 = 0$

Geometric dimensions and load: $\mathbf{a}_2 = [10; 0; 3], \mathbf{a}_3 = [0; 12; -1], \mathbf{b}_1 = [-3; 0; -1], \mathbf{b}_2 = [2; 0; -2], \mathbf{b}_3 = [0.5; 2; -0.5]$, $\mathbf{x} = [5;0;6]$, $Q = 10$

$$\det \mathbf{S}(e_3) = +259471267245962556774573760480620 0e_3^{20} + 4074712631973790923339757827869508 0e_3^{19} +2843499659683590045680763996461990 42e_3^{18} + 1043229864111708871262572335104466928e_3^{17} + 1732614932964338744872879189400671929e_3^{16} -542766881892285288015899750149587120e_3^{15} - 5567651819227402148936076223248701980e_3^{14} - 1576761363412478821780276805557441240e_3^{13} +917022406032514719296016814007203587 9e_3^{12} - 8092071189847190468672649452990127312e_3^{11} + 6328343891843260664260241570918397586e_3^{10} -3160923867178565924047127213997797736e_3^{9} + 9057083176573806751846290625467170 99e_3^{8} - 6453299637744944727902142653266177 20e_3^{7} +2074388963364285918741704647231580 00e_3^{6} + 2473835610620840296868632877487600 0e_3^{5} + 1050928834260313444512152554817312 5e_3^{4}$$

Configuration	(e_0, e_1, e_2, e_3)	(ρ_1, ρ_2, ρ_3)	(τ_1, τ_2, τ_3)	$\mathbf{H}_r$
1	$(1, 0, +4.5333086033392681121, 0)$	$(10.96, 8.63, 13.09)$	$(+9.16, +6.88, 0)$	<>
2	$(1, 0, -0.2137852705182155638, 0)$	$(4.69, 3.09, 13.33)$	$(+7.67, +5.72, 0)$	>
3	$(1, 0, -0.1028510973048352173$ $+0.6194328478319967566I, 0)$	$(6.41, 5.51, 13.51)$	$(+1.60 - 4.82I, +0.18 + 6.58I, 0)$	<>
4	$(1, 0, -0.1028510973048352173$ $-0.6194328478319967566I, 0)$	$(6.41, 5.51, 13.51)$	$(+1.60 + 4.82I, +0.18 - 6.58I, 0)$	<>
5	$(0, +1.8939390744332848908, 0, 1)$	$(4.78, 7.99, 16.70)$	$(+7.15, +5.38, 0)$	<>
7	$(0, -0.1075017293765457598, 0, 1)$	$(9.92, 6.56, 16.07)$	$(+15.41, +12.71, 0)$	<>
6	$(0, -1.6229484022580992952$ $+1.3215936809813085774I, 0, 1)$	$(9.95, 3.88, 16.71)$	$(+6.04 - 0.26I, +4.50 + 2.01I, 0)$	<>
8	$(0, -1.6229484022580992952$ $-1.3215936809813085774I, 0, 1)$	$(9.95, 3.88, 16.71)$	$(+6.04 + 0.26I, +4.50 - 2.01I, 0)$	<>
...	...	...	...	...

coefficient of the monomial y^2 in p_1 is always nonzero, so that $\bar{x}_4$ certainly extends to $(\bar{x}_4, \bar{y}_4)$. The same happens if $A_{20} = 0$ and $\Gamma_0 \neq 0$. In this case, in fact, the leading terms in y of p_1 and p_{23} are, respectively, $[-(B_{011} + C_{101})x + A_{01}]y$ and $[B_{020}(B_{011} + C_{101})x - B_{020}C_{001} + B_{011}C_{010}]y^2$, and the vanishing of Γ_0 is a necessary condition for the corresponding leading coefficients to be zero. When, conversely, $A_{20} = 0$ and $\Gamma_0 = 0$, p_{123} provides, among its roots, the value of x for which such coefficients vanish, i.e.

$$\bar{x}_4 = \frac{B_{020}C_{001} - B_{011}C_{010}}{B_{020}(B_{011} + C_{101})} = \frac{-A_{01}}{(B_{011} + C_{101})} \tag{A6}$$

In this case, $\bar{x}_4$ does not ordinarily extend to a solution $(\bar{x}_4, \bar{y}_4)$, unless the coefficients of the monomials of degree zero in p_1 and degree less than or equal to one in p_{23} simultaneously vanish, for $x = \bar{x}_4$. This requires, however, the satisfaction of rather complicated and unlikely conditions in the geometric and orientation parameters. Further particular instances may be analyzed on a case by case basis.

Finally, a solution $(\bar{x}_4, \bar{y}_4)$ of p_1 and p_{23} does not extend to a full solution in V when the coefficients of the monomial z in p_2 and p_3 vanish for $(x, y) = (\bar{x}_4, \bar{y}_4)$ (which requires $\Gamma_1 = 0$), but p_2 and p_3 remain nonzero, i.e., $\Gamma_2 \neq 0 \neq \Gamma_3$.

Appendix B: Special Solution Sets of the IGP With Assigned Position: The Case $e_0 = 0$

Let Π_{ij}, $i \neq j$, be the plane parallel to $\mathbf{k}$ passing through A_i and A_j. When $\mathbf{x} \in \Pi_{ij}$ and the coordinates frames $Oxyz$ and $Gx'y'z'$ are chosen generically, Eqs. (29) and (30) provide, among the 24 solutions of the IGP with assigned position, eight configurations in which the plane GB_iB_j lies upon Π_{ij}. In such cases, the ith and the jth cable hold the entire load and the hth cable, $i \neq h \neq j$, is uncharged. These configurations coincide with the solutions of the IGP of a 2–2 CDPR comprising the platform, the ith and the jth cable, and which may work in both its operation modes [2].

If the coordinates frames $Oxyz$ and $Gx'y'z'$ are chosen, instead, so that $(\mathbf{b}_i \times \mathbf{b}_j) \times [(\mathbf{a}_i - \mathbf{a}_j) \times \mathbf{k}] = 0$, the degree of $\det \mathbf{S}(e_3)$ lowers to 20 and Eq. (30) is not able to provide all solutions of the problem. In particular, it only supplies four configurations for which GB_iB_j and Π_{ij} are superimposed. The four "missing" solutions may be obtained by setting $e_0 = 0$ and $e_3 = 1$, by eliminating e_2 or e_3 from a set of five relations chosen within Eqs. (21), (23), (27), and (28), and by finally retaining only the solutions for which all equations are satisfied. Table 4 reports a numeric example. For the sake of brevity, only the configurations for which the robot behaves as a 2–2 CDPR are listed (for complex solutions, cable lengths are conventionally computed as $\sqrt{\mathbf{s}_i^H \mathbf{s}_i}$, $i = 1, \ldots, 3$).

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