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Multiscale Haar Orthonormal Matrices with the Corresponding Riesz Products and a Characterization of Cantor-Type Languages

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ABSTRACT. *We introduce a class of multiscale orthonormal matrices* $H(m)$ *of order* $m \times m$, $m = 2, 3, \ldots$. *For* $m = 2^N$, $N = 1, 2, \ldots$, *we get the well known Haar wavelet system. The term "multiscale" indicates that the construction of* $H(m)$ *is achieved in different scales by an iteration process, determined through the prime integer factorization of* m *and by repetitive dilation and translation operations on matrices. The new Haar transforms allow us to detect the underlying ergodic structures on a class of Cantor-type sets or languages. We give a sufficient condition on finite data of length* m, *or step functions determined on the intervals* $[k/m, (k+1)/m)$, $k = 0, \ldots, m-1$ *of* $[0, 1)$, *to be written as a Riesz-type product in terms of the rows of* $H(m)$. *This allows us to approximate in the weak-* topology continuous measures by Riesz-type products.*

1. Introduction

A matrix is a concise and useful way of treating linear transforms and an extremely important concept in time series analysis (see [5]). In order to analyze data, we prefer linear transforms whose corresponding matrices have the ability to handle a large amount of information with fast computations. Sparse matrices (matrices with a small number of nonzero elements), have the ability to reduce computational cost (see [7, 10]). Thus, in [1] and [2] we introduced new classes of sparse invertible matrices, capable of revealing local information and suitable for providing multiscale analysis on finite data.

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The initial idea of this work emerged from the observation that the Gram Schmidt orthonormalization process of the following sparse matrices (see [1] and [2]):

$$U(1)=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, U(2)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, U(3)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \dots,$$

derives the Haar matrices:

$$H(1)=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, H(2)=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$$H(3)=\begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \dots .$$

Since the sparse matrices introduced in [1] are created by a multiscale construction, we wanted to check if we can get a similar construction to create Haar matrices. In Section 2, we build a class of orthonormal matrices $H(m)$ of order $m \times m$ $(m = 2, 3, \dots)$, using an iteration in scales, determined by the prime integer factorization of m. The matrices $H(m)$ can be considered as a generalization of the usual Haar matrices, since the rows of $H(m)$ are unbalanced Haar wavelets, as introduced in [4, 9], and [12].

The construction of $H(m)$, where $m = p_1 p_2 \cdots p_N$ is the prime integer factorization of m, $p_1 \geq p_2 \geq \ldots \geq p_N$, starts with a matrix $H(p_1)$, whose all rows, except for the first row, have zero mean. $H(p_1 p_2 \cdots p_k)$ is obtained from $H(p_1 p_2 \cdots p_{k-1})$ by joining two matrices, derived by a dilation and a translation process on $H(p_1 p_2 \cdots p_{k-1})$. As a result, we get a multiresolution analysis and a Haar transform:

$$\{t_n : n = 1, \dots, m\} \leftrightarrow \{\langle t, h_n \rangle : n = 1, \dots, m\},$$

where $\langle ., . \rangle$ is the usual inner product and h_n are the rows of $H(m)$.

In Section 3, we use the Haar transform to identify a Cantor-type language. A *Cantor-type language* of length N in an alphabet $A = \{\alpha_0, \alpha_1, \dots, \alpha_{p-1}\}$ of p letters, is the set of all words $\{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_N : \varepsilon_i \in A' \subset A, i = 1, \dots, N\}$. The corresponding Cantor set on $[0, 1)$ is the set $\left\{x = \sum_{i=1}^{N} \varepsilon_i p^{-i}, \varepsilon_i \in B\right\}$, where $B = \{i \in \{0, \dots, p-1\} : a_i \in A'\}$.

In Theorem 2 we prove that any Cantor language can be identified:

(a) By the sequence $\{\langle t, h_1\rangle, \ldots, \langle t, h_p\rangle\}$ of the first p Haar coefficients of t, where t is the indicator sequence (see Definition 6) of the Cantor language and

(b) by the set of zeros of the Haar transform of t.

In Section 4 we deal with Riesz-type products based on Haar matrices $H(m)$. We prove that to any step function $f(x)$ on [0, 1) satisfying $f(x) = t_n, x \in \left[\frac{n-1}{m}, \frac{n}{m}\right), n = 1, \ldots, m$, there corresponds a unique sequence of numbers $\{a_n : n = 1, \ldots, m\}$ and a representation:

$$f(x) = \prod_{k=1}^{m} (1 + a_k h_k(x)) \ ,$$

where $h_k(x) = h_{k,n},\ x \in \left[\frac{n-1}{m}, \frac{n}{m}\right),\ k, n = 1, \ldots, m$ and $h_{k,n}$ is the (k, n) entry of the matrix $H(m)$, provided that $\langle f, h_k\rangle \neq 0$. In other words, we prove that any data $\{t_n : n = 1, \ldots, m\}$ satisfying $\langle t, h_n\rangle \neq 0$, can be expressed as a product:

$$t_n = \prod_{k=1}^{m} \left(1 + a_k h_{k,n}\right) \ ,$$

called *Haar-Riesz product* associated with the coefficients $\{a_n\}$. Thus, we introduce a non linear transform:

$$\{t_n : n = 1, \ldots, m\} \leftrightarrow \{a_n : n = 1, \ldots, m\} \ ,$$

which can be implemented by a fast computational algorithm.

We should note here that the original Riesz's construction associated with a sequence $\{a_n\}$, was to point that the pointwise limit of the functions:

$$F_N(x) = \int_0^x \prod_{n=1}^{N} \left(1 + a_n \cos\left(2\pi 4^n t\right)\right) \, dt$$

is a continuous function F of bounded variation in $[0, 2\pi]$, whose Fourier-Stieltjes coefficients do not vanish at infinity. Riesz's construction was the source of powerful ideas for producing many examples of measures with desired properties, by replacing $\cos(2\pi t)$ with trigonometric polynomials (see [11]), or other generating functions (see [6]). Recently, the authors of [8] presented a new perspective for constructing Riesz products. From their point of view, the original Riesz product emerged from a Ruelle Perron-Frobenious operator acting on the well known low pass filter function of wavelets.

The common thread among the aforementioned approaches is a multiscale Riesz product defined in Benedetto-Bernstein and Konstantinidis work (see [3]). Their definition consists of a homomorphism $T : G \to G$, G being a locally compact Abelian group with Haar measure m and a real valued function H on G called generating function, such that:

$$d\mu_N = \prod_{n=1}^{N} \Big(1 + a_n H\big(T^{n-1}x\big)\Big) \, dm$$

converges weak-* to a continuous measure. They prove a dichotomy theorem and examine the support properties of measures based on this construction. Thus, our approach to make multiscale Haar-Riesz products is different and hopefully suitable for examining the characteristic properties of singular measures as well as hidden ergodic structures. We will treat this problem in a future occasion.

The article is organized in the following sections:

In Section 2, Definitions 1–2, we introduce new dilation and translation operations on matrices. In Lemma 2 we prove that these operators are orthonormal. In Definition 5 we construct the matrix $H(m)$ in terms of a recursion equation. Theorem 1 states that the matrices $H(m)$ are orthonormal.

In Section 3, Proposition 1, we compute the Haar coefficients of the indicator sequence t of a Cantor language in terms of the first p Haar coefficients of t. In Theorem 2 we compute the set of zeros of the Haar transform of t and we present the reconstruction formula for t.

In Section 4, we define the Haar-Riesz product corresponding to the matrix $H(m)$. In Proposition 2 we compute the Haar-Riesz coefficients and in Theorem 3 we prove their uniqueness.

2. The Iteration Process for the Construction of the Haar Matrices $H(m)$

In order to construct the orthonormal matrices $H(m)$, we need to determine new dilation and translation operators on the space of matrices. We use the following notation.

Notation. Let $\mathsf{M}_{n,m}$ be the space of matrices of order $n \times m$ over the field of complex numbers. If $n = m$, then $\mathsf{M}_{n,m}$ is abbreviated to M_n. A matrix $M \in \mathsf{M}_{n,m}$ is orthonormal, if its rows form an orthonormal set. We denote $M_i = \{M_{ij} : j = 1, \dots, m\}$ to be the i row of the matrix $M \in \mathsf{M}_{n,m}$. The support of the row M_i is: $\operatorname{supp}\{M_i\} = \{j = 1, \dots, m : M_{i,j} \neq 0\}$. Finally, we denote by $[x]$ the lowest integer which is greater than or equal to a real number x.

Let $p = 2, 3, \dots$, we define the following operators D_p and T_p on the space $\mathsf{M}_{n,m}$.

Definition 1. Let $D_p : \mathsf{M}_{n,m} \to \mathsf{M}_{n,pm}$ be the following dilation operator:

$$D_p(M) = \left\{ M_{i,\left[\frac{j}{p}\right]}, \quad i = 1, \dots, n, \quad j = 1, \dots, pm \right\} .$$

Example 1. $D_2\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \end{pmatrix}, D_3\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 & 4 & 4 \end{pmatrix}.$

Definition 2. Let $T_p : \mathsf{M}_{k,l} \to \mathsf{M}_{pk,pl}$ be the following translation operator:

$$T_p(M) = \left\{ \begin{cases} M_{\left[\frac{i}{p}\right],\mathrm{Mod}(j-1,l)+1}, & \text{whenever } \mathrm{Mod}(i-1,p)+1 = \left[\frac{j}{l}\right] \\ 0, & \text{otherwise} \end{cases} , i = 1, \dots, pk,\ j = 1, \dots, pl \right\} .$$

Example 2. $T_3\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{11} & b_{12} \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$

Remark 1. The operator $D_p(M)$ creates p replicas of any column of the matrix M, while the matrix $T_p(M)$, $M \in \mathsf{M}_{k,l}$ has:

(i) k generator rows:

$$\big(T_p(M)\big)_{rp+1} = \begin{cases} M_{(r+1),j} & j = 1, \ldots, l \\ 0, & j = l+1, \ldots, pl \end{cases}, \quad r = 0, \ldots, k-1 ,$$

(ii) each row $(T_p(M))_{rp+s}$, $s = 2, \ldots, p$ is an $(s-1)l$-translation of the row $\big(T_p(M)\big)_{rp+1}$.

Lemma 1. *The operators D_p, T_p satisfy the following properties:*

(i) $D_p D_q = D_{pq}$.

(ii) $D_p T_q = T_q D_p$.

(iii) *If the rows of the matrix $M \in M_{n,m}$ form an orthonormal set, then both operators $p^{-1/2} D_p$ and T_p preserve orthonormality.*

Proof.
(i) and (ii) are straightforward applications of Definitions 1 and 2.
(iii) If the rows of M form an orthonormal set, then:

$$\frac{1}{p}\Big\langle \big(D_p(M)\big)_i , \big(D_p(M)\big)_j \Big\rangle = \frac{1}{p}\sum_{l=1}^{pm} M_{i,\left[\frac{l}{p}\right]} M_{j,\left[\frac{l}{p}\right]} = \sum_{l=1}^{m} M_{i,l} M_{j,l} = \delta_{i,j} .$$

In order to prove that T_p is orthonormal we use Remark 1:

$$\Big\langle \big(T_p(M)\big)_i , \big(T_p(M)\big)_j \Big\rangle = \begin{cases} \Big\langle M_{\left[\frac{i}{p}\right]}, M_{\left[\frac{j}{p}\right]} \Big\rangle, & |i-j| = cp \\ 0, & |i-j| \neq cp \end{cases}, \quad c = 1, \ldots, n-1$$

and the result follows as a consequence of the orthonormality of M. □

Definition 3. Let $S : M_{n,m} \times M_{k,m} \to M_{n+k,m}$ be the following block matrix operator:

$$S(M, N) = \left\{ \begin{cases} M_{i,j}, & i = 1, \ldots n, \quad j = 1, \ldots m \\ N_{(i-n),j}, & i = n+1, \ldots, n+k, \quad j = 1, \ldots m \end{cases} \right\} .$$

Example 3. $S\left(\begin{pmatrix} 1\,1\,1\,0 \\ 0\,0\,0\,2 \end{pmatrix}, \begin{pmatrix} 1\,0\,0\,0 \\ 0\,2\,0\,0 \\ 0\,0\,3\,0 \end{pmatrix}\right) = \begin{pmatrix} 1\,1\,1\,0 \\ 0\,0\,0\,2 \\ 1\,0\,0\,0 \\ 0\,2\,0\,0 \\ 0\,0\,3\,0 \end{pmatrix}.$

Definition 4. Let $p = 2, 3, \ldots$, we define the following matrix $\Psi^p = (\psi^p_{ij})$ of order $(p-1) \times p$:

$$\psi^p_{ij} = \begin{cases} \frac{1}{\sqrt{p-i}} \frac{1}{\sqrt{p-i+1}}, & \text{whenever} \quad 1 \le j \le p-i \\ -\frac{\sqrt{p-i}}{\sqrt{p-i+1}}, & \text{whenever} \quad j = p-i+1 , \\ 0, & \text{whenever} \quad p-i+1 < j \le p \end{cases} \qquad i = 1, \ldots, p-1 . \quad (2.1)$$

Example 4. $\Psi^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\Psi^3 = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$,

$$\Psi^5 = \begin{pmatrix} \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 2. *The matrix Ψ^p satisfies the following properties:*

(i) $\sum_{j=1}^{p} \psi_{ij}^{p} = 0$, *for any* $i = 1, \dots, p-1$.

(ii) $\psi_i^p \psi_j^p = \psi_{i,1}^p \psi_j^p$, *whenever* $i < j$.

(iii) *The matrix* $S\big(\frac{1}{\sqrt{p}}(1, \dots, 1)_{1\times p}, \Psi^p\big)$ *is orthonormal.*

(iv) *Let* $r, q = 2, 3, \dots$ *and let* $i < j$, *then:*

$$\left(T_r\left(D_q(\Psi^p)\right)\right)_i \left(T_r\left(D_q(\Psi^p)\right)\right)_j = c \quad \left(T_r\left(D_q(\Psi^p)\right)\right)_j ,$$

$$\text{where } c = \begin{cases} \psi^p_{\left[\frac{i}{r}\right],1}, & j-i = sr \\ 0, & j-i \neq sr \end{cases}, \quad s = 0, \dots, p-2.$$

Proof.
(i) Obvious, see (2.1).
(ii) Let $i < j$, then $p - j + 1 \leq p - i$. Since $\psi_{i,n}^p$ has the same non zero value for all $n \leq p - i$ and since $\psi_{j,n}^p = 0$ for all $n > p - j + 1$, we have: $\psi_i^p \psi_j^p = \psi_{i,1}^p \psi_j^p$.
(iii) Elementary application of (ii) and Equation (2.1).
(iv) Let $M^{q,p} = D_q(\Psi^p)$. We suppose that $i < j$ and we use Remark 1 to deduce that

$$\text{supp}\left\{T_r\left(M^{q,p}\right)_i\right\} \bigcap \text{supp}\left\{T_r\left(M^{q,p}\right)_j\right\} = \emptyset$$

whenever $j - i \neq sr,\ s = 0, \dots, p-2$. If $j - i = sr$, then:

$$\text{supp}\left\{T_r\left(M^{q,p}\right)_i\right\} \bigcap \text{supp}\left\{T_r\left(M^{q,p}\right)_j\right\} = \text{supp}\left\{M_{\left[\frac{i}{r}\right]}\right\} \bigcap \text{supp}\left\{M_{\left[\frac{j}{r}\right]}\right\}.$$

Since $M_{\left[\frac{i}{r}\right]} \quad M_{\left[\frac{j}{r}\right]} = \psi^p_{\left[\frac{i}{r}\right],1} \ \psi^p_{\left[\frac{j}{r}\right]}$ as a consequence of part (ii) and Equation (2.1), the result follows. □

From now on, we consider the prime integer factorization of $m > 0$:

$$m = p_1 p_2 \cdots p_N , \tag{2.2}$$

where $p_1 \geq p_2 \geq \ldots \geq p_N$.

Definition 5. Let m be as in (2.2). For any $n = 1, \dots, N$ we define a sequence of block matrices $H^m(n)$ of order $\left(\prod_{i=1}^n p_i\right) \times \left(\prod_{i=1}^n p_i\right)$:

$$H^m(n) = \begin{cases} S\left(\frac{1}{\sqrt{p_1}}(1, \dots, 1)_{1\times p_1}, \Psi^{p_1}\right), & n = 1 \\ S\left(\frac{1}{\sqrt{p_n}} D_{p_n}\big(H^m(n-1)\big), T_{(p_1\cdots p_{n-1})}\big(\Psi^{p_n}\big)\right), & n = 2, \dots, N\ . \end{cases}$$

For the case $n = N$ we use the notation $H(m)$.

Example 5. Let $m = 12$, then $p_1 = 3$, $p_2 = 2$, $p_3 = 2$ and we have:

$$H^{12}(1) = \begin{pmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\sqrt{\frac23} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \end{pmatrix}, \quad H^{12}(2) = \begin{pmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & \frac{1}{2\sqrt3} & -\frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \\ \frac12 & \frac12 & -\frac12 & -\frac12 & 0 & 0 \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix},$$

$$H^{12}(3) = \begin{pmatrix} \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & \frac{1}{2\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} \\ \frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & \frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & -\frac{1}{2\sqrt2} & 0 & 0 & 0 & 0 \\ \frac12 & \frac12 & -\frac12 & -\frac12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac12 & \frac12 & -\frac12 & -\frac12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}.$$

Theorem 1. *The matrices $H^m(n)$, $n = 1, \dots, N$ are orthonormal.*

In order to prove Theorem 1 we need the following lemmas.

Lemma 3. *Let $n = 2, \dots, N$, then*

$$\frac{1}{\sqrt{p_n}} D_{p_n}\left(H^m(n-1)\right) = S\left(A_0, A_1, \dots, A_{n-2}\right)\ ,$$

where $A_0 = \frac{1}{\sqrt{p_n \cdots p_2}} D_{(p_n\cdots p_2)}\big(H^m(1)\big)$ *and* $A_k = \frac{1}{\sqrt{p_n\cdots p_{k+2}}} T_{(p_1\cdots p_k)}\big(D_{(p_n\cdots p_{k+2})}\big(\Psi^{p_{k+1}}\big)\big)$, $k > 0$.

Proof. We use Lemma 1 (i)–(ii) and Definition 5 to get:

$$
\begin{aligned}
\frac{1}{\sqrt{p_n}} D_{p_n}\left(H^m(n-1)\right) &= \frac{1}{\sqrt{p_n}} D_{p_n}\left(S\left(\frac{1}{\sqrt{p_{n-1}}} D_{p_{n-1}}\left(H^m(n-2)\right), T_{(p_1\cdots p_{n-2})}\left(\Psi^{p_{n-1}}\right)\right)\right) \\
&= S\left(\frac{1}{\sqrt{p_n p_{n-1}}} D_{(p_n p_{n-1})}\left(H^m(n-2)\right), A_{n-2}\right) \\
&= S\left(\frac{1}{\sqrt{p_n p_{n-1} p_{n-2}}} D_{(p_n p_{n-1} p_{n-2})}\left(H^m(n-3)\right), A_{n-3}, A_{n-2}\right) \\
&= \ldots = S\left(A_0, A_1, \ldots, A_{n-2}\right) . \qquad \square
\end{aligned}
$$

Lemma 4. *Let $1 \leq n < l \leq m$, then $h_n h_l = h_{n,l_0} h_l$, where h_{l,l_0} is the first nonzero element of the l-row of the matrix $H(m)$.*

Proof. $H(m) = S(A_0, \ldots, A_{N-2}, A_{N-1})$ as a result of Lemma 3.

Step 1. Let $n < l$ and let $h_n, h_l \in A_0$, then: $h_{n,i} h_{l,i} = \psi^{p_1}_{n,\left[\frac{i}{p_2\cdots p_N}\right]} h_{l,i} = \psi^{p_1}_{n,1} h_{l,i}$.

Step 2. Let $n < l$ and let $h_n, h_l \in A_k$, $(k = 1, \ldots, N-1)$, then Lemma 2 (iv) yields the result.

Step 3. If h_n is a row of the submatrix A_k and h_l is a row of the submatrix A_m where $k < m$, then either $\text{supp}\{h_l\} \bigcap \text{supp}\{h_n\} = \emptyset$ or $\text{supp}\{h_l\} \bigcap \text{supp}\{h_n\} = \text{supp}\{h_l\}$. Since the row h_n has the same entries within the support of the row h_l, we have $h_n\ h_l = c\ h_l$, where

$$
c = \begin{cases} h_{n,l_0}, & \text{whenever } \text{supp}\{h_l\} \bigcap \text{supp}\{h_n\} = \text{supp}\{h_l\} \\ 0, & \text{whenever } \text{supp}\{h_l\} \bigcap \text{supp}\{h_n\} = \emptyset \end{cases} . \qquad \square
$$

Proof of Theorem 1. We proceed by induction. The matrix $H^m(1)$ is orthonormal as a result of Lemma 2 (iii). Let $H^m(n-1)$ $(n = 2, \ldots, N-1)$ be an orthonormal matrix, then the rows of both matrices $p_n^{-1/2} D_{p_n}\ (H^m(n-1))$ and $T_{(p_1\ldots p_{n-1})}\ (\Psi^{p_n})$ form orthonormal sets as a consequence of Lemma 1 (iii) and the inductive hypothesis, thus it suffices to prove that $\langle h_k, h_l\rangle = 0$ whenever $h_k \in p_n^{-1/2} D_{p_n}\ (H^m(n-1))$ and $h_l \in T_{(p_1\ldots p_{n-1})}\ (\Psi^{p_n})$. Indeed, we have:

$$
\langle h_k, h_l\rangle = \sum_{r=1}^{m} h_{k,r} h_{r,l} = c \sum_{j=1}^{p_n} \psi^{p_n}_{\left[\frac{r}{p_1\cdots p_{n-1}}\right],j} = 0 ,
$$

so the matrix $H^m(n)$ is orthonormal. $\square$

Remark 2. The multiresolution structure arised from $H(p^N)$, where p is a prime number and $N = 2, 3, \ldots$.

Let V_N be the space of all real-valued sequences of length p^N and let h_i be the i-row of the Haar matrix $H(p^N)$, then any element $t = \{t(n), n = 1, \ldots, m\} \in V_N$ can be written as:

$$
t(n) = \sum_{i=1}^{p^N} \langle t, h_i\rangle h_{n,i} .
$$

For any $j = 0, \ldots, N-1, k = 1, \ldots, p-1$, we define the subspaces $W_{j,k} = \text{span}\{h_{kp^j+s} : s = 1, \ldots, p^j\}$. Let V_0 be the space of constant sequences, then we have the following decomposition:

$$
V_N = V_0 \oplus_{j=0}^{N-1} \oplus_{k=1}^{p-1} W_{j,k} .
$$

Example 6. Let $m = 3^3$, then $V_m = V_0 + W_{0,1} + W_{0,2} + W_{1,1} + W_{1,2} + W_{2,1} + W_{2,2}$, where:

$$\begin{aligned} V_0 &= \text{span}\{h_1\},\ W_{0,1} = \text{span}\{h_2\},\ W_{0,2} = \text{span}\{h_3\}\ , \\ W_{1,1} &= \text{span}\{h_4, h_5, h_6\},\ W_{1,2} = \text{span}\{h_7, h_8, h_9\}\ , \\ W_{2,1} &= \text{span}\{h_{10}, \dots, h_{18}\},\ W_{2,2} = \text{span}\{h_{19}, \dots, h_{27}\}\ . \end{aligned}$$

3. Haar Coefficients of Cantor-Type Sets

For the rest of the text we assume that h_i is a row of the Haar matrix $H(m)$ and p is a prime number. A *Cantor-type language* of length N in an alphabet $A = \{a_0, a_1, \dots, a_{p-1}\}$ of p letters, is the set of all words $\{\varepsilon_1\varepsilon_2\dots\varepsilon_N : \varepsilon_i \in A' \subset A, i = 1, \dots, N\}$:

(i) A' has at least two elements.

(ii) $a_0 \in A', a_1 \notin A'$.

The corresponding Cantor set on [0, 1) is the set $\left\{x = \sum_{n=1}^{N} \varepsilon_n p^{-n} : \varepsilon_n \in B\right\}$, where

$$B = \left\{i \in \{0, \dots, p-1\} : a_i \in A'\right\}\ . \tag{3.1}$$

Obviously: $0 \in B,\ 1 \notin B$.

Definition 6. We call indicator sequence of a Cantor-type language, the sequence $t = \{t_1, t_2, \dots, t_{p^N}\}$ satisfying:

$$t_n = \begin{cases} 1, & \text{whenever } n = 1 + \sum_{i=1}^{N} \varepsilon_i p^{N-i}, \varepsilon_i \in B \\ 0, & \text{otherwise} \end{cases} .$$

Example 7. Let $p = 5$, $N = 2$, $A = \{a_0, \dots, a_4\}$, $A' = \{a_0, a_2, a_4\}$, then:

$$t = \{1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1\}\ .$$

The indicator sequence t emerges from an iteration process presented below.

Let V_i be the set of all real sequences of length p^i $(i = 1, \dots, N-1)$. We define the mapping:

$$X_i : V_i \to V_{i+1}, \left(X_i(e)\right)_n = \begin{cases} 0, & \text{whenever } n = 1 + \sum_{j=1}^{i} \varepsilon_j p^{i-j}, \varepsilon_i \notin B \\ e_{\text{Mod}(n-1,p^i)+1}, & \text{otherwise} \end{cases} \quad n = 1, \dots, p^{i+1}\ ,$$

where B is defined in (3.1). If $g = \{g_1, \dots, g_p\}$:

$$g_n = \begin{cases} 1, & \text{whenever } \alpha_{n-1} \in A' \\ 0, & \text{whenever } \alpha_{n-1} \in A - A' \end{cases} ,$$

then the indicator sequence t satisfies:

$$t = X_{N-1}X_{N-2}\dots X_1(g)\ .$$

We list the following properties of t:

Lemma 5.
(i) *Let* $e = X_{j-1}\dots X_1(g)$, $(j = 2, \dots, N)$, *g being defined above, then:*

$$t_n = \begin{cases} 0, & \text{whenever } n = 1 + \sum_{i=1}^{N} \varepsilon_i p^{N-i}, \varepsilon_i \notin B \text{ for any } i = 1, \dots, N-j \\ e_{\mathrm{Mod}(n-1, p^j)+1}, & \text{otherwise} \end{cases}, \quad n = 1, \dots, p^N .$$

(ii) *If* $\sum_{n=1}^{p} g_n = c,\ 2 \le c < p$, *then:* $\sum_{n=1}^{p^j} e_n = c^j$.

(iii) $t_n = 0$ *for all n satisfying:* $p^{i-1} + 1 \le n \le 2p^{i-1},\ i = 1, \dots, N$.

Proof.

(i) and (ii) are elementary.

(iii) Since $1 \notin B$ [see (3.1)], we have $t_n = 0$ whenever $\varepsilon_i = 1$, i.e., $t_n = 0$ whenever $p^{i-1} + 1 \le n \le 2p^{i-1}$. □

Lemma 6. *Let t be the indicator sequence of a Cantor language, then:*

(a) $\langle t, h_r \rangle \neq 0$ *for any* $r = 1, \dots, p$.

(b) *Let* $j = 1, \dots, N-1$, $s = 1, \dots, p^j$. *If* $\langle t, h_{p^j+s} \rangle \neq 0$, *then* $\langle t, h_{kp^j+s} \rangle \neq 0$ *for any* $k = 1, \dots, p-1$.

(c) $\langle t, h_{p^j+s} \rangle = 0 \Leftrightarrow t_n = 0$ *for all n* satisfying $n = (s-1)p^{N-j} + 1, \dots, sp^{N-j}$.

(d) $\langle t, h_{p^j+s} \rangle = 0 \Leftrightarrow \langle t, h_{kp^j+s} \rangle = 0$ for any $k = 1, \dots, p-1$.

Proof.

(a) The case $r = 1$ is obvious (see Lemma 5 (ii) for $j = N$). For $r = 2, \dots, p$, we have:

$$\langle t, h_r \rangle = h_{r,1} \sum_{l=1}^{(p-r+1)p^{N-1}} t_l - h_{r,(p-r+1)p^{N-1}+1} \sum_{l=(p-r+1)p^{N-1}+1}^{(p-r+2)p^{N-1}} t_l \,. \tag{3.2}$$

First case: If $(p-r+1) \notin B$, then Lemma 5 (i) implies that the last term in the right-hand side of Equation (3.2) vanishes, so: $\langle t, h_r \rangle = h_{r,1} \sum_{l=1}^{(p-r+1)p^{N-1}} t_l > 0$.

Second case: If $(p-r+1) \in B$, then we apply Lemma 5 (i) in Equation (3.2) and we have:

$$\langle t, h_r \rangle = \big((p-r+1)h_{r,1} - h_{r,(p-r+1)p^{N-1}+1}\big) \sum_{l=1}^{p^{N-1}} (X_{N-2} \dots X_1(g))_l \ ,$$

so $\langle t, h_r \rangle < 0$ as a result of Lemma 5 (iii).

(b) We suppose that $\langle t, h_{p^j+s} \rangle \neq 0$. Since $\mathrm{supp}\{h_{kp^j+s}\} \subseteq \{n : n = (s-1)p^{N-j} + 1, \dots, sp^{N-j}\}$ for any $k = 1, \dots, p-1$ and since the row h_{kp^j+s} is an sp^{N-j} translation of the row h_{kp^j+1}, we have:

$$\begin{aligned} \langle t, h_{kp^j+s} \rangle &= \sum_{l=(s-1)p^{N-j}+1}^{sp^{N-j}} (X_{N-j-1} \dots X_1(g))_l \, h_{kp^j+s,l} = \sum_{l=1}^{p^{N-j}} (X_{N-j-1} \dots X_1(g))_l \, h_{kp^j+1,l} \\ &= h_{kp^j+1,1} \sum_{l=1}^{(p-k)p^{N-j-1}} (X_{N-j-1} \dots X_1(g))_l - h_{kp^j+1,(p-k)p^{N-j-1}+1} \sum_{l=(p-k)p^{N-j-1}+1}^{(p-k+1)p^{N-j-1}} (X_{N-j-1} \dots X_1(g))_l \ . \end{aligned}$$

We use the same methodology as in part (a) to deduce that for $p-k \notin B$ we get $\left\langle t, h_{kp^j+s}\right\rangle > 0$, whereas for $p-k \in B$ we get $\left\langle t, h_{kp^j+s}\right\rangle < 0$.

(c) It suffices to examine the case:

$$\left\langle t, h_{p^j+s}\right\rangle = 0 \Rightarrow t_n = 0 \text{ for all } n \text{ satisfying: } n = (s-1)p^{N-j}+1, \ldots, sp^{N-j} .$$

If $\sum_{l=(s-1)p^{N-j}+1}^{sp^{N-j}} t_l \neq 0$, then the proof presented in part (b) indicates that $\left\langle t, h_{p^j+s}\right\rangle \neq 0$, which is a contradiction.

(d) Obvious, combine parts (b) and (c). □

Proposition 1. *For any $k = 1, \ldots, p-1$, the indicator sequence t* of a Cantor-type language satisfies:

$$\langle t, h_{k+1}\rangle = \frac{c^j}{\sqrt{p^j}}\left\langle t, h_{kp^j+s}\right\rangle,\ j = 1, \ldots, N-1, s = 1, \ldots, p^j ,$$

provided that $\left\langle t, h_{kp^j+s}\right\rangle \neq 0$. Notice that c is the cardinality of the support of the set B defined in (3.1).

Proof. Let $\left\langle t, h_{kp^j+s}\right\rangle \neq 0$, then Lemma 6 (d) indicates that $\left\langle t, h_{kp^j+s}\right\rangle = \left\langle t, h_{kp^j+1}\right\rangle$, thus it suffices to prove: $\langle t, h_{k+1}\rangle = \frac{c^j}{\sqrt{p^j}}\left\langle t, h_{kp^j+1}\right\rangle$. We denote by $M^{p,j} = \frac{1}{\sqrt{p^{N-j-1}}} D_{p^{N-j-1}}(\Psi^p)$ and we observe that $h_{kp^j+1} = \left(M^{p,j}\right)_k$. Moreover, $h_{kp^j+1} = 0$ for all $n > p^{N-j}$. Now, we calculate:

$$\begin{aligned}
\langle t, h_{k+1}\rangle &= \sum_{r=1}^{p^N} t_r h_{k+1,r} = \frac{1}{\sqrt{p^{N-1}}}\sum_{r=1}^{p^N} t_r \left(D_{p^{N-1}}(\Psi^p)\right)_{k,r} \\
&= \frac{1}{\sqrt{p^{N-1}}}\sum_{q=0}^{p^{N-j}-1}\sum_{v=1}^{p^j} t_{qp^j+v}\left(D_{p^{N-1}}(\Psi^p)\right)_{k,qp^j+v} = \frac{1}{\sqrt{p^j}}\sum_{q=0}^{p^{N-j}-1}\sum_{v=1}^{p^j} t_{qp^j+v}\left(D_{p^j}\left(M^{p,j}\right)\right)_{k,qp^j+v} \\
&= \frac{1}{\sqrt{p^j}}\sum_{q=0}^{p^{N-j}-1}\sum_{v=1}^{p^j} t_{qp^j+v}\left(M^{p,j}\right)_{k,q+1} = \frac{1}{\sqrt{p^j}}\sum_{q=1}^{p^{N-j}}\left(M^{p,j}\right)_{k,q}\sum_{v=1}^{p^j} t_{(q-1)p^j+v} \\
&= \frac{1}{\sqrt{p^j}} \sum_{\substack{q=1 \\ q=1+\sum_{i=1}^{N-j}\varepsilon_i p^{N-j-i},\, \varepsilon_i \notin \text{B for any } i < N-j}}^{p^{N-j}} \left(M^{p,j}\right)_{k,q}\sum_{v=1}^{p^j}\left(X_{j-1}\ldots X_1(g)\right)_v \\
&= \frac{1}{\sqrt{p^j}}\sum_{q=1}^{p^{N-j}}\left(M^{p,j}\right)_{k,q}\left(X_{N-j-1}\ldots X_1(g)\right)_q \sum_{v=1}^{p^j}\left(X_{j-1}\ldots X_1(g)\right)_v \\
&= \frac{c_j}{\sqrt{p^j}}\sum_{q=1}^{p^{N-j}}\left(M^{p,j}\right)_{k,q}\left(X_{N-j-1}\ldots X_1(g)\right)_q = \frac{c_j}{\sqrt{p^j}}\left\langle t, h_{kp^j+1}\right\rangle .
\end{aligned}$$

□

Theorem 2. *Let Q be the set of zeros of the Haar coefficients of the indicator sequence t of a Cantor language, then:*

(a)

$$Q = \bigcup_{j=1}^{N-1}\left(R_j + S_j\right) ,$$

where $R_j = \{rp^j : r = 0, \ldots, p-2\}$ *and* $S_j = \left\{k = p^j + 1 + \sum_{s=0}^{j-1} \varepsilon_s p^{j-1-s} :\right.$ *at least one* $\left.\varepsilon_s \notin B\right\}$.

(b)

$$t_n = \sum_{i=0}^{N-1} \sum_{\substack{m=p^i+1 \\ m \notin (R_i+S_i)}}^{p^{i+1}} \frac{\sqrt{p^i}}{c^i} \left\langle t, h_{\left[\frac{m}{p^i}\right]} \right\rangle h_{n,m} ,$$

where c is the cardinality of the support of the set B defined in (3.1).

Proof.

(a) Let $Q_j = \{p^j + 1 \leq k \leq p^{j+1} : \langle t, h_k \rangle = 0\}$, $j = 0, \ldots, N-1$. Lemma 6 (a) indicates that $Q_0 = \emptyset$. Lemma 6 (d) implies that:

$$Q_j = \left\{s + rp^j : s \in S_j, r = 0, \ldots, (p-2)\right\} ,$$

where $S_j = \left\{p^j + 1 \leq s \leq 2p^{j+1} : \left\langle t, h_{p^j+s} \right\rangle = 0\right\}$. Lemma 6 (c) indicates that $\left\langle t, h_{p^j+s} \right\rangle = 0$ if and only if $t_n = 0$ for all n satisfying $n = (s-1)p^{N-j} + 1, \ldots, p^{N-j}$ and so by Lemma 5 (i), S_j can be written as:

$$S_j = \left\{ k = p^j + 1 + \sum_{s=0}^{j-1} \varepsilon_s p^{j-1-s} : \text{at least one } \varepsilon_s \notin B \right\} .$$

(b) It is clear that

$$t_n = \sum_{i=0}^{N-1} \sum_{m=p^i+1}^{p^{i+1}} \langle t, h_m \rangle h_{n,m} = \sum_{i=0}^{N-1} \sum_{\substack{m=p^i+1 \\ m \notin (R_i+S_i)}}^{p^i+1} \langle t, h_m \rangle h_{n,m} .$$

Proposition 1 completes the proof. □

4. Haar-Riesz Products Associated to the Matrices $H(m)$

Let $m = 2, 3, \ldots,$ we call Haar-Riesz product associated to the sequence of complex numbers $a = \{a_n : n = 1, \ldots, m\}$ the expression:

$$t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}) ,$$

where h_k are rows of the matrix *H(m)*. In Proposition 2 we show the relation between t and a and in Theorem 3 we present an iteration process for the computation of a. First, we need the following lemma.

Lemma 7. *Let* $1 \leq r_1 < r_2 < \ldots < r_q \leq m$ *be a strictly increasing sequence of positive integers, then:*

$$\prod_{n=1}^{q} h_{r_n} = \left(\prod_{n=1}^{q-1} h_{r_n, q_0} \right) h_{r_q} ,$$

where h_{r_q,q_0} is the first nonzero element of the r_q-row of the matrix H(m).

Proof. Immediate consequence of Lemma 4. □

Proposition 2. *Let $\{a_1, \ldots, a_m\}$ be the sequence of Haar-Riesz coefficients associated to the Haar-Riesz product $t_n = \prod_{k=1}^{m}\left(1 + a_k h_{k,n}\right)$. If $\langle t, h_1\rangle \neq 0$, then:*

$$\langle t, h_s\rangle = \begin{cases} a_1 + \sqrt{m}, & s = 1 \\ a_s \prod_{k=1}^{s-1}\left(1 + a_k h_{k,s_0}\right), & s = 2, \ldots, m \end{cases},$$

where h_{s,s_0} is the first nonzero element of the row h_s.

Proof. $t_n = \prod_{k=1}^{m}\left(1 + a_k h_{k,n}\right)$

$$= 1 + \sum_{k=1}^{m} a_k h_{k,n} + \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,n} h_{k_2,n} + \ldots + (a_1 \ldots a_m)(h_{1,n} \ldots h_{m,n}).$$

We apply Lemma 7 and we have:

$$t_n = 1 + \sum_{k=1}^{m} a_k h_{k,n} + \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,k_2^0} h_{k_2,n} + \ldots + (a_1 \ldots a_m)\left(\prod_{j=1}^{m-1} h_{k_j,k_m^0}\right) h_{m,n},$$

where h_{k_i,k_0^i} is the first nonzero element of the row h_{k_i}.

If $s = 1$, then $\langle t, h_1\rangle = \langle 1, h_1\rangle + \sum_{k=1}^{m} a_k \langle h_k, h_1\rangle + 0 + \ldots + 0 = \sqrt{m} + a_1$.

Let $s > 1$. Define h_{s,s_0} the first nonzero element of the row h_s, then the orthonormality of the matrix *H(m)* implies that:

$$\begin{aligned}
\langle t, h_s\rangle &= a_s + \sum_{k_1=1}^{m-1}\sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,k_2^0} \delta_{k_2,s} + \ldots + (a_1 \ldots a_s)\left(\prod_{j=1}^{s-1} h_{k_j,s_0}\right) \\
&= a_s\left(1 + \sum_{k_1=1}^{s-1} a_{k_1} h_{k_1,s_0} + \sum_{k_1=1}^{m-2}\sum_{k_2=k_1+1}^{m-1} a_{k_1} a_{k_2}\left(\prod_{j=1}^{2} h_{k_j,s_0}\right) + \ldots + (a_1 \ldots a_{s-1})\left(\prod_{j=1}^{s-1} h_{k_j,s_0}\right)\right) \\
&= a_s \prod_{k=1}^{s-1}\left(1 + a_k h_{k,s_0}\right).
\end{aligned}$$

□

Theorem 3. *Let $t = \{t_1, \ldots, t_m\}$ be a sequence of complex numbers such that $\langle t, h_i\rangle \neq 0$, for any $i = 1, \ldots, m$, then there exists a unique sequence of coefficients $\{a_n : n = 1, \ldots, m\}$ satisfying:*

$$t_n = \prod_{k=1}^{m}\left(1 + a_k h_{k,n}\right).$$

Moreover, the coefficients $\{a_n\}$ satisfy:

$$a_n = \begin{cases} \langle t, h_1\rangle - \sqrt{m} & n = 1 \\ \frac{\langle t, h_n\rangle}{\prod_{k=1}^{n-1}\left(1 + a_k h_{k,n_0}\right)}, & n = 2, \ldots, m \end{cases},$$

where h_{n,n_0} is the first nonzero entry of the row h_n.

Proof. Obvious, see Proposition 2. The fact that the matrix $H(m)$ is orthonormal ensures the uniqueness of the coefficients. □

Remark 3. The assumption that all inner products must be non zero, can be relaxed as follows: Given t as above and $\varepsilon > 0$, there exists a Haar-Riesz product such that $\left|t_n - \prod_{k=1}^{m}\left(1 + a_k h_{k,n}\right)\right| < \varepsilon$. In fact, replace t with a data t' such that $|t - t'| < \varepsilon$ and whose all inner products are nonzero.

Proposition 3. *Any continuous positive measure* μ *on* $[0, 1]$ *can be approximated in the weak-* topology by a sequence of Haar-Riesz products* $\{\mu_m, m = 2, 3, \dots\}$*:*

$$d\mu_m = \prod_{k=1}^{m} (1 + a_k h_k(x))\ dx\ ,$$

where $h_k(x) = h_{k,n}$, $x \in \left[\frac{n-1}{m}, \frac{n}{m}\right)$, $k, n = 1, \dots, m$, $h_{k,n}$ *is the* (k, n) *entry of the matrix* *H(m)* and a_k are the corresponding coefficients.

Proof. Apply Theorem 3 and Remark 3 on $t = \{t_k : k = 1, \dots, m\}$, where $t_m = \left\{\int_{k/m}^{(k+1)/m} d\mu, k = 1, \dots, m\right\}$. □

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