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Third-Order Hydrodynamic Loads on an Oscillating Vertical Cylinder in Water

The third-harmonic component of the third-order hydrodynamic loads on a vertical circular cylinder oscillating in water is calculated by a conventional perturbation method within the framework of a potential theory. Although the third-order forces are expressed in terms of the first, second, and third-order components of the velocity potential, the latter is not directly required for the calculation. It is replaced by a properly defined linearized radiation potential via Haskind-like theorem. The results of the study are applicable to the analysis of high-frequency resonances of deepwater offshore structures under earthquake excitation or under steep waves (ringing problem).

Introduction

The problem considered is directly related to the prediction of nonlinear loads on cylindrical structures (also offshore structures) founded on the sea bottom in seismically active regions. The high-frequency components of nonlinear hydrodynamic forces due to earthquake-induced motions of the structure may excite structural responses at natural frequencies substantially higher than the dominant frequencies of an earthquake spectrum.

The present study has also been motivated by the high-frequency dynamical effects (so-called ringing) which have recently been observed on some deepwater offshore structures such as tension-leg platforms and gravity-based towers.

The association of the nonlinear radiation problem with the ringing problem seems to be, at first glance, astonishing. The existing theoretical analyses (Faltinsen et al., 1995; Malenica and Molin, 1995) look for the reasons of ringing in the third-harmonic components of diffraction loads, which become important when an incoming wave steepens. However, the experimental simulations of the phenomenon (Stansberg, 1997; and Scolan et al., 1997) show that the structure may oscillate with a dominating wave frequency before a transient high-frequency ringing response occurs. Thus, the high-frequency nonlinear loads may result not only from the wave diffraction, but from the radiated wave field as well. In that sense, a third-order solution obtained in the present work may be viewed as a complementary one to the diffraction solution by Malenica and Molin.

The hydrodynamic loads induced by the motion of bottom founded cylindrical structures in water have been studied for more than five decades in the context of earthquake engineering. The early investigations, concerned with cylindrical tanks and piers, assumed the structure to be rigid and water to be incompressible. Later investigations were concerned with flexible towers (Liaw and Chopra, 1973) and with the influence of water compressibility on hydrodynamic loads (Nilrat, 1980). Until the late eighties, the linear computational models dominated (see, e.g., Goyal et al., 1989), followed then by some examples of nonlinear computations in the time domain (Wang and Chwang, 1989; Chen and Huang, 1997). The nonlinear forces in the frequency domain have not been calculated so far.

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It is obvious that the nonlinear radiation of waves by a moving cylinder appears also in the analysis of second-order wave loads on floating systems; but the calculated second-order forces are either due to the diffraction of an incident wave or due to the interaction among diffraction and radiation potentials, or due to the interaction between the slow-drift motion and waves (Molin, 1994). This is not surprising, since despite the fact that the second-order pressure due to the horizontal oscillations of the cylinder does exist, the resultant second-order force vanishes.

Therefore, in order to compute the nonlinear forces on a horizontally oscillating axisymmetric structure, one has to carry out the perturbation analysis up to third order in the frequency domain. Such an analysis has been completed in the present work. It is focused on the third-harmonic force and on those components of the radiation potential which contribute to this force.

Problem Formulation

Consider the radiation of nonlinear gravity waves in water of constant depth h due to the forced oscillatory motion of a circular cylinder extending from the bottom S_{Bot} to the free surface S_F. The origin of a fixed coordinate system (x, y, z) is located at the undisturbed free surface $z = 0$ and the vertical z-axis is positive upward (Fig. 1). It is assumed that the cylinder is rigid.

The oscillation of the cylinder axis are described by the displacement function $u(t) = u_0 \cos \omega t$ in the direction of the x-coordinate. Assuming that $u_0 < R$, R being the cylinder radius, one can describe the instantaneous position of the cylinder surface S_B (Fig. 2) in cylindrical coordinates (r, ϑ, z) by

$$r = u(t) \cos \vartheta + \sqrt{R^2 - u^2(t) \sin^2\vartheta} = R + \xi(\vartheta, t) \quad (1)$$

The hydrodynamic forces $\bar{F}$ on the cylinder can be calculated through integration of the pressure p over the instantaneous wetted cylinder surface $S_B(t)$. Under assumption that potential theory is applicable, the pressure p can be expressed in terms of the radiation potential ϕ by the Bernoulli equation. This leads to

$$\bar{F} = \int_{S_B(t)} \left(-\rho g z - \rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho (\nabla \phi)^2 \right) \bar{n} dS \qquad (2)$$

where $\bar{n}$ denotes generalized unit normal vector pointing into

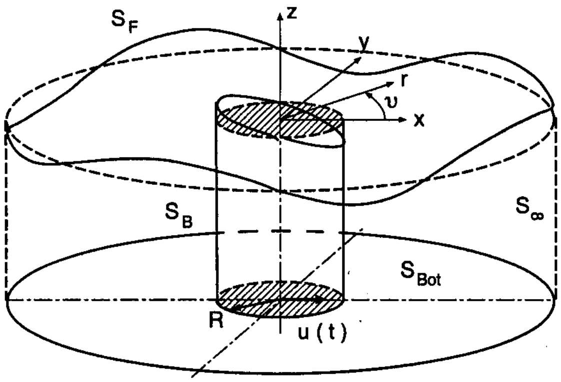

Fig. 1 Definition sketch

the cylinder, ρ is the water density, and g is the gravitational acceleration.

The governing equations for ϕ are

$$\nabla^2\phi = 0 \quad (R + \xi < r < \infty, 0 \le \vartheta < 2\pi, -h < z < \eta) \quad (3)$$

$$\phi_{tt} + g\phi_z + 2\nabla\phi\cdot\nabla\phi_t + \tfrac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi)^2 = 0 \quad (z = \eta) \quad (4)$$

$$\phi_r = \xi_t + \nabla\phi\cdot\nabla\xi \quad (r = R + \xi) \quad (5)$$

$$\phi_z = 0 \quad (z = -h) \quad (6)$$

and an appropriate radiation condition must be satisfied at infinity.

The free-surface elevation η is described in an implicit manner

$$\eta = -\frac{1}{g}(\phi_t + \tfrac{1}{2}(\nabla\phi)^2) \quad (z = \eta) \quad (7)$$

In our analysis we assume that the relevant characteristic length of the structure (radius R) is comparable to the wavelength λ, and thus $kR = O(1)$, k being the wavenumber $k = 2\pi/\lambda$. Assuming that the amplitude u_0 of forced oscillation of the cylinder is small in comparison to R, i.e., $u_0/R \ll 1$, we define $\epsilon = ku_0 \ll 1$ as a small parameter.

Calculation of Forces—Basic Steps

The following procedure is applied to the calculation of forces.

Firstly, assuming that $u(t)$, ϕ, and η are small quantities we expand ϕ and η at the exact free surface, and ϕ at the actual position of the cylinder in Taylor series about $z = 0$ and $r = R$, respectively.

Then, we introduce the perturbation series with respect to the previously defined small parameter ϵ

$$u = \epsilon u^{(1)}$$

$$\xi = \epsilon\xi^{(1)} + \epsilon^2\xi^{(2)} + \epsilon^3\xi^{(3)} + O(\epsilon^4)$$

$$\phi = \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \epsilon^3\phi^{(3)} + O(\epsilon^4)$$

$$\eta = \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + \epsilon^3\eta^{(3)} + O(\epsilon^4) \quad (8)$$

where the components of the forcing function ξ follow from the Taylor expansion of (1) and are given by

$$\epsilon\xi^{(1)} = u_0\Re\{e^{-i\omega t}\}\cos\vartheta$$

$$\epsilon^2\xi^{(2)} = -\frac{u_0^2}{8R}(1 - \cos 2\vartheta) - \frac{u_0^2}{8R}(1 - \cos 2\vartheta)\Re\{e^{-i2\omega t}\}$$

$$\epsilon^3\xi^{(3)} = 0 \quad (9)$$

Inserting the series (8) into the boundary value problems (3)–(6), we obtain a sequence of linear boundary value problems in Eulerian description. For instance, at third order, we have

$$\nabla^2\phi^{(3)} = 0 \quad \text{(in fluid)} \quad (10)$$

$$\begin{aligned}\phi_{tt}^{(3)} + g\phi_z^{(3)} &= -2\frac{\partial}{\partial t}(\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}) + \frac{1}{g}\phi_t^{(1)}(\phi_{ttz}^{(2)} \\ &\quad + g\phi_{zz}^{(2)}) + \frac{1}{g}(\phi_{ttz}^{(1)} + g\phi_{zz}^{(1)})(\phi_t^{(2)} + \tfrac{1}{2}(\nabla\phi^{(1)})^2 \\ &\quad - \frac{1}{g}\phi_t^{(1)}\phi_{tz}^{(1)}) + \frac{1}{g}\phi_t^{(1)}\frac{\partial^2}{\partial t\partial z}(\nabla\phi^{(1)})^2 \\ &\quad - \tfrac{1}{2}\nabla\phi^{(1)}\cdot\nabla(\nabla\phi^{(1)})^2 \quad (\text{on } z = 0) \quad (11)\end{aligned}$$

$$\begin{aligned}\phi_r^{(3)} &= \nabla\xi^{(1)}\cdot\nabla\phi^{(2)} + \nabla\xi^{(2)}\cdot\nabla\phi^{(1)} - \xi^{(2)}\phi_{rr}^{(1)} - \xi^{(1)}\phi_{rr}^{(2)} \\ &\quad + \xi^{(1)}\nabla\xi^{(1)}\cdot\nabla\phi_r^{(1)} - \tfrac{1}{2}(\xi^{(1)})^2\phi_{rrr}^{(1)} \quad (\text{on } r = R) \quad (12)\end{aligned}$$

$$\phi_z^{(3)} = 0 \quad (\text{on } z = -h) \quad (13)$$

Third-order hydrodynamic pressure follows from the Bernoulli equation

$$p^{(3)} = -\rho(\phi_t^{(3)} + \nabla\phi^{(1)}\cdot\nabla\phi^{(2)}) \quad (14)$$

The solution of the boundary value problems at each order of approximation considered would complete the second step of the force calculation procedure.

The next step is the transformation of the quantities, given in Eulerian description in terms of fixed coordinates (r, ϑ, z), into a local coordinate system moving with the cylinder. This enables us to carry out exact integration over the instantaneous cylinder surface in a simple manner. For the points of the cylinder surface, the mapping $(r, \vartheta, z) \mapsto (R, \vartheta', z)$ (Fig. 2) is given by

$$\begin{aligned}r &= \sqrt{R^2 + u^2(t) + 2Ru(t)\cos\vartheta'} \\ &= R + \epsilon u^{(1)}\cos\vartheta' + \frac{(\epsilon u^{(1)})^2}{2R}\sin^2\vartheta' + O(\epsilon^3) \quad (15)\end{aligned}$$

$$\begin{aligned}\vartheta &= \arctan\left(\frac{R\sin\vartheta'}{u(t) + R\cos\vartheta'}\right) \\ &= \vartheta' - \frac{\epsilon u^{(1)}}{R}\sin\vartheta' + \frac{(\epsilon u^{(1)})^2}{2R^2}\sin 2\vartheta' + O(\epsilon^3) \quad (16)\end{aligned}$$

With the use of (15) and (16), the pressure p and the exact free surface η can be expressed in terms of local coordinates (R, ϑ', z) via Taylor expansion. In order to retain all relevant terms, the expansion up to second order is necessary.

The unit normal vector has simple time-independent components in the local coordinate system

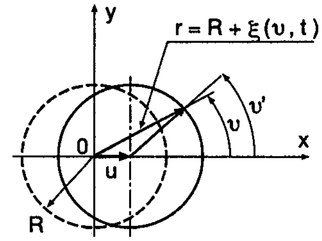

Fig. 2 Instantaneous position of the cylinder surface

$$\bar{n}(\vartheta') = [-\cos\vartheta', -\sin\vartheta', 0]^T \quad (17)$$

For the sake of simplicity, the components corresponding to the hydrodynamic moments have been neglected.

Denoting by $\tilde{p}(R, \vartheta', z, t)$ the truncated Taylor expansion of $p(r, \vartheta, z, t)$ and by $\tilde{\eta}(R, \vartheta', t)$ the similar expansion for $\eta(r, \vartheta, t)$, we can express the forces by

$$\bar{F}(t) = \int_{-h}^{\tilde{\eta}} \int_0^{2\pi} \tilde{p}(R, \vartheta', z, t)\bar{n}(\vartheta')Rd\vartheta'dz \quad (18)$$

The hydrostatic component of $\tilde{p}$ can be integrated exactly. For the other components the integration with respect to z coordinate can be approximated by

$$\int_{-h}^{\tilde{\eta}} [\sim]dz \approx \int_{-h}^{0} [\sim]dz + \tilde{\eta}[\sim]_{|z=0} + \tfrac{1}{2}\tilde{\eta}^2[\sim]_{|z=0} \quad (19)$$

Inserting (15) and (16) in (18), using Bernoulli equation, invoking (8) and (19), we can separate out the force components up to third order

$$\bar{F}(t) = \bar{F}^{(0)} + \epsilon\bar{F}^{(1)} + \epsilon^2\bar{F}^{(2)} + \epsilon^3\bar{F}^{(3)} + O(\epsilon^4) \quad (20)$$

where $\bar{F}^{(0)}$ denotes a buoyancy force.

The calculation is straightforward in principle (though tedious), but care must be taken to retain all terms of relevant order. One may prove that the second-order force $F^{(2)}$ is identically equal to zero for the problem considered.

The third-order force is obtained in terms of the first, second, and third-order components of the velocity potential. The boundary value problems for these components admit the following solutions in terms of complex quantities:

$$\begin{aligned} \epsilon[\phi, \bar{F}]^{(1)} &= \Re\{[\varphi, \mathscr{F}]^{(1)}e^{-i\omega t}\} \\ \epsilon^2[\phi, \bar{F}]^{(2)} &= [\bar{\varphi}, \bar{\mathscr{F}}]^{(2)} + \Re\{[\varphi, \mathscr{F}]^{(2)}e^{-i2\omega t}\} \\ \epsilon^3[\phi, \bar{F}]^{(3)} &= \Re\{[\varphi, \mathscr{F}]_1^{(3)}e^{-i\omega t}\} + \Re\{[\varphi, \mathscr{F}]_3^{(3)}e^{-i3\omega t}\} \end{aligned} \quad (21)$$

The first-harmonic third-order potential is only a small $O(\epsilon^2)$ correction to the first-order potential and will not be considered in further analysis. Consequently, the time-independent second-order potential which contributes only to this third-order component can also be neglected.

The complex amplitude of the third-harmonic third-order hydrodynamic force is given by

$$\mathscr{F}^{(3)} = \mathscr{F}_1^{(3)} + \mathscr{F}_2^{(3)} + \mathscr{F}_3^{(3)} \quad (22)$$

where $\mathscr{F}_1^{(3)}$ results from triple products of first-order potentials

$$\begin{aligned} \mathscr{F}_1^{(3)} = \frac{\rho R u_0}{4}\int_0^{2\pi}\int_{-h}^{0}\Bigg\{ &-\nabla\varphi^{(1)}\nabla\varphi_r^{(1)}\cos\vartheta' \\ &+ \nabla\varphi^{(1)}\nabla\varphi_\vartheta^{(1)}\frac{\sin\vartheta'}{R} + iu_0\omega\Bigg[\varphi_{rr}^{(1)}\cos 2\vartheta' \\ &- \varphi_{zz}^{(1)}\sin^2\vartheta' + \left(\varphi_\vartheta^{(1)} - \frac{R}{2}\varphi_{r\vartheta}^{(1)}\right)\frac{\sin 2\vartheta'}{R^2}\Bigg]\Bigg\}\bar{n}dzd\vartheta'_{\substack{|r=R\\ \vartheta=\vartheta'}} \\ &- \tfrac{1}{4}\rho R\int_0^{2\pi}\Bigg\{\frac{i\omega}{2g}[\varphi^{(1)}(\nabla\varphi^{(1)})^2 + \nu^2(\varphi^{(1)})^3] \\ &+ u_0\nu\varphi^{(1)}\left(\varphi_r^{(1)}\cos\vartheta' - \varphi_\vartheta^{(1)}\frac{\sin\vartheta'}{R}\right)\Bigg\}\bar{n}d\vartheta'_{\substack{|r=R,\vartheta=\vartheta'\\ z=0}} \end{aligned} \quad (23)$$

$\mathscr{F}_2^{(3)}$ comes from products of first-order and second-order quantities

$$\begin{aligned} \mathscr{F}_2^{(3)} = \rho R\int_0^{2\pi}\int_{-h}^{0}\Bigg\{&-\tfrac{1}{2}\nabla\varphi^{(1)}\cdot\nabla\varphi^{(2)} \\ &+ iu_0\omega\left(\varphi_r^{(2)}\cos\vartheta' - \varphi_\vartheta^{(2)}\frac{\sin\vartheta'}{R}\right)\Bigg\}\bar{n}dzd\vartheta'_{\substack{|r=R\\ \vartheta=\vartheta'}} \\ &- \rho R\int_0^{2\pi}\{\nu\varphi^{(1)}\varphi^{(2)}\}\,\bar{n}d\vartheta'_{\substack{|r=R,\vartheta=\vartheta'\\ z=0}} \end{aligned} \quad (24)$$

and $\mathscr{F}_3^{(3)}$ is due to the third-order potential

$$\mathscr{F}_3^{(3)} = 3i\omega\rho R\int_0^{2\pi}\int_{-h}^{0}\{\varphi_3^{(3)}\}\bar{n}dzd\vartheta'_{\substack{|r=R\\ \vartheta=\vartheta'}} \quad (25)$$

where $\nu = \omega^2/g$. In comparison to third-order diffraction forces (Malenica and Molin, 1995), some new terms due to the motion of the cylinder appear in the components $\mathscr{F}_1^{(3)}$ and $\mathscr{F}_2^{(3)}$.

Solution for the Radiation Potentials

Although the third-order force is given in terms of the radiation potentials up to third order, only the first-order and the second-order radiation potentials are explicitly required. The contribution from the third-order potential will be calculated in an indirect manner, as shown in the next section.

The well-known solution for the first-order radiation potential (Liaw and Chopra, 1973) is given as an infinite series of radial eigenfunctions (here Hankel and modified Bessel functions of the first order)

$$\varphi^{(1)} = \{a_0H_1^{(1)}(k_1r)Z_0(k_1z) + \sum_{l=1}^{\infty} a_lK_1(\kappa_lr)Z_l(\kappa_lz)\}\cos\vartheta \quad (26)$$

where k_1 and $i\kappa_l$ are real and imaginary roots of the dispersion relation $\omega^2/g = k\tanh kh$.

The eigenfunctions Z_l corresponding to k_1 and $i\kappa_l$ are

$$Z_0(k_1z) = \frac{\cosh k_1(z+h)}{\cosh k_1h}, \quad Z_l(\kappa_lz) = \frac{\cos\kappa_l(z+h)}{\cos\kappa_lh} \quad (27)$$

and the expansion coefficients a_l are given by

$$a_0 = \frac{-2iu_0\omega\sinh 2k_1h}{k_1(2k_1h + \sinh 2k_1h)H_1'(k_1R)} \quad (28)$$

$$a_l = \frac{-2iu_0\omega\sin 2\kappa_lh}{\kappa_l(2\kappa_lh + \sin 2\kappa_lh)K_1'(\kappa_lR)} \quad (29)$$

The second-harmonic component of the radiation potential relevant here can be calculated either by Weber integral transform or by an eigenfunction expansion method. We use the latter one and decompose the total second-harmonic potential into

$$\varphi^{(2)} = \varphi_L^{(2)} + \varphi_F^{(2)} \quad (30)$$

where the "locked" component $\varphi_L^{(2)}$ satisfies the inhomogeneous free-surface boundary condition

$$\varphi_{Lz}^{(2)} - 4\nu\varphi_L^{(2)} = \sum_{n=0,2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} A_{npq}(r)\cos n\vartheta \quad (\text{on } z=0) \quad (31)$$

and the "forced" component $\varphi_F^{(2)}$ satisfies the homogeneous free-surface boundary condition

$$\varphi_{Fz}^{(2)} - 4\nu\varphi_F^{(2)} = 0 \quad (\text{on } z=0) \quad (32)$$

Both potentials satisfy the Laplace equation in fluid, no-flow condition on the bottom and jointly the inhomogeneous boundary condition on the wetted cylinder surface

$$\varphi_{Lr}^{(2)} + \varphi_{Fr}^{(2)} = \sum_{n=0,2} [C_n(z) + \sum_{p=0}^{\infty} D_{np}(r, z)] \cos n\vartheta \quad (\text{on } r = R) \quad (33)$$

The function $A_{npq}(r)$ is due to quadratic contribution from the first-order potential (26). $C_n(z)$ and $D_{np}(r, z)$ are due to the second-order forcing function $\xi^{(2)}$ (see Eq. (9)), and to the products of the first-order potential and the first-order forcing function $\xi^{(1)}$ (or their derivatives).

The asymptotic analysis of the second-order boundary value problem reveals that the far-field behavior of $\varphi^{(2)}$ is

$$\varphi^{(2)} \sim \frac{\Theta_F^{(2)}(\vartheta)}{\sqrt{k_2 r}} \frac{\cosh k_2(z+h)}{\cosh k_2 h} e^{ik_2 r} + O(r^{-1}) \quad \text{at} \quad r \to \infty \quad (34)$$

where k_2 is the second-order wavenumber satisfying the dispersion relation $4\omega^2/g = k_2 \tanh k_2 h$. It follows that the total second-order radiation potential satisfies usual Sommerfeld radiation condition.

Turning back to the "locked" potential, we may try to find the solution using the method of eigenfunction expansion developed by Huang and Eatock Taylor (1996) for diffraction problems. Proceeding similarly, we obtain the solution in terms of the eigenfunctions $Z_m(\beta_m z)$ satisfying the homogeneous boundary conditions on the free surface and on the bottom

$$\varphi_L^{(2)} = \sum_{n=0,2} \sum_{m=0}^{\infty} Z_m(\beta_m z) \cos n\vartheta \times \int_R^{\infty} \{\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{mq}(\beta_m^2 - \alpha_p^2) A_{npq}(\rho)\} \mathrm{G}_{nm}(r, \rho) d\rho \quad (35)$$

where the eigenvalues β_m (real and imaginary) fulfill the relation $4\nu = \beta_m \tanh \beta_m h$, and $\mathrm{G}_{nm}(r, \rho)$ denotes the Green function of the Bessel equation

$$\mathrm{G}_{nm} = \begin{cases} i\rho H_n^{(1)}(\beta_m \rho) J_n(\beta_m r) & \text{for} \quad r < \rho \\ i\rho H_n^{(1)}(\beta_m r) J_n(\beta_m \rho) & \text{for} \quad r > \rho \end{cases} \quad (36)$$

The coefficients v_{mq} are given by

$$v_{mq} = \int_{-h}^{0} V_q Z_m dz \Big/ \int_{-h}^{0} Z_m^2 dz \quad (37)$$

with

$$V_q(\alpha_q z) = \frac{1}{\nu} \frac{\cosh \alpha_q(z+h)}{\cosh \alpha_q h}, \quad Z_m(\beta_m z) = \frac{\cosh \beta_m(z+h)}{\cosh \beta_m h} \quad (38)$$

in which another set of eigenvalues α_q satisfies the relation $\alpha_q \tanh \alpha_q h = 5\nu$.

The "forced" potential $\varphi_F^{(2)}$ can be expressed in terms of the same eigenfunctions as the "locked" component

$$\varphi_F^{(2)} = \sum_{n=0,2} \sum_{m=0}^{\infty} a_{nm} H_n(\beta_m r) Z_m(\beta_m z) \cos n\vartheta \quad (39)$$

in which $H_n = H_n^{(1)}(k_2 r)$ for $m = 0$, and $H_n = K_n(\beta_m r)$ for $m > 0$. The solution (39) fulfills the Laplace equation and all required boundary conditions, except for the boundary condition on the wetted cylinder surface (33). Satisfying this condition, we are able to calculate the unknown expansion coefficients a_{nm}.

Denoting

$$C_n(z) + \sum_{p=0}^{\infty} D_{np}(r, z) = E_n(r, z)$$

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{mq}(\beta_m^2 - \alpha_p^2) A_{npq}(\rho) = B_{nm}(\rho)$$

and inserting (35) into (33), we eventually obtain

$$a_{nm} = \frac{\int_{-h}^{0} E_n(R, z) Z_m(\beta_m z) dz}{\beta_m H_n'(\beta_m R) \int_{-h}^{0} Z_m^2(\beta_m z) dz} - \int_R^{\infty} \frac{B_{nm}(\rho)}{\beta_m H_n'(\beta_m R)} \frac{\partial \mathrm{G}_{nm}(r, \rho)}{\partial r} d\rho \quad (40)$$

Force Due to Third-Order Potential

Instead of calculating the third-order radiation potential directly, we determine its contribution to the third-order force (25) in an indirect manner introducing assisting third-harmonic linearized radiation potentials $\Psi^{(\cdot)} = \Re e\{\psi^{(\cdot)} e^{-3i\omega t}\}$, which on the cylinder surface satisfy

$$\bar{\psi}_n = -\left(\frac{\partial \psi^{(x)}}{\partial r}, \frac{\partial \psi^{(y)}}{\partial r}, 0\right) = -(\cos \vartheta', \sin \vartheta', 0)^T = \bar{n} \quad (41)$$

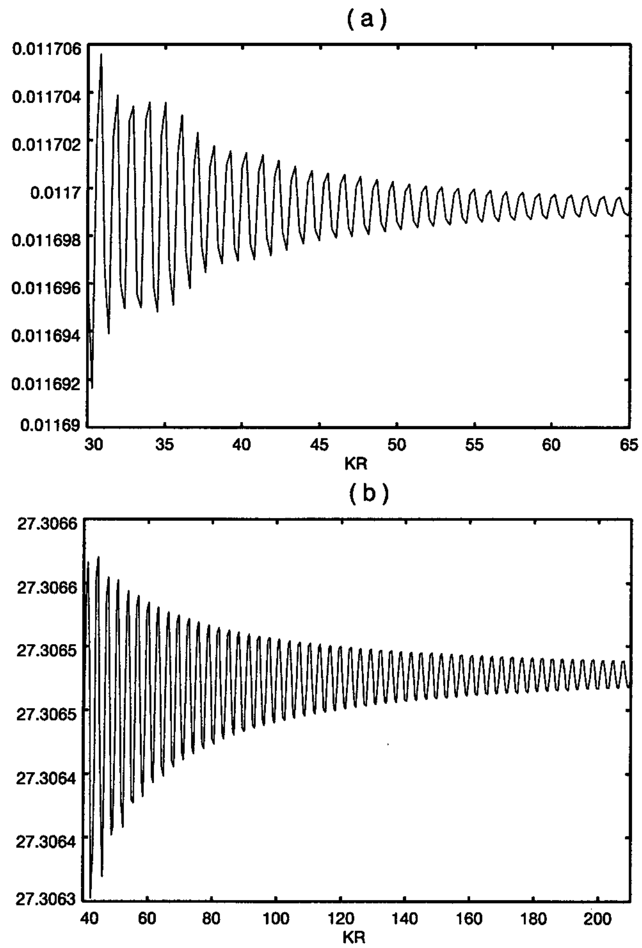

Fig. 3 Convergence of the modulus of the free-surface integral for (a) second space harmonic component of $\varphi_L^{(2)}$, (b) third-order horizontal force $\mathscr{F}_3^{(3)}$. $H = 10R$; $kR = 1$.

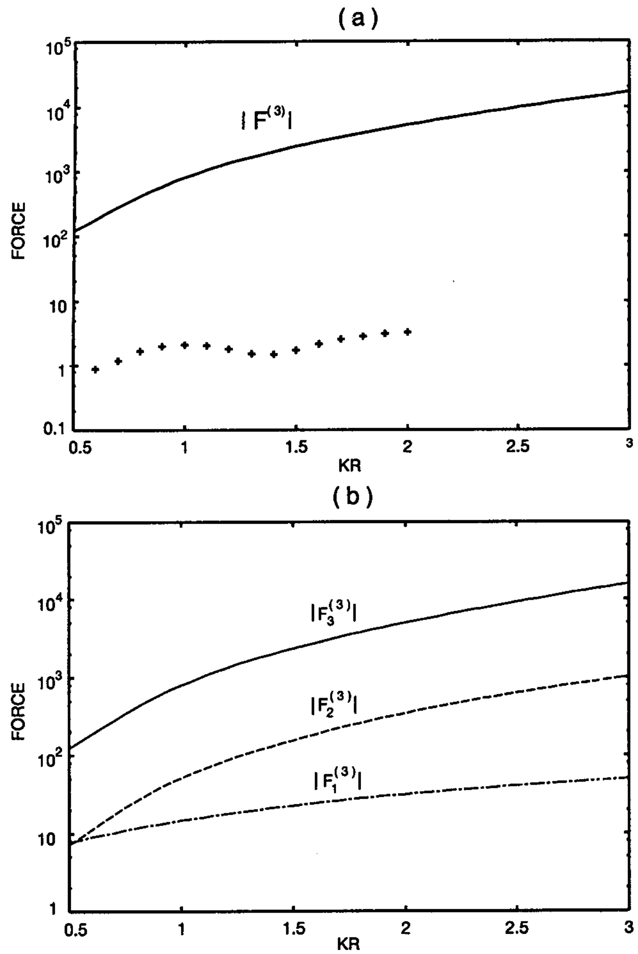

Fig. 4 Third-order horizontal radiation force. (*a*) Dimensionless magnitude of the total force ($|\mathscr{F}^{(3)}|/(\rho g u_0^3)$), (*b*) three dimensionless components. Points in (*a*) give the values of the diffraction force ($|\mathscr{F}^{(3)}|/(\rho g A^3)$) computed by Malenica and Molin (1995). $H = 10R$.

The analysis is in principle the same as in the paper by Molin (1979) with minor differences concerning the boundary condition on the wetted cylinder surface. Using Green's theorem, we obtain the final formula for the component $\mathscr{F}_3^{(3)}$ of the third-order force

$$\mathscr{F}_3^{(3)} = 3i\omega\rho\left\{R\int_0^{2\pi}\int_{-h}^{0}(-\varphi_{3r}^{(3)}\overline{\psi})\,dz d\vartheta'\Big|_{\substack{r=R\\ \vartheta=\vartheta'}} + \int_0^{2\pi}\int_R^{\infty}\alpha^{(3)}\overline{\psi} r dr d\vartheta'\Big|_{\substack{z=0\\ \vartheta=\vartheta'}}\right\} \quad (42)$$

where $\alpha^{(3)}$ denotes the expression on the right-hand side of the free-surface boundary condition (11) in a complex form. Similarly, $\varphi_{3r}^{(3)}$ can be replaced by the expression on the right-hand side of the boundary condition on the cylinder surface (also in complex notation).

Numerical Results

The numerical results for the third-order horizontal radiation force on an offshore cylinder ($R = 10$ m, $H = 100$ m) are presented. As in previous works (Chau and Eatock Taylor, 1992; Malenica and Molin, 1995; Huang and Eatock Taylor, 1996), the main difficulty in the numerical implementation is associated with the calculation of integrals over the free surface, which have highly oscillatory integrands (in r). Such integrals appear in the expressions for $\varphi^{(2)}$ and $\mathscr{F}_3^{(3)}$ (see Eqs. (35), (39), and (42)). In order to calculate them efficiently with the preservation of sufficient accuracy, we make use of the richness of experience gathered in the aforementioned works.

Having no possibility to validate our computer program by comparison with other results (there are no such results), we carefully check during computations the convergence of integrals involved in the second-order radiation potential, then the convergence of the potential itself, and finally the convergence of the integrals appearing in the formula for the third-order force (42).

The typical convergence tests for integrals are presented in Fig. 3. Figure 3(*a*) shows the convergence of the free-surface integral involved in the calculation of $\varphi_L^{(2)}$ for the most "dangerous" combination of eigenfunctions in the integrand (a tripple product of Hankel functions). Figure 3(*b*) shows the convergence of the free-surface integral appearing in the third component $\mathscr{F}_3^{(3)}$ of the third-order force.

The results for the total dimensionless third-order force ($|\mathscr{F}^{(3)}|/(\rho g u_0^3)$) and its three components ($\mathscr{F}_i^{(3)}/(\rho g u_0^3)$) are given in Fig. 4. One can observe that the component resulting from the third-order potential is the biggest one. The third-order force increases with increasing dimensionless frequency (kR); but one should notice that the wave steepness, which can be assessed by the value of the small parameter ϵ, also increases in that case. In Fig. 5, the magnitudes of the first-order force and of the total third-order forces are compared for the constant $\epsilon = 0.1$ and for the constant amplitude $u_0 = 0.33$ m of forced horizontal oscillations. Both forces are nondimensionalized by ($\rho g V$) with V being displaced volume of water. One can see

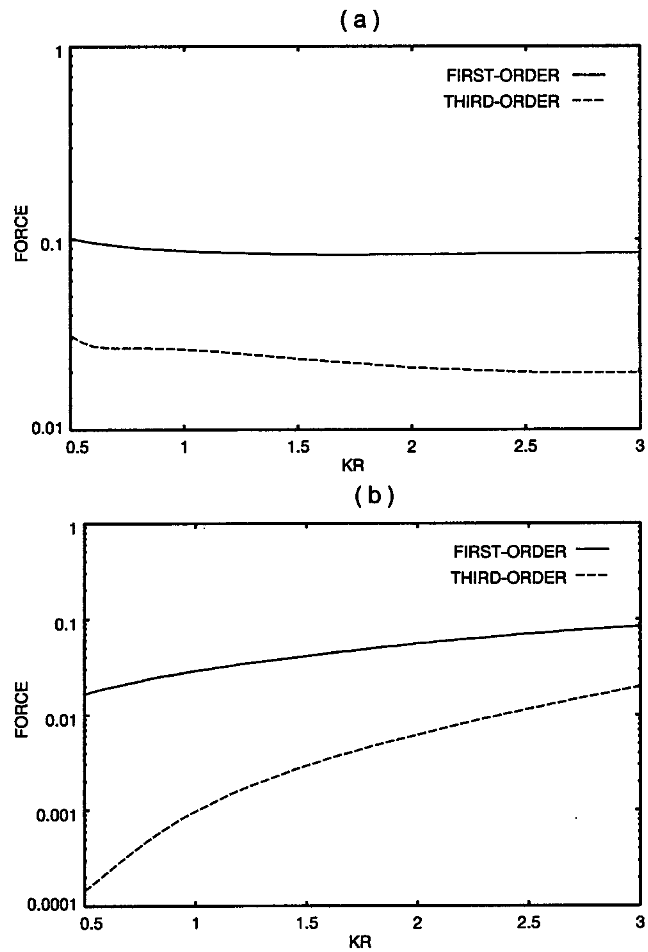

Fig. 5 Dimensionless magnitudes of the first-order (——) and third-order (-----) horizontal radiation forces for (*a*) constant $\epsilon = 0.1$, (*b*) constant $u_0 = 0.33$ m. $H = 10R$.

that the ratio of both forces remains approximately constant for a constant value of ϵ. However, the third-order force increases faster than the first-order force when the oscillation amplitude is kept constant.

Concluding Remarks

The main objective of the work was the calculation of the third harmonic of the nonlinear hydrodynamic force on a vertical cylinder oscillating horizontally in water. To complete the task, the second-order radiation potential had to be derived, whereas an indirect method could be applied to avoid the calculation of the third-order radiation potential.

The magnitude of the third-order radiation force is proportional to third power of the oscillation amplitude of the cylinder $|\mathscr{F}_R^{(3)}| = \rho g |\chi_R(i\omega)| u_0^3$. The analogous force due to diffraction is proportional to third power of the incident wave amplitude $|\mathscr{F}_D^{(3)}| = \rho g |\chi_D(i\omega)| A^3$ (see Malenica and Molin, 1995).

In applications to earthquake engineering, u_0 corresponds to the response amplitude of the structure in water at the dominant frequency of an earthquake excitation. For floating or bottom founded offshore structures under waves, u_0 corresponds to the response amplitude of the structure in horizontal direction. This amplitude can, in linear approximation, be related to the incident wave amplitude A through an appropriate frequency response function $H(i\omega)$: $|\mathscr{F}_R^{(3)}| = \rho g |\chi_R(i\omega)| \cdot |H(i\omega)|^3 A^3$. Therefore, the analysis of the relevance of third-order radiation loads for this application requires the comparison of $|\chi_R(i\omega)| \cdot |H(i\omega)|^3$ with $|\chi_D(i\omega)|$. Some values of the function $\chi_D = |\mathscr{F}_D^{(3)}|/\rho g A^3$ computed by Malenica and Molin (1995, Fig. 7) are additionally presented as points in Fig. 4(a). Comparing these values with the present results for $|\chi_R(i\omega)| = |\mathscr{F}_R^{(3)}|/\rho g u_0^3$ (Fig. 4(a)), we may notice that $|\chi_R(i\omega)| \gg |\chi_D(i\omega)|$ in the frequency range considered. This allows to conclude that third-order radiation loads cannot be neglected if the frequency response function of the structure is not small for dominant frequencies of a wave spectrum.

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