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# Simulation Procedure for Divergent Light Halos

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We present a new procedure for the simulation of divergent light halos. The procedure uses rotational symmetries to make a selected sampling of events that greatly improves the efficiency of the algorithm. We can typically generate a simulated display in minutes using a personal computer. The theory behind the procedure also gives a quantitative explanation of the observational fact that a divergent light halo display depends on the distance between the light source and the observer. © 2002 Optical Society of America

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## *Introduction*

There are very few attempts to make Monte Carlo simulations of divergent light halos<sup>1-3</sup>. With a divergent light halo it is not as with a parallel light halo, always possible to find a location for the scattering crystal such that the scattered ray will hit the eye of the observer. With divergent light there is a geometrical constraint that relates the locations of the light source, the observer, and the scattering crystal. The procedure described below will find, if possible, where to locate the crystal such that the scattered ray hits the eye of the observer. It will also take into account the intensity dependence on the distances involved.

### *The procedure*

We represent the ray emitted from the source by a vector. The scattered ray is represented by a second vector. The first geometrical constraint is that the sum of these vectors has a length that is equal to the distance observer-source which we normalise to 1. By symmetry we are only allowed to perform rotations of these vectors around two axes: a vertical axis and an axis parallel to the crystal symmetry axis, the tilt axis. (We do not treat here the case of Parry oriented crystals.) This is the second constraint. We will try to determine these rotations and the lengths of the vectors such that the scattered ray ends up in the eye of the observer.

### *Theory*

Assume that we have a vertical hexagonal crystal that we tilt in a mathematically positive direction around the horizontal  $x$  axis. Further assume that we have a generated a ray with random direction  $\mathbf{k}_1$  from the source that has been raytraced through the crystal and exits with direction  $\mathbf{k}_2$ . Both these vectors are unit vectors. The initial and scattered ray can then be represented mathematically by

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{k}_1 t_1, & t_1 > 0 \\ \mathbf{r}_2 &= \mathbf{k}_2 t_2, & t_2 > 0\end{aligned}\tag{1}$$

where  $t_1$  and  $t_2$  are parameters, the *parametric distances*.

Consider a sphere with unit radius centered at the origin. We imagine two coordinate systems on the sphere, one "equatorial"  $(\alpha, \delta)$  and one "ecliptical"  $(\lambda, \beta)$ . The "vernal point" of the two coordinate systems is on the  $x$  axis. The tilt will then correspond to the "obliquity"  $\varepsilon$  of the "ecliptic".

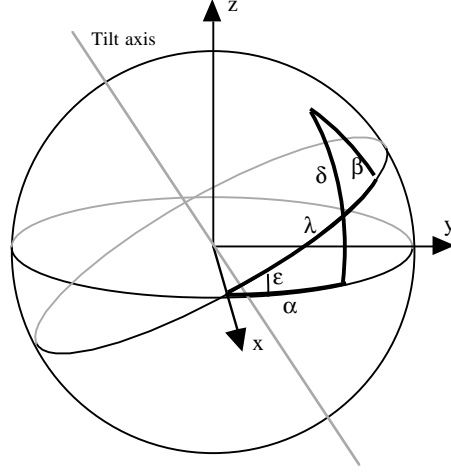


Fig. 1. The coordinate systems

The vector from the source to a hitpoint on the sphere is

$$\mathbf{k} = \mathbf{k}_1 t_1 + \mathbf{k}_2 t_2, \quad |\mathbf{k}|^2 = 1 \quad (1)$$

The hitpoint has equatorial coordinates  $(\alpha, \delta)$  given by the relations

$$\begin{aligned} k_x &= \cos \delta \cos \alpha \\ k_y &= \cos \delta \sin \alpha \\ k_z &= \sin \delta \end{aligned} \quad (2)$$

Assume that the observer sits in  $(0, \delta_0)$ . If we rotate the observer around the z axis, his "latitude" will vary in the interval  $[\beta_{0,max}, \beta_{0,min}]$  where

$$\begin{aligned} \beta_{0,max} &= \min(\delta_0 + \varepsilon, \pi - (\delta_0 + \varepsilon)) \\ \beta_{0,min} &= \max(\delta_0 - \varepsilon, \pi - (\delta_0 - \varepsilon)) \end{aligned} \quad (3)$$

When we vary  $t_1$  and  $t_2$ , the hitpoint will trace out a great circle on the sphere. However, because of the condition  $|\mathbf{k}|^2 = 1$ ,  $t_1$  and  $t_2$  are not independent. We have

$$\begin{aligned}
|\mathbf{k}|^2 &= 1 \\
&= (\mathbf{k}_1 t_1 + \mathbf{k}_2 t_2)^2 \\
&= \mathbf{k}_1^2 t_1^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2 t_1 t_2 + \mathbf{k}_2^2 t_2^2 \\
&= t_1^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2 t_1 t_2 + t_2^2
\end{aligned} \tag{4}$$

which gives

$$t_2 = -at_1 \pm \left(1 - t_1^2(1 - a^2)\right)^{1/2}, \quad a = \mathbf{k}_1 \cdot \mathbf{k}_2 \tag{5}$$

If  $a \geq 0$  we only have the one solution  $t_2 = -at_1 + \left(1 - t_1^2(1 - a^2)\right)^{1/2}$ . If  $a < 0$  we have for  $t_1 > 1$  both solutions in (5). Further we have

$$t_1 \in [0, t_{1,max}], \quad t_{1,max} = \begin{cases} 1 & \text{for } a \geq 0 \\ 1/(1 - a^2)^{1/2} & \text{for } a < 0 \end{cases} \tag{6}$$

In the  $t_1, t_2$  plane the solution traces out part of a tilted ellipse, centered at the origin and passing through the points (0,1) and (1,0). The ellipse degenerates into a straight line for  $a = 1$ .

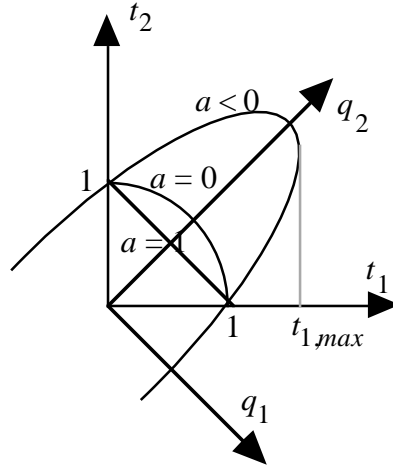


Fig. 2. The q-t-plane

To get a single-valued relation between the variables we introduce two new quantities  $q_1$  and  $q_2$  by

$$q_1 = t_1 - t_2, \quad q_2 = t_1 + t_2 \tag{7}$$

where

$$q_1 \in [-1, 1] \text{ and } q_2 = \left( \frac{2}{1+a} - q_1^2 \frac{1-a}{1+a} \right)^{1/2} \quad (8)$$

The hitpoint has ecliptic coordinates  $(\lambda, \beta)$  given by

$$\begin{aligned} \sin \beta &= k_z \cos \varepsilon - k_y \sin \varepsilon \\ \cos \beta \sin \lambda &= k_z \sin \varepsilon + k_y \cos \varepsilon \\ \cos \beta \cos \lambda &= k_x \end{aligned} \quad (9)$$

Using (1) and (5) we have

$$\mathbf{k} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)q_1 + \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)q_2 \quad (10)$$

The endpoints of the great circle are given by inserting  $\mathbf{t} = (0, 1)$  and  $\mathbf{t} = (t_{1,max}, 0)$  corresponding to  $\mathbf{q} = (-1, 1)$  and  $\mathbf{q} = (1, 1)$  respectively. Then (9) and (10) will give us a starting value,  $\beta_{start}$ , and ending value,  $\beta_{end}$  for the latitude of the great circle. If  $\beta$  is a monotonous function of  $\lambda$ , we know that the great circle latitude is confined in the interval  $[\beta_{start}, \beta_{end}]$ . However, if  $\beta$  is not monotonous, there will be an extremal value of  $\beta$  given by

$$\frac{d\beta}{dq_1} = 0 \quad (11)$$

From (9) we have

$$\begin{aligned} \sin \beta &= \frac{1}{2}[(k_{1z} + k_{2z})\cos \varepsilon - (k_{1y} + k_{2y})\sin \varepsilon]q_2 \\ &\quad + \frac{1}{2}[(k_{1z} - k_{2z})\cos \varepsilon - (k_{1y} - k_{2y})\sin \varepsilon]q_1 \end{aligned} \quad (12)$$

or using (10)

$$\sin \beta = A(C - Dq_1^2)^{1/2} + Bq_1 \quad (13)$$

with

$$\begin{aligned}
A &= \frac{1}{2} [(k_{1z} + k_{2z}) \cos \varepsilon - (k_{1y} + k_{2y}) \sin \varepsilon] \\
B &= \frac{1}{2} [(k_{1z} - k_{2z}) \cos \varepsilon - (k_{1y} - k_{2y}) \sin \varepsilon] \\
C &= \frac{2}{1+a} \\
D &= \frac{1-a}{1+a}
\end{aligned} \tag{14}$$

Then from (11) the extremal value of  $\beta$  is realised by

$$\hat{q}_1 = \pm \left( \frac{C}{D(1 + A^2 D/B^2)} \right)^{1/2} \tag{15}$$

There is an extremal value within the allowed part of the great circle if  $|\hat{q}_1| \leq 1$ .  $\hat{q}_1$  has the same sign as  $ABD$ . From (10) we get the extremal value  $\hat{\beta} = \beta(\hat{q}_1)$ . Then the interval in latitude in which the hitpoint can vary, will be the largest of the intervals  $[\hat{\beta}, \beta_{start}]$  and  $[\hat{\beta}, \beta_{end}]$ .

We now find the intersection set  $\Omega_\beta$  between the intervals of the latitude of the observer and the latitudes of the great circle. This can, in the non-monotonous case, result in two disjoint intervals of  $\beta$ . If the intersection is zero, it is not possible to make the hitpoint coincide with the observer. Finding this intersection and implementing it in program code is a quite tricky task. The intersection region in latitude  $\beta$  will correspond to a certain region,

$$\Omega_q = \{q_1; \beta(q_1) \in \Omega_\beta\}, \tag{16}$$

possibly disjoint, in  $q_1$ . The equation for solving  $q_1$  as a function of  $\beta$  is (12) with solution

$$q_1 = \frac{B \sin \beta}{A^2 D + B^2} \pm \left( \frac{B^2 \sin^2 \beta}{(A^2 C + B^2)^2} + \frac{A^2 C - \sin^2 \beta}{A^2 D + B^2} \right)^{1/2}, \quad |q_1| \leq 1 \tag{17}$$

If there is an extremum latitude within the allowed part of the great circle there will be two solutions otherwise only one. We have to check the validity of the solutions by inserting them back in (13).

We now want to select a scattering crystal anywhere in space along the line

$$\mathbf{r} = \mathbf{k}_1 t_1, \quad t_1 \in [0, t_{1,max}] \quad (18)$$

and such that  $q_1$  belongs to the allowed region  $\Omega_q$ . However, we have a problem because the relation between  $q_1$  and  $t_1$  is not single-valued for  $\mathbf{k}_1 \cdot \mathbf{k}_2 = a < 0$ . If we trace the possible corresponding regions in  $t_1$  we get something quite complicated. We avoid this problem by using a single-valued variable  $t'_1$  such that

$$t'_1 = \begin{cases} t_1 & \text{for } q_1 \in [-1, Q] \\ 2t_{1,max} - t_1 & \text{for } q_1 \in [Q, 1] \end{cases} \quad (19)$$

where

$$Q = q_1(t_{1,max}) = \begin{cases} 1 & \text{for } a \geq 0 \\ \left( \frac{1+a}{1-a} \right)^{1/2} & \text{for } a < 0 \end{cases} \quad (20)$$

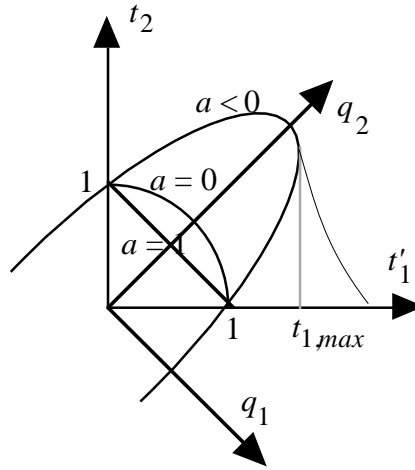


Fig 3. The definition of  $t'_1$

This will "fold out" the region  $t_1 \in [1, t_{1,max}]$  and remove the redundancy. The region  $\Omega_q$  will now map one-to-one on a region

$$\Omega_t = \{t'_1; q_1(t'_1) \in \Omega_q\} \quad (21)$$

We generate a uniform random number  $t'$  in  $\Omega_t$ . Actually the probability should be proportional to the square of the distance of the crystal from the origin, the number of crystals in

a spherical shell being proportional to the square of this distance, but as the illumination of the crystal is inversely proportional to the square of the same distance we can combine the two and generate numbers uniformly which is a great simplification.  $t'$  will correspond to a unique point  $\mathbf{q}$  in the  $q$  plane. This will in turn by (9) and (10) correspond to a unique hitpoint  $(\lambda, \beta)$ .

We now find a rotation  $\alpha$  of the observer around the  $z$  axis and a rotation  $\Lambda$  around the tilt axis such that the hitpoint coincides with the observer. The observer will then have equatorial coordinates  $(\alpha, \delta_0)$  and the hitpoint will have ecliptic coordinates  $(\lambda + \Lambda, \beta)$ . The condition is that these points coincide. Using well-known relations between an equatorial and an ecliptic system<sup>4</sup> we have the equations:

$$\begin{aligned}\sin \beta &= \sin \delta_0 \cos \varepsilon - \cos \delta_0 \sin \alpha \sin \varepsilon \\ \cos \beta \sin(\lambda + \Lambda) &= \sin \delta_0 \sin \varepsilon + \cos \delta_0 \sin \alpha \cos \varepsilon \\ \cos \beta \cos(\lambda + \Lambda) &= \cos \delta_0 \cos \alpha\end{aligned}\tag{22}$$

We can get  $\sin \alpha$  from the first of these equations where it is now possible to choose randomly between with two possible values for the angle,  $\alpha_1$  and  $\alpha_2 = \pi - \alpha_1$ . Inserting the chosen value in the remaining equations gives a corresponding rotation angle  $\Lambda$ .

Finally we perform two rotations on  $\mathbf{k}_2$ , the direction of the scattered ray, first a rotation  $\Lambda$  around the tilt axis, then a rotation  $-\alpha$  around the  $z$  axis. The rotated vector will be the direction of a scattered ray that hits the observer.

### *Intensity*

We have already taken into account the intensity decrease from the source to crystal path. We now also want to take into account the intensity decrease because of the crystal-observer distance. The light coming from the crystal will behave as if it passed a small hole and will thus exhibit diffraction. If the observer is close to the crystal the light beam will be more or less within the limits of the pupil of the observer. Far away only part of the beam will enter the eye. We can make a simple model for this. We assume that the crystal can be approximated by a



circular aperture with diameter  $d$ . The intensity integrated over the pupil area at distance  $D$  from the crystal is then

$$I_{\infty} \int_0^{x_{max}} \left| \frac{J_1(x)}{x} \right|^2 2\pi x \cdot dx \quad (23)$$

where

$$x_{max} = \frac{2\pi}{\lambda} \cdot \frac{d}{2} \cdot \sin \frac{p}{2D} \quad (24)$$

$\lambda$  is the wave-length of the light and  $p$  the pupil diameter.  $J_1(x)$  is a Bessel function of the first kind<sup>5</sup>. It turns out that this integrated intensity can be quite well approximated by the simple expression

$$\frac{\left( \frac{pd}{\lambda D} \right)^2}{\left( \frac{pd}{\lambda D} \right)^2 + 1} \quad (25)$$

where we have normalised the intensity to 1 when  $D = 0$ .

Finally we use that  $D = D_L t_2$ , where  $D_L$  is the (real) distance source-observer and we can write (25) in terms of the parametric distance as

$$\frac{\left( \frac{pd}{\lambda D_L} \right)^2}{\left( \frac{pd}{\lambda D_L} \right)^2 + t_2^2} \quad (26)$$

Given a fixed pupil size and wave-length this expression depends on the parameter  $D_L/d$ . It means that the halo display will change as we approach it, precisely what is observed in the field. If we photograph the halo display with a camera aperture larger than the pupil, this can by (26) be seen to be equivalent of moving physically closer to the halo. Observing a divergent halo display far from the source will enhance the influence of crystals near the eye, the display will be more similar to a parallel light halo.

## *Concluding remarks*

It is interesting that the vector  $-\mathbf{k}_2 t_2$  will be the parametric location in space of the scattering crystal relative to the observer. Saving a file with such vectors makes it possible to reconstruct the halo display in three dimensions using for instance stereo pictures.

In most field situations, the ground will cut off rays coming from crystals that would be located below ground level. This can easily be implemented by excluding rays with  $k_{1z} t_1$  less than some fixed value.

For tilts that are exactly zero the program will not be able to find the rotations as they are then dependent, only their sum can be determined. We can avoid this problem by giving the tilt a very small non-zero value in this case.

We have implemented the described routine together with a raytracing program that can be run on a personal computer (iMac). Preliminary simulation runs show that the program can rapidly and efficiently reproduce field observations of divergent halos with good accuracy. Also the variation of the appearance of the halo display with distance from the source agrees well with our model to describe the intensity. We will present these results in a separate publication. The source code in PASCAL of the computer algorithm is available on the web at <http://www.thep.lu.se/~larsg/>

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