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*Research Article*

# Hierarchical Fixed Point Problems in Uniformly Smooth Banach Spaces

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We propose some relaxed implicit and explicit viscosity approximation methods for hierarchical fixed point problems for a countable family of nonexpansive mappings in uniformly smooth Banach spaces. These relaxed viscosity approximation methods are based on the well-known viscosity approximation method and hybrid steepest-descent method. We obtain some strong convergence theorems under mild conditions.

## 1. Introduction

Let $X$ be a real Banach space and $U$ the unit sphere of $X$; that is, $U = \{x \in X : \|x\| = 1\}$. Recall that $X$ is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in U$; in this case, $X$ is also said to have a Gâteaux differentiable norm. $X$ is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of $X$ is said to be the Fréchet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. In addition, we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of $X$ as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X,\ \|x\| = 1, \|y\| = \tau\right\}. \tag{2}$$

It is known that $X$ is uniformly smooth if and only if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$.

Let $X$ be a real Banach space and let $J$ denote the normalized duality mapping from $X$ to $2^{X^*}$ given by

$$J(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{3}$$

where $X^*$ denotes the dual space of $X$ and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of the mapping $T$. It is well known that if $X$ is smooth, then $J$ is single-valued and norm-to-weak$^*$ continuous, whereas if $X$ is a Banach space with a uniformly Gâteaux differentiable norm, then $J$ is single-valued and norm-to-weak$^*$ uniformly continuous on bounded subsets of $X$. Further, if $X$ is a uniformly smooth Banach space, then $J$ is single-valued and norm-to-norm uniformly continuous on bounded subsets of $X$. In what follows, we still denote by $J$ the single-valued normalized duality mapping.

Let $C$ be a nonempty closed convex subset of $X$. Recall that a mapping $T : C \to C$ is said to be $L$-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{4}$$

In particular, if $L = 1$, then $T$ is said to be nonexpansive; that is,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{5}$$

We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence.

*Definition 1.* Let $A : C \to X$ be a mapping of $C$ into $X$. Then $A$ is said to be

(i) accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge 0, \tag{6}$$

where $J$ is the normalized duality mapping;

(ii) $\alpha$-strongly accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge \alpha\|x - y\|^2, \tag{7}$$

for some $\alpha \in (0, 1)$;

(iii) pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \le \|x - y\|^2; \tag{8}$$

(iv) $\beta$-strongly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \le \beta\|x - y\|^2, \tag{9}$$

for some $\beta \in (0, 1)$;

(v) $\lambda$-strictly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \le \|x - y\|^2 - \lambda\|x - y - (Ax - Ay)\|^2, \tag{10}$$

for some $\lambda \in (0, 1)$.

In a real smooth Banach space $X$ we say that an operator $A$ is strongly positive [1] if there exists a constant $\overline{\gamma} > 0$ with the property

$$\begin{gathered}\langle Ax, J(x)\rangle \ge \overline{\gamma}\|x\|^2, \\ \|aI - bA\| = \sup_{\|x\|\le 1} |\langle (aI - bA)x, J(x)\rangle|, \\ a \in [0, 1], \quad b \in [-1, 1],\end{gathered} \tag{11}$$

where $I$ is the identity mapping.

Recently, the problem of convergence of implicit iterative algorithms to a common fixed point for a family of nonexpansive mappings and its extensions to Hilbert spaces or Banach spaces have been considered by many authors; see [1–9] and the references therein.

Yao et al. [10] introduced the following Halpern-type implicit iterative algorithm,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad \forall n \ge 1, \tag{12}$$

and proved a strong convergence theorem under suitable conditions.

On the other hand, let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $A : C \to H$ be a nonlinear mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0, \quad \forall x \in C. \tag{13}$$

If we assume that $C$ is the fixed point set of a nonexpansive mapping $T$ and $S$ is another nonexpansive mapping (not necessarily with fixed points), the problem (13) becomes the VIP of finding $x^* \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^*\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T), \tag{14}$$

introduced first by Moudafi and Maingé in [11], which is called hierarchical fixed point problem.

In particular, whenever $\mathrm{Fix}(S) \ne \emptyset$, all elements of $\mathrm{Fix}(S)$ are solutions of VIP (14). If $S$ is a $\rho$-contraction (i.e., $\|Sx - Sy\| \le \rho\|x - y\|$ for some $\rho \in (0, 1)$), the set of solutions of VIP (14) is a singleton and it is well known as a viscosity problem, which was first introduced by Moudafi [12] and then developed by several authors [13, 14].

Very recently, Cai and Bu [1] investigated a general hierarchical fixed point problem for a countable family of continuous pseudocontractions, which covers as a special case of the problem considered in [10]. For this purpose, they first established strong convergence of an implicit iterative scheme for solving a hierarchical fixed point problem for a continuous pseudocontractive mapping in a uniformly smooth Banach space.

In this paper, let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $X$ such that $C \pm C \subset C$. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$ and let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. Let $F : C \to C$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$ and let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator. First of all, we introduce a relaxed implicit viscosity scheme for solving a hierarchical fixed point problem for a nonexpansive mapping $T$:

$$x_t = (I - \theta_t F) T x_t + \theta_t [f(x_t) - t(Af(x_t) - T x_t)], \tag{15}$$

where $\lim_{t\to 0}\theta_t = 0$. It is proven that as $t \to 0, \{x_t\}$ converges strongly to a point $z \in \mathrm{Fix}(T)$, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP:

$$\langle (F - f)z, J(z - p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \tag{16}$$

On the other hand, let $\{T_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive mappings from $C$ to itself such that $\Omega =$

$\bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. We propose a relaxed implicit viscosity iterative algorithm for solving a hierarchical fixed point problem for a countable family of nonexpansive mappings $\{T_n\}$:

$$y_n = \alpha_n f(y_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n y_n,$$
$$x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n A) T_n y_n, \quad \forall n \geq 0, \tag{17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, and $\{\sigma_n\}$ are four sequences in $(0,1)$. It is proven that under mild conditions $\{x_n\}$ converges strongly to a point $z \in \Omega$, which is the unique solution in $\Omega$ to the VIP:

$$\langle (A - f) z, J(z - p) \rangle \leq 0, \quad \forall p \in \Omega. \tag{18}$$

Furthermore, we also propose a relaxed explicit viscosity iterative algorithm for solving another hierarchical fixed point problem for a countable family of nonexpansive mappings $\{T_n\}$:

$$x_0 \in C \quad \text{chosen arbitrarily,}$$
$$x_{n+1} = (I - \beta_n F) T_n x_n + \beta_n [f(x_n) - \alpha_n (A f(x_n) - T_n x_n)], \quad \forall n \geq 0, \tag{19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. It is proven that under appropriate assumptions, $\{x_n\}$ converges strongly to a point $z \in \Omega$, which is the unique solution in $\Omega$ to the VIP:

$$\langle (F - f) z, J(z - p) \rangle \leq 0, \quad \forall p \in \Omega. \tag{20}$$

The above relaxed viscosity algorithms are based on the well-known viscosity approximation method (see, e.g., [4–6, 9]) and hybrid steepest-descent method (see, e.g., [14–17]). Our results extend, improve, supplement, and develop the recent results announced by many authors.

## 2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 2 can be found in [18]. Lemma 3 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

**Lemma 2.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{21}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;

(iii) $\gamma_n \geq 0$ $(\forall n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Then* $\limsup_{n\to\infty} s_n = 0$.

**Lemma 3.** *In a smooth Banach space $X$, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x + y) \rangle, \quad \forall x, y \in X. \tag{22}$$

Let LIM be a continuous linear functional on $l^{\infty}$ and $(a_0, a_1, \ldots) \in l^{\infty}$. We write $\mathrm{LIM}\, a_n$ instead of $\mathrm{LIM}((a_0, a_1, \ldots))$. LIM is said to be Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}\, 1 = 1$ and $\mathrm{LIM}\, a_{n+1} = \mathrm{LIM}\, a_n$ for all $(a_0, a_1, \ldots) \in l^{\infty}$. It is well known that for Banach limit LIM the following holds:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies that $\mathrm{LIM}\, a_n \leq \mathrm{LIM}\, c_n$;

(ii) $\mathrm{LIM}\, a_{n+N} = \mathrm{LIM}\, a_n$ for any fixed positive integer $N$;

(iii) $\liminf_{n\to\infty} a_n \leq \mathrm{LIM}\, a_n \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^{\infty}$.

It is easy to see that there holds the following conclusion.

**Lemma 4** (see [19]). *Let $(a_0, a_1, \ldots) \in l^{\infty}$. If $\mathrm{LIM}\, a_n = 0$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$.*

Recall that a Banach space $X$ is said to satisfy Opial's condition, if whenever $\{x_n\}$ is a sequence in $X$ which converges weakly to $x$ as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x. \tag{23}$$

**Lemma 5** (Demiclosedness principle; see [20, Theorem 10.3]). *Let $X$ be a reflexive Banach space satisfying Opial's condition, $C$ a nonempty closed convex subset of $X$, and $T : C \to C$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on $C$, where $I$ is the identity mapping; that is, if $\{x_n\}$ is a sequence of $C$ such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$.*

The following lemma can be derived by the standard argument and hence its proof will be omitted.

**Lemma 6.** *Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$ and let $F : C \to X$ be a mapping.*

(i) *If $F : C \to X$ is $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \geq 1$, then $I - F$ nonexpansive and $F$ is Lipschitz continuous with constant $1 + 1/\lambda$;*

(ii) *If $F : C \to X$ is $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$, then for any fixed $\tau \in (0,1)$, $I - \tau F$ is contractive with coefficient $1 - \tau(1 - \sqrt{(1-\alpha)/\lambda})$.*

## 3. Relaxed Implicit Viscosity Scheme for Hierarchical Fixed Point Problem for a Nonexpansive Mapping

In this section, we introduce our relaxed implicit viscosity scheme for solving hierarchical fixed point problem for a nonexpansive mapping and show the strong convergence theorem. First, we list several useful and helpful lemmas.

**Lemma 7** (see [21]). *Let $X$ be a Banach space, $C$ a nonempty closed and convex subset of $X$, and $T : C \to C$ a continuous* *and strong pseudocontraction. Then $T$ has a unique fixed point in $C$.*

**Lemma 8** (see [19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property $a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\beta_n$, $\forall n \ge 0$, where $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbf{R}$ such that* (i) $\sum_{n=0}^{\infty} \gamma_n = \infty$ *and* (ii) $\limsup_{n\to\infty}\beta_n \le 0$. *Then $\{a_n\}$ converges to zero as $n \to \infty$.*

**Lemma 9** (see [22]). *Let $C$ be a nonempty closed convex subset of a real Banach space $X$ and $T : C \to C$ a continuous pseudocontractive map. We denote $B = (2I-T)^{-1}$. Then the following holds.*

(i) *The map $B$ is a nonexpansive self-mapping on $C$.*

(ii) *If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $\lim_{n\to\infty}\|x_n - Bx_n\| = 0$.*

**Lemma 10** (see [23]). *Assume that $A$ is a strongly positive linear bounded operator on a smooth Banach space $X$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

We now state and prove our first result.

**Theorem 11.** *Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $X$ such that $C \pm C \subset C$. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$ and $F : C \to C$ $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$. Let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, \gamma_0)$, $\gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$. Let $\{x_t\}$ be defined by*

$$x_t = (I - \theta_t F) T x_t + \theta_t \left[ f(x_t) - t\left(Af(x_t) - Tx_t\right)\right], \quad (24)$$

*where $\lim_{t\to 0}\theta_t = 0$. Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point $z$ of $T$, which is the unique solution in $\mathrm{Fix}(T)$ to the VIP:*

$$\langle (F - f) z, J(z - p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \quad (25)$$

*Proof.* First, we claim that $\gamma_0 < \alpha$. Indeed, it is known that strongly accretive constant $\alpha \in (0,1)$ and strictly pseudocontractive constant $\lambda \in (0,1)$. Moreover, observe that

$$\sqrt{(1-\alpha)\lambda} < 1 \Longleftrightarrow 1 - \alpha < \sqrt{\frac{1-\alpha}{\lambda}} \Longleftrightarrow \gamma_0 < \alpha. \quad (26)$$

Let us show that the net $\{x_t\}$ is defined well. As a matter of fact, we define the mapping $S_t : C \to C$ as follows:

$$S_t x = (I - \theta_t F) Tx + \theta_t \left[ f(x) - t\left(Af(x) - Tx\right)\right], \quad \forall x \in C. \quad (27)$$

Since $\lim_{t\to 0}\theta_t = 0$, we may assume, without loss of generality, that $\theta_t \in (0,1)$ for all $t \in (0, \epsilon_0)$, where $\epsilon_0 = \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}$. Utilizing Lemmas 6 and 10, we obtain that for each $t \in (0, \epsilon_0)$

$$\begin{aligned}
&\langle S_t x - S_t y, J(x-y)\rangle \\
&\quad = \langle (I - \theta_t F) Tx - (I - \theta_t F) Ty, J(x-y)\rangle \\
&\qquad + \theta_t \langle (I - tA) f(x) - (I - tA) f(y), J(x-y)\rangle \\
&\qquad + \theta_t t \langle Tx - Ty, J(x-y)\rangle \\
&\quad \le \|(I - \theta_t F) Tx - (I - \theta_t F) Ty\| \|x - y\| \\
&\qquad + \theta_t \|(I - tA) f(x) - (I - tA) f(y)\| \|x - y\| \\
&\qquad + \theta_t t \|Tx - Ty\| \|x - y\| \\
&\quad \le \left(1 - \theta_t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \|Tx - Ty\| \|x - y\| \\
&\qquad + \theta_t (1 - t\overline{\gamma}) \|f(x) - f(y)\| \|x - y\| + t\theta_t \|x - y\|^2 \\
&\quad \le (1 - \theta_t\gamma_0) \|x - y\|^2 + \theta_t (1 - t\overline{\gamma}) \beta \|x - y\|^2 \\
&\qquad + t\theta_t \|x - y\|^2 \\
&\quad = \left[1 - \theta_t\left(\gamma_0 - \beta - t\left(1 - \overline{\gamma}\beta\right)\right)\right] \|x - y\|^2 \\
&\quad \le \left[1 - \theta_t\left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right] \|x - y\|^2 \\
&\quad = \left(1 - \frac{1}{2}\theta_t(\gamma_0 - \beta)\right) \|x - y\|^2.
\end{aligned} \quad (28)$$

It follows that for each $t \in (0, \epsilon_0)$, $S_t : C \to C$ is a continuous and strongly pseudocontractive mapping with pseudocontractive coefficient $1 - (1/2)\theta_t(\gamma_0 - \beta)$. Hence, by Lemma 7 we know that there exists a unique fixed point in $C$, denoted by $x_t$, which uniquely solves the fixed point equation:

$$x_t = (I - \theta_t F) T x_t + \theta_t \left[ f(x_t) - t\left(Af(x_t) - Tx_t\right)\right]. \quad (29)$$

Let us show the uniqueness of the solution of VIP (25). Suppose both $z_1 \in \mathrm{Fix}(T)$ and $z_2 \in \mathrm{Fix}(T)$ are solutions to VIP (25). Then we have

$$\begin{aligned}
&\langle (F - f) z_1, J(z_1 - z_2)\rangle \le 0, \\
&\langle (F - f) z_2, J(z_2 - z_1)\rangle \le 0.
\end{aligned} \quad (30)$$

Adding up the above two inequalities, we obtain

$$\langle (F - f) z_1 - (F - f) z_2, J(z_1 - z_2)\rangle \le 0. \quad (31)$$

Note that

$$\begin{aligned}
&\langle (F-f)z_1-(F-f)z_2, J(z_1-z_2)\rangle \\
&= \langle Fz_1-Fz_2, J(z_1-z_2)\rangle \\
&\quad -\langle f(z_1)-f(z_2), J(z_1-z_2)\rangle \\
&\geq \alpha\|z_1-z_2\|^2-\beta\|z_1-z_2\|^2 \\
&= (\alpha-\beta)\|z_1-z_2\|^2 \geq 0.
\end{aligned} \tag{32}$$

Taking into account $\alpha-\beta>0$, we have $z_1=z_2$, and hence the uniqueness is proved. We use $\tilde{z}$ to denote the unique solution of VIP (25).

Next, we prove that $\{x_t : t\in(0,\epsilon_0)\}$ is bounded. Indeed, we note that $0<\theta_t<1, \forall t\in(0,\epsilon_0)$. Take a fixed $p\in \mathrm{Fix}(T)$ arbitrarily. Utilizing Lemma 10 we deduce that for all $t\in(0,\epsilon_0)$

$$\begin{aligned}
&\|x_t-p\|^2 \\
&= \langle (I-\theta_tF)Tx_t+\theta_t[f(x_t)-t(Af(x_t)-Tx_t)] \\
&\quad -p, J(x_t-p)\rangle \\
&= \langle (I-\theta_tF)Tx_t-(I-\theta_tF)Tp, J(x_t-p)\rangle \\
&\quad +\theta_t\langle (I-tA)f(x_t)-(I-tA)f(p), J(x_t-p)\rangle \\
&\quad +\theta_t t\langle Tx_t-p, J(x_t-p)\rangle-\theta_t\langle (F-f)p, J(x_t-p)\rangle \\
&\quad +\theta_t t\langle (I-Af)p, J(x_t-p)\rangle \\
&\leq \|(I-\theta_tF)Tx_t-(I-\theta_tF)Tp\|\|x_t-p\| \\
&\quad +\theta_t\|(I-tA)f(x_t)-(I-tA)f(p)\|\|x_t-p\| \\
&\quad +\theta_t t\|Tx_t-p\|\|x_t-p\|+\theta_t\|(F-f)p\|\|x_t-p\| \\
&\quad +\theta_t t\|(I-Af)p\|\|x_t-p\| \\
&\leq \left(1-\theta_t\left(1-\sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|Tx_t-Tp\|\|x_t-p\| \\
&\quad +\theta_t(1-t\overline{\gamma})\|f(x_t)-f(p)\|\|x_t-p\| \\
&\quad +\theta_t t\|Tx_t-p\|\|x_t-p\|+\theta_t\|(F-f)p\|\|x_t-p\| \\
&\quad +\theta_t t\|(I-Af)p\|\|x_t-p\| \\
&\leq (1-\theta_t\gamma_0)\|x_t-p\|^2+\theta_t(1-t\overline{\gamma})\beta\|x_t-p\|^2 \\
&\quad +\theta_t t\|x_t-p\|^2+\theta_t\|(F-f)p\|\|x_t-p\| \\
&\quad +\theta_t t\|(I-Af)p\|\|x_t-p\| \\
&= [1-\theta_t(\gamma_0-\beta-t(1-\overline{\gamma}\beta))]\|x_t-p\|^2 \\
&\quad +\theta_t\|(F-f)p\|\|x_t-p\|+\theta_t t\|(I-Af)p\|\|x_t-p\| \\
&\leq \left[1-\theta_t\left(\gamma_0-\beta-\frac{\gamma_0-\beta}{2(1-\overline{\gamma}\beta)}(1-\overline{\gamma}\beta)\right)\right] \\
&\quad \times\|x_t-p\|^2+\theta_t\|(F-f)p\|\|x_t-p\| \\
&\quad +\theta_t t\|(I-Af)p\|\|x_t-p\| \\
&= \left(1-\frac{1}{2}\theta_t(\gamma_0-\beta)\right)\|x_t-p\|^2+\theta_t\|(F-f)p\|\|x_t-p\| \\
&\quad +\theta_t t\|(I-Af)p\|\|x_t-p\|,
\end{aligned} \tag{33}$$

which immediately yields

$$\begin{aligned}
\|x_t-p\| &\leq \frac{2}{\gamma_0-\beta}(\|(F-f)p\|+t\|(I-Af)p\|) \\
&\leq \frac{2}{\gamma_0-\beta}\left(\|(F-f)p\|+\|A\|^{-1}\|(I-Af)p\|\right).
\end{aligned} \tag{34}$$

Thus $\{x_t : t\in(0,\epsilon_0)\}$ is bounded.

Assume that $\{t_n\}\subset(0,\epsilon_0)$ and $t_n\to 0$ as $n\to\infty$. Set $\theta_n=\theta_{t_n}$ and $x_n := x_{t_n}$, and define $\mu : C\to \mathbf{R}$ by $\mu(x)=\mathrm{LIM}\|x_n-x\|^2, \forall x\in C$, where LIM is a Banach limit on $l^\infty$. Let

$$K=\left\{x\in C:\mu(x)=\min_{y\in C}\mathrm{LIM}\|x_n-y\|^2\right\}. \tag{35}$$

We see easily that $K$ is a nonempty closed convex subset of $X$. Note that $\|x_n-Tx_n\|=\theta_n\|f(x_n)-t_n(Af(x_n)-Tx_n)-FTx_n\|\to 0$ as $n\to\infty$. In terms of Lemma 9, we know that the mapping $B=(2I-T)^{-1} : C\to C$ is nonexpansive and $\mathrm{Fix}(T)=\mathrm{Fix}(B)$ and $\lim_{n\to\infty}\|x_n-Bx_n\|=0$, where $I$ denotes the identity operator. It follows that

$$\begin{aligned}
\mu(Bx) &= \mathrm{LIM}\|x_n-Bx\|^2=\mathrm{LIM}\|Bx_n-Bx\|^2 \\
&\leq \mathrm{LIM}\|x_n-x\|^2=\mu(x),
\end{aligned} \tag{36}$$

which implies that $B(K)\subset K$; that is, $K$ is invariant under $B$. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mapping, $B$ has a fixed point, say $z\in K$. Since $z$ is also a minimizer of $\mu$ over $C$, we have that, for $t\in(0,\epsilon_0)$ and $x\in C$,

$$\begin{aligned}
0 &\leq \frac{\mu(z+t(x-Fz))-\mu(z)}{t} \\
&= \mathrm{LIM}\frac{\|x_n-z+t(Fz-x)\|^2-\|x_n-z\|^2}{t} \\
&= \mathrm{LIM}\Big(\big(\langle x_n-z, J(x_n-z+t(Fz-x))\rangle \\
&\quad +t\langle Fz-x, J(x_n-z+t(Fz-x))\rangle \\
&\quad -\|x_n-z\|^2\big)t^{-1}\Big).
\end{aligned} \tag{37}$$

Since $X$ is uniformly smooth, we conclude that the duality mapping $J$ is norm-to-norm uniformly continuous on any bounded subset of $X$. Letting $t \to 0$, we find that the two limits above can be interchanged and obtain

$$\mathrm{LIM}\left\langle x - Fz, J\left(x_n - z\right)\right\rangle \leq 0, \quad \forall x \in C. \tag{38}$$

On the other hand, we have

$$\begin{aligned} & x_n - z \\ & = \left(I - \theta_n F\right) T x_n - \left(I - \theta_n F\right) T z \\ & \quad + \theta_n\left[\left(I - t_n A\right) f\left(x_n\right) - \left(I - t_n A\right) f(z) + t_n\left(T x_n - z\right)\right] \\ & \quad + \theta_n (f - F) z + \theta_n t_n (I - A f) z. \end{aligned} \tag{39}$$

It follows that

$$\begin{aligned} & \left\|x_n - z\right\|^2 \\ & \quad = \left\langle\left(I - \theta_n F\right) T x_n - \left(I - \theta_n F\right) T z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n\left[\left\langle\left(I - t_n A\right)\left(f\left(x_n\right) - f(z)\right), J\left(x_n - z\right)\right\rangle\right. \\ & \qquad \qquad \left. + t_n\left\langle T x_n - z, J\left(x_n - z\right)\right\rangle\right] \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\left\langle(I - A f) z, J\left(x_n - z\right)\right\rangle \\ & \quad \leq \left(1 - \theta_n \gamma_0\right)\left\|T x_n - T z\right\|\left\|x_n - z\right\| \\ & \qquad + \theta_n\left[\left(1 - t_n \bar{\gamma}\right)\left\|f\left(x_n\right) - f(z)\right\|\left\|x_n - z\right\|\right. \\ & \qquad \qquad \left. + t_n\left\|T x_n - z\right\|\left\|x_n - z\right\|\right] \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\|(I - A f) z\|\left\|x_n - z\right\| \\ & \quad \leq \left(1 - \theta_n \gamma_0\right)\left\|x_n - z\right\|^2 \\ & \qquad + \theta_n\left[\left(1 - t_n \bar{\gamma}\right) \beta\left\|x_n - z\right\|^2 + t_n\left\|x_n - z\right\|^2\right] \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\|(I - A f) z\|\left\|x_n - z\right\| \\ & \quad = \left[1 - \theta_n\left(\gamma_0 - \beta - t_n(1 - \bar{\gamma} \beta)\right)\right]\left\|x_n - z\right\|^2 \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\|(I - A f) z\|\left\|x_n - z\right\| \\ & \quad \leq \left[1 - \theta_n\left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \bar{\gamma} \beta)}(1 - \bar{\gamma} \beta)\right)\right]\left\|x_n - z\right\|^2 \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\|(I - A f) z\|\left\|x_n - z\right\| \\ & \quad = \left(1 - \frac{1}{2} \theta_n\left(\gamma_0 - \beta\right)\right)\left\|x_n - z\right\|^2 \\ & \qquad + \theta_n\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \\ & \qquad + \theta_n t_n\|(I - A f) z\|\left\|x_n - z\right\|. \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} & \left\|x_n - z\right\|^2 \\ & \quad \leq \frac{2}{\gamma_0 - \beta} \\ & \qquad \times\left(\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle + t_n\|(I - A f) z\|\left\|x_n - z\right\|\right). \end{aligned} \tag{41}$$

Combining (38) and (41), we get

$$\mathrm{LIM}\left\|x_n - z\right\|^2 \leq \frac{2}{\gamma_0 - \beta} \mathrm{LIM}\left\langle(f - F) z, J\left(x_n - z\right)\right\rangle \leq 0, \tag{42}$$

which leads to $\mathrm{LIM}\|x_n - z\|^2 = 0$. Hence there exists a subsequence which is still denoted as $\{x_n\}$ such that $x_n \to z$ as $n \to \infty$.

Next, we prove that $z$ solves VIP (25). Since

$$x_t = \left(I - \theta_t F\right) T x_t + \theta_t\left[f\left(x_t\right) - t\left(A f\left(x_t\right) - T x_t\right)\right], \tag{43}$$

we can deduce that

$$x_t - T x_t = \theta_t\left(f\left(x_t\right) - F T x_t\right) + \theta_t t\left(T x_t - A f\left(x_t\right)\right). \tag{44}$$

Since $T$ is nonexpansive, $I - T$ is accretive. So, from the accretivity of $I - T$, it follows that, for any fixed $p \in \mathrm{Fix}(T)$,

$$\begin{aligned} 0 & \leq \left\langle(I - T) x_t - (I - T) p, J\left(x_t - p\right)\right\rangle \\ & = \left\langle(I - T) x_t, J\left(x_t - p\right)\right\rangle \\ & = \theta_t\left\langle f\left(x_t\right) - F T x_t, J\left(x_t - p\right)\right\rangle \\ & \quad + \theta_t t\left\langle T x_t - A f\left(x_t\right), J\left(x_t - p\right)\right\rangle \\ & = \theta_t\left\langle(f - F) x_t, J\left(x_t - p\right)\right\rangle \\ & \quad + \theta_t\left\langle F x_t - F T x_t, J\left(x_t - p\right)\right\rangle \\ & \quad + \theta_t t\left\langle T x_t - A f\left(x_t\right), J\left(x_t - p\right)\right\rangle. \end{aligned} \tag{45}$$

This implies that

$$\begin{aligned} & \left\langle(F - f) x_t, J\left(x_t - p\right)\right\rangle \\ & \quad \leq \left\langle F x_t - F T x_t, J\left(x_t - p\right)\right\rangle + t\left\langle T x_t - A f\left(x_t\right), J\left(x_t - p\right)\right\rangle. \end{aligned} \tag{46}$$

Now replacing $t$ with $t_n$, letting $n \to \infty$, and noticing the boundedness of $\{Tx_{t_n} - Af(x_{t_n})\}$ and the fact that $Fx_{t_n} - FTx_{t_n} \to Fz - FTz = 0$ for $z \in \text{Fix}(T)$, we have that

$$\langle (F - f) z, J(z - p)\rangle \leq 0, \quad \forall p \in \text{Fix}(T). \tag{47}$$

That is, $z \in \text{Fix}(T)$ is a solution of VIP (25). Then $z = \tilde{z}$. In summary, we infer that each cluster point of $\{x_n\}$ is equal to $z$ as $t_n \to 0$. This completes the proof. □

## 4. Relaxed Viscosity Algorithms for Hierarchical Fixed Point Problems for a Countable Family of Nonexpansive Mappings

In this section, we propose relaxed implicit and explicit viscosity algorithms for solving hierarchical fixed point problems for a countable family of nonexpansive mappings and show strong convergence theorems. For this purpose, we will use the following lemmas in the sequel.

**Lemma 12** (see [24]). *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $T_1, T_2, \ldots$ be a sequence of mappings of $C$ into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of $C$. Moreover, let $T$ be a mapping of $C$ into itself defined by $Ty = \lim_{n\to\infty} T_n y$, for all $y \in C$. Then* $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.

**Lemma 13** (see [1, Lemma 2.6]). *Let $C$ be a nonempty closed convex subset of a real Banach space $X$ which has uniformly Gateaux differentiable norm. Let $T : C \to C$ be a continuous pseudocontractive mapping with* $\text{Fix}(T) \neq \emptyset$ *and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0, 1)$ and Lipschitzian constant $L > 0$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $C \pm C \subset C$ and that $\{x_t\}$ converges strongly to $z \in$* $\text{Fix}(T)$ *as $t \to 0$, where $x_t$ is defined by $x_t = tf(x_t) + (I - tA)Tx_t$. Suppose that $\{x_n\} \subset C$ is bounded and that* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. *Then* $\limsup_{n\to\infty}\langle (f - A)z, J(x_n - z)\rangle \leq 0$.

**Theorem 14.** *Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $X$ such that $C \pm C \subset C$. Let $\{T_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings from $C$ to itself such that $\Omega = \bigcap_{i=0}^{\infty}$* $\text{Fix}(T_i) \neq \emptyset$. *Let $F : C \to C$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma} \in (\beta, 1 + \beta)$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned} y_n &= \alpha_n f(y_n) + \beta_n x_n \\ &\quad + ((1 - \beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n y_n, \end{aligned} \tag{48}$$

$$x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n A) T_n y_n, \quad \forall n \geq 0,$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\epsilon_n\}$, and $\{\sigma_n\}$ are four sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$;

(ii) $\lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0, \limsup_{n\to\infty}(\sigma_n/\alpha_n) < \infty$, *and* $\sum_{n=0}^{\infty}(\alpha_n/(\alpha_n + \beta_n)) = \infty$.

*Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|T_{n+1}x - T_n x\| < \infty$ for any bounded subset $D$ of $C$, let $T$ be a mapping of $C$ into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, and suppose that* $\text{Fix}(T) = \bigcap_{i=0}^{\infty} \text{Fix}(T_i)$. *Then, $\{x_n\}$ converges strongly to a point $z$ of $\Omega$ such that $z$ is a unique solution in $\Omega$ to the VIP:*

$$\langle (f - A) z, J(p - z)\rangle \leq 0, \quad \forall p \in \Omega. \tag{49}$$

*Proof.* By condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since $A$ is a $\overline{\gamma}$-strongly positive linear bounded operator on $C$, from (11) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in C, \|u\| = 1\}. \tag{50}$$

Observe that

$$\begin{aligned} \langle ((1 - \beta_n) I - \alpha_n A) u, J(u)\rangle &= 1 - \beta_n - \alpha_n \langle Au, J(u)\rangle \\ &\geq 1 - \beta_n - \alpha_n \|A\| \\ &\geq 0. \end{aligned} \tag{51}$$

It follows that

$$\begin{aligned} &\|(1 - \beta_n) I - \alpha_n A\| \\ &\quad = \sup\{\langle ((1 - \beta_n) I - \alpha_n A) u, J(u)\rangle : u \in C, \|u\| = 1\} \\ &\quad = \sup\{1 - \beta_n - \alpha_n \langle Au, J(u)\rangle : u \in C, \|u\| = 1\} \\ &\quad \leq 1 - \beta_n - \alpha_n\overline{\gamma}. \end{aligned} \tag{52}$$

Next, we show that $\{y_n\}$ is well defined. For each $n \geq 0$, define a mapping $S_n : C \to C$ by

$$S_n x = \alpha_n f(x) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A)(I - \epsilon_n F) T_n x, \quad \forall x \in C. \tag{53}$$

For every $x, y \in C$, we have

$$\begin{aligned} &\langle S_n x - S_n y, J(x - y)\rangle \\ &\quad = \alpha_n \langle f(x) - f(y), J(x - y)\rangle \\ &\qquad + \langle ((1 - \beta_n) I - \alpha_n A) \\ &\qquad\quad \times ((I - \epsilon_n F) T_n x - (I - \epsilon_n F) T_n y), J(x - y)\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\beta\|x-y\|^2+(1-\beta_n-\alpha_n\overline{\gamma})\\
&\quad\times\|(I-\epsilon_nF)T_nx-(I-\epsilon_nF)T_ny\|\|x-y\|\\
&\leq \alpha_n\beta\|x-y\|^2\\
&\quad+(1-\beta_n-\alpha_n\overline{\gamma})(1-\epsilon_n\gamma_0)\|T_nx-T_ny\|\|x-y\|\\
&\leq \alpha_n\beta\|x-y\|^2+(1-\beta_n-\alpha_n\overline{\gamma})\|x-y\|^2\\
&=[1-\beta_n-\alpha_n(\overline{\gamma}-\beta)]\|x-y\|^2,
\end{aligned} \tag{54}$$

where $\gamma_0 = 1-\sqrt{(1-\alpha)/\lambda}$. Therefore, $S_n$ is a continuous strong pseudocontraction for each $n \geq 0$. By Lemma 7, we see that there exists a unique fixed point $y_n$ for each $n \geq 0$ such that

$$y_n=\alpha_nf(y_n)+\beta_nx_n+((I-\beta_n)I-\alpha_nA)(I-\epsilon_nF)T_ny_n. \tag{55}$$

That is, the sequence $\{y_n\}$ is well defined. Next, we prove that $\{x_n\}$ is bounded. Take a fixed $p \in \Omega$ arbitrarily. Taking into account $\lim_{n\to\infty}(\epsilon_n/\alpha_n)=0$, we may assume that there exists a constant $\tau \in (0,1)$ such that $\epsilon_n \leq \tau\alpha_n$ for all $n \geq 0$. Then we have

$$\begin{aligned}
&\|y_n-p\|^2\\
&=\alpha_n\langle f(y_n)-Ap,J(y_n-p)\rangle\\
&\quad+\beta_n\langle x_n-p,J(y_n-p)\rangle\\
&\quad+\langle((1-\beta_n)I-\alpha_nA)((I-\epsilon_nF)T_ny_n-(I-\epsilon_nF)p),\\
&\qquad J(y_n-p)\rangle\\
&\quad-\epsilon_n\langle((1-\beta_n)I-\alpha_nA)Fp,J(y_n-p)\rangle\\
&\leq\alpha_n\langle f(y_n)-f(p),J(y_n-p)\rangle\\
&\quad+\alpha_n\langle f(p)-Ap,J(y_n-p)\rangle\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|\\
&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|(I-\epsilon_nF)T_ny_n-(I-\epsilon_nF)p\|\\
&\quad\times\|y_n-p\|+\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|Fp\|\|y_n-p\|\\
&\leq\alpha_n\beta\|y_n-p\|^2+\alpha_n\langle f(p)-Ap,J(y_n-p)\rangle\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|\\
&\quad+(1-\beta_n-\alpha_n\overline{\gamma})(1-\epsilon_n\gamma_0)\|T_ny_n-p\|\|y_n-p\|\\
&\quad+\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|Fp\|\|y_n-p\|\\
&\leq\alpha_n\beta\|y_n-p\|^2+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-p\|^2\\
&\quad+\alpha_n\langle f(p)-Ap,J(y_n-p)\rangle\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|+\epsilon_n\|Fp\|\|y_n-p\|\\
&=(1-\beta_n-\alpha_n(\overline{\gamma}-\beta))\|y_n-p\|^2\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|\\
&\quad+\alpha_n\|f(p)-Ap\|\|y_n-p\|+\epsilon_n\|Fp\|\|y_n-p\|\\
&\leq(1-\beta_n-\alpha_n(\overline{\gamma}-\beta))\|y_n-p\|^2\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|\\
&\quad+(\alpha_n+\epsilon_n)(\|f(p)-Ap\|+\|Fp\|)\|y_n-p\|\\
&\leq(1-\beta_n-\alpha_n(\overline{\gamma}-\beta))\|y_n-p\|^2\\
&\quad+\beta_n\|x_n-p\|\|y_n-p\|\\
&\quad+\alpha_n(1+\tau)(\|f(p)-Ap\|+\|Fp\|)\|y_n-p\|,
\end{aligned} \tag{56}$$

which implies that

$$\begin{aligned}
\|y_n-p\|\leq&\frac{\beta_n}{\beta_n+\alpha_n(\overline{\gamma}-\beta)}\|x_n-p\|\\
&+\frac{\alpha_n(\overline{\gamma}-\beta)}{\beta_n+\alpha_n(\overline{\gamma}-\beta)}\\
&\cdot\frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}.
\end{aligned} \tag{57}$$

Therefore, we have

$$\begin{aligned}
&\|x_{n+1}-p\|\\
&=\|\sigma_nf(y_n)+(I-\sigma_nA)T_ny_n-p\|\\
&=\|\sigma_n(f(y_n)-f(p))+(I-\sigma_nA)T_ny_n\\
&\qquad-(I-\sigma_nA)T_np+\sigma_n(f(p)-Ap)\|\\
&\leq\sigma_n\|f(y_n)-f(p)\|\\
&\quad+\|(I-\sigma_nA)(T_ny_n-T_np)\|+\sigma_n\|f(p)-Ap\|\\
&\leq\sigma_n\beta\|y_n-p\|+(1-\sigma_n\overline{\gamma})\|y_n-p\|+\sigma_n\|f(p)-Ap\|\\
&=(1-\sigma_n(\overline{\gamma}-\beta))\|y_n-p\|+\sigma_n\|f(p)-Ap\|\\
&\leq(1-\sigma_n(\overline{\gamma}-\beta))\\
&\quad\times\left[\frac{\beta_n}{\beta_n+\alpha_n(\overline{\gamma}-\beta)}\|x_n-p\|+\frac{\alpha_n(\overline{\gamma}-\beta)}{\beta_n+\alpha_n(\overline{\gamma}-\beta)}\right.\\
&\qquad\left.\cdot\frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_n \|f(p)-Ap\| \\
&\leq (1-\sigma_n(\overline{\gamma}-\beta)) \\
&\quad \times \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\} \\
&\quad +\sigma_n \|f(p)-Ap\| \\
&= (1-\sigma_n(\overline{\gamma}-\beta)) \\
&\quad \times \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\} \\
&\quad +\sigma_n(\overline{\gamma}-\beta)\frac{\|f(p)-Ap\|}{\overline{\gamma}-\beta} \\
&\leq \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}, \frac{\|f(p)-Ap\|}{\overline{\gamma}-\beta}\right\} \\
&\leq \max\left\{\|x_n-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\}.
\end{aligned} \tag{58}
$$

By induction, we get

$$
\begin{aligned}
&\|x_n-p\| \\
&\quad \leq \max\left\{\|x_0-p\|, \frac{(1+\tau)(\|f(p)-Ap\|+\|Fp\|)}{\overline{\gamma}-\beta}\right\}, \\
&\qquad \forall n\geq 0.
\end{aligned} \tag{59}
$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{T_n y_n\}$. We observe that

$$
\begin{aligned}
\|y_n-T_ny_n\| &= \|\alpha_n(f(y_n)-AT_ny_n)+\beta_n(x_n-T_ny_n) \\
&\quad -\epsilon_n((1-\beta_n)I-\alpha_nA)FT_ny_n\| \\
&\leq \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad +\epsilon_n\|((1-\beta_n)I-\alpha_nA)FT_ny_n\| \\
&\leq \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad +\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|FT_ny_n\| \\
&\leq \alpha_n\|f(y_n)-AT_ny_n\|+\beta_n\|x_n-T_ny_n\| \\
&\quad +\epsilon_n\|FT_ny_n\|,
\end{aligned} \tag{60}
$$

which go together with condition (i) and $\epsilon_n \leq \tau\alpha_n, \forall n\geq 0$, implying that

$$
\lim_{n\to\infty}\|y_n-T_ny_n\| = 0. \tag{61}
$$

On the other hand, we have

$$
\|y_n-Ty_n\| \leq \|y_n-T_ny_n\|+\|T_ny_n-Ty_n\|. \tag{62}
$$

Utilizing Lemma 12, we immediately derive

$$
\lim_{n\to\infty}\|y_n-Ty_n\| = 0. \tag{63}
$$

Let $x_t = tf(x_t)+(I-tA)Tx_t$. Utilizing [1, Lemma 2.5] and Lemma 13, we conclude that $\{x_t\}$ converges strongly to $z\in \mathrm{Fix}(T)=\bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i)=\Omega$ and

$$
\limsup_{n\to\infty}\langle (f-A)z, J(y_n-z)\rangle \leq 0. \tag{64}
$$

Finally, we show that $x_n\to z$ as $n\to\infty$. We observe that

$$
\begin{aligned}
&\|y_n-z\|^2 \\
&= \alpha_n\langle f(y_n)-Az, J(y_n-z)\rangle+\beta_n\langle x_n-z, J(y_n-z)\rangle \\
&\quad +\langle((1-\beta_n)I-\alpha_nA)(T_ny_n-z), J(y_n-z)\rangle \\
&\quad -\epsilon_n\langle((1-\beta_n)I-\alpha_nA)FT_ny_n, J(y_n-z)\rangle \\
&\leq (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2+\beta_n\|x_n-z\|\|y_n-z\| \\
&\quad +\alpha_n\langle f(y_n)-f(z), J(y_n-z)\rangle \\
&\quad +\alpha_n\langle f(z)-Az, J(y_n-z)\rangle \\
&\quad +\epsilon_n\|((1-\beta_n)I-\alpha_nA)FT_ny_n\|\|y_n-z\| \\
&\leq (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2 \\
&\quad +\beta_n\|x_n-z\|\|y_n-z\|+\alpha_n\beta\|y_n-z\|^2 \\
&\quad +\alpha_n\langle f(z)-Az, J(y_n-z)\rangle \\
&\quad +\epsilon_n(1-\beta_n-\alpha_n\overline{\gamma})\|FT_ny_n\|\|y_n-z\|
\end{aligned}
$$

$$\le (1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|^2+\frac{\beta_n}{2}\|x_n-z\|^2+\frac{\beta_n}{2}\|y_n-z\|^2$$
$$+\alpha_n\beta\|y_n-z\|^2+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle$$
$$+\epsilon_n\|FT_ny_n\|\|y_n-z\|$$
$$=\left(1-\frac{\beta_n}{2}-\alpha_n(\overline{\gamma}-\beta)\right)\|y_n-z\|^2+\frac{\beta_n}{2}\|x_n-z\|^2$$
$$+\alpha_n\langle f(z)-Az, J(y_n-z)\rangle+\epsilon_n\|FT_ny_n\|\|y_n-z\|, \quad (65)$$

which implies that

$$\|y_n-z\|^2$$
$$\le\frac{\beta_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\|x_n-z\|^2$$
$$+\frac{2\alpha_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\langle f(z)-Az, J(y_n-z)\rangle$$
$$+\frac{2\epsilon_n}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\|FT_ny_n\|\|y_n-z\|$$
$$=\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2+\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}$$
$$\times\left(\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right). \quad (66)$$

Furthermore, utilizing Lemma 3 from the last relation we have

$$\|x_{n+1}-z\|^2$$
$$=\|\sigma_n(f(y_n)-f(z))+(I-\sigma_nA)T_ny_n$$
$$-(I-\sigma_nA)T_nz+\sigma_n(f(z)-F(z))\|^2$$
$$\le\|\sigma_n(f(y_n)-f(z))$$
$$+(I-\sigma_nA)T_ny_n-(I-\sigma_nA)T_nz\|^2$$
$$+2\sigma_n\langle f(z)-Az, J(x_{n+1}-z)\rangle$$
$$\le[\sigma_n\beta\|y_n-z\|+(1-\sigma_n\overline{\gamma})\|T_ny_n-T_nz\|]^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\le[\sigma_n\beta\|y_n-z\|+(1-\sigma_n\overline{\gamma})\|y_n-z\|]^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$=(1-\sigma_n(\overline{\gamma}-\beta))^2\|y_n-z\|^2$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\le\|y_n-z\|^2+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$\le\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2$$
$$+\frac{2\alpha_n(\overline{\gamma}-\beta)}{(\beta_n+2\alpha_n(\overline{\gamma}-\beta))}$$
$$\times\left(\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right)$$
$$+2\sigma_n\|f(z)-Az\|\|x_{n+1}-z\|$$
$$=\left(1-\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}\right)\|x_n-z\|^2+\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}$$
$$\times\left\{\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right.$$
$$+\frac{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}{\overline{\gamma}-\beta}\cdot\frac{\sigma_n}{\alpha_n}$$
$$\left.\cdot\|f(z)-Az\|\|x_{n+1}-z\|\right\}. \quad (67)$$

We note that

$$\frac{2\alpha_n(\overline{\gamma}-\beta)}{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}>\frac{2\alpha_n(\overline{\gamma}-\beta)}{2\beta_n+2\alpha_n}=(\overline{\gamma}-\beta)\frac{\alpha_n}{\alpha_n+\beta_n}. \quad (68)$$

Therefore, condition (ii) leads to $\sum_{n=0}^{\infty}(2\alpha_n(\overline{\gamma}-\beta)/(\beta_n+2\alpha_n(\overline{\gamma}-\beta)))=\infty$. In addition, since $\alpha_n\to 0, \beta_n\to 0, (\epsilon_n/\alpha_n)\to 0$, and $\limsup_{n\to\infty}(\sigma_n/\alpha_n)<\infty$, we get the following from (64)

$$\limsup_{n\to\infty}\left\{\frac{\langle f(z)-Az, J(y_n-z)\rangle}{\overline{\gamma}-\beta}+\frac{\epsilon_n}{\alpha_n}\cdot\frac{\|FT_ny_n\|\|y_n-z\|}{\overline{\gamma}-\beta}\right.$$
$$+\frac{\beta_n+2\alpha_n(\overline{\gamma}-\beta)}{\overline{\gamma}-\beta}\cdot\frac{\sigma_n}{\alpha_n}$$
$$\left.\cdot\|f(z)-F(z)\|\|x_{n+1}-z\|\right\}\le 0. \quad (69)$$

Applying Lemma 2, we have $x_n\to z$ as $n\to\infty$. This completes the proof. □

*Remark 15.* Put $\alpha_n = \sigma_n = 1/n$ and $\beta_n = \epsilon_n = 1/n^2$. Then $\{\alpha_n\}, \{\beta_n\}, \{\epsilon_n\}$, and $\{\sigma_n\}$ satisfy conditions (i) and (ii) of Theorem 14. But we note that $\alpha_n/\beta_n = n \to \infty$.

*Remark 16.* In the iterative scheme of Theorem 14, the first iterative step $y_n = \alpha_n f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)(I - \epsilon_n F)T_n y_n$ is a predictor step and the second iterative step $x_{n+1} = \sigma_n f(y_n) + (I - \sigma_n A)T_n y_n$ is a corrector step. Hence our iteration process is the predictor-corrector method.

*Remark 17.* Theorem 14 extends and improves Theorem 3.1 of [10] to a great extent in the following aspects:

(i) $u$ is replaced by a fixed contractive mapping;

(ii) one continuous pseudocontractive mapping (including nonexpansive mapping) is replaced by a countable family of nonexpansive mappings;

(iii) condition $\alpha_n/\beta_n \to 0$ is weakened to the one $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$;

(iv) we add a strongly positive linear bounded operator $A$ and a strongly accretive and strictly pseudocontractive mapping $F$ in our iterative algorithm.

**Theorem 18.** *Let $C$ be a nonempty closed convex subset of a uniformly smooth Banach space $X$ which has the weakly sequentially continuous duality mapping $J$. Assume that $C \pm C \subset C$. Let $\{T_i\}_{i=0}^{\infty}$ be a countable family of nonexpansive mappings from $C$ to itself such that $\Omega = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : C \to C$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, \gamma_0), \gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = (I - \beta_n F) T_n x_n + \beta_n [f(x_n) - \alpha_n (Af(x_n) - T_n x_n)], \quad \forall n \geq 0, \tag{70}$$

*where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\beta_{n-1}/\beta_n = 1$.

*Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|T_{n+1}x - T_n x\| < \infty$ for any bounded subset $D$ of $C$, let $T$ be a mapping of $C$ into itself defined by $Tx = \lim_{n\to\infty}T_n x$ for all $x \in C$, and suppose that $\mathrm{Fix}(T) = \bigcap_{i=0}^{\infty}\mathrm{Fix}(T_i)$. Then, $\{x_n\}$ converges strongly to a point $z$ of $\Omega$ such that $z$ is a unique solution in $\Omega$ to the VIP:*

$$\langle (F - f) z, J(z - p)\rangle \leq 0, \quad \forall p \in \Omega. \tag{71}$$

*Proof.* First, since $A$ is a $\overline{\gamma}$-strongly positive linear bounded operator on $C$, from (11) we have

$$\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in C, \|u\| = 1\}. \tag{72}$$

Let us show that $\{x_n\}$ is bounded. Indeed, since $\lim_{n\to\infty}\alpha_n = 0$, without loss of generality, we may assume that $0 < \alpha_n \leq \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}, \forall n \geq 0$. Take $p \in \Omega$. Then it follows that $p = T_n p, \forall n \geq 0$, and

$$\begin{aligned} x_{n+1} - p &= (I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p \\ &\quad + \beta_n [(I - \alpha_n A) f(x_n) - (I - \alpha_n A) f(p) \\ &\quad + \alpha_n (T_n x_n - p)] \\ &\quad + \beta_n (f - F) p + \beta_n \alpha_n (I - Af) p. \end{aligned} \tag{73}$$

Hence we deduce the following $0 < \alpha_n \leq \min\{(\gamma_0 - \beta)/2(1 - \overline{\gamma}\beta), \|A\|^{-1}\}$ that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\quad = \|(I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p \\ &\qquad + \beta_n [(I - \alpha_n A) f(x_n) - (I - \alpha_n A) f(p) \\ &\qquad + \alpha_n (T_n x_n - p)] \\ &\qquad + \beta_n (f - F) p + \beta_n \alpha_n (I - Af) p\| \\ &\quad \leq \|(I - \beta_n F) T_n x_n - (I - \beta_n F) T_n p\| \\ &\qquad + \beta_n [\|I - \alpha_n A\| \|f(x_n) - f(p)\| + \alpha_n \|T_n x_n - p\|] \\ &\qquad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ &\quad \leq (1 - \beta_n \gamma_0) \|x_n - p\| \\ &\qquad + \beta_n [(1 - \alpha_n \overline{\gamma}) \beta \|x_n - p\| + \alpha_n \|x_n - p\|] \\ &\qquad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ &\quad = [1 - \beta_n (\gamma_0 - \beta - \alpha_n (1 - \overline{\gamma}\beta))] \|x_n - p\| \\ &\qquad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ &\quad \leq \left[1 - \beta_n \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)} (1 - \overline{\gamma}\beta)\right)\right] \|x_n - p\| \\ &\qquad + \beta_n \|(f - F) p\| + \beta_n \alpha_n \|(I - Af) p\| \\ &\quad \leq \left(1 - \frac{1}{2}\beta_n (\gamma_0 - \beta)\right) \|x_n - p\| \\ &\qquad + \beta_n (\|(f - F) p\| + \|(I - Af) p\|) \\ &\quad = \left(1 - \frac{1}{2}\beta_n (\gamma_0 - \beta)\right) \|x_n - p\| \\ &\qquad + \frac{1}{2}\beta_n (\gamma_0 - \beta) \frac{2(\|(f - F) p\| + \|(I - Af) p\|)}{\gamma_0 - \beta} \\ &\quad \leq \max\left\{\|x_n - p\|, \frac{2(\|(f - F) p\| + \|(I - Af) p\|)}{\gamma_0 - \beta}\right\}. \end{aligned} \tag{74}$$

By induction

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{2\left(\|(f-F)p\| + \|(I-Af)p\|\right)}{\gamma_0 - \beta}\right\}, \quad \forall n \geq 0. \tag{75}$$

This implies that $\{x_n\}$ is bounded and so are $\{T_n x_n\}$, $\{f(x_n)\}$ and $\{FT_n x_n\}$.

Now we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{76}$$

Indeed, first of all, (70) can be rewritten as follows:

$$\begin{aligned} y_n &= (I - \alpha_n A) f(x_n) + \alpha_n T_n x_n, \\ x_{n+1} &= (I - \beta_n F) T_n x_n + \beta_n y_n, \quad \forall n \geq 0. \end{aligned} \tag{77}$$

Observe that

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &= \|(I - \alpha_n A) f(x_n) + \alpha_n T_n x_n \\ &\quad - (I - \alpha_{n-1} A) f(x_{n-1}) - \alpha_{n-1} T_{n-1} x_{n-1}\| \\ &= \|\alpha_n (T_n x_n - T_{n-1} x_{n-1}) \\ &\quad + (\alpha_n - \alpha_{n-1})(T_{n-1} x_{n-1} - Af(x_{n-1})) \\ &\quad + (I - \alpha_n A) f(x_n) - (I - \alpha_n A) f(x_{n-1})\| \\ &\leq \alpha_n \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + \|I - \alpha_n A\| \|f(x_n) - f(x_{n-1})\| \\ &\leq \alpha_n (\|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + (1 - \alpha_n \overline{\gamma}) \beta \|x_n - x_{n-1}\| \\ &\leq \alpha_n (\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + (1 - \alpha_n \overline{\gamma}) \beta \|x_n - x_{n-1}\| \\ &= (\beta - \alpha_n (\overline{\gamma}\beta - 1)) \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\quad + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\|, \end{aligned} \tag{78}$$

and hence

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|(I - \beta_n F) T_n x_n + \beta_n y_n \\ &\quad - (I - \beta_{n-1} F) T_{n-1} x_{n-1} - \beta_{n-1} y_{n-1}\| \\ &\leq \|\beta_n (y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - FT_{n-1} x_{n-1}) \\ &\quad + (I - \beta_n F) T_n x_n - (I - \beta_n F) T_{n-1} x_{n-1}\| \\ &\leq \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0) \|T_n x_n - T_{n-1} x_{n-1}\| \\ &\leq \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)(\|T_n x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &\leq \beta_n \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &\leq \beta_n [(\beta - \alpha_n (\overline{\gamma}\beta - 1)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\ &\qquad \times \|T_{n-1} x_{n-1} - Af(x_{n-1})\| \\ &\qquad + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\|] \\ &\quad + |\beta_n - \beta_{n-1}| \|y_{n-1} - FT_{n-1} x_{n-1}\| \\ &\quad + (1 - \beta_n \gamma_0)(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &\leq \beta_n [(\beta - \alpha_n (\overline{\gamma}\beta - 1)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M \\ &\qquad + \alpha_n \|T_n x_{n-1} - T_{n-1} x_{n-1}\|] + |\beta_n - \beta_{n-1}| M \\ &\quad + (1 - \beta_n \gamma_0)(\|x_n - x_{n-1}\| + \|T_n x_{n-1} - T_{n-1} x_{n-1}\|) \\ &= [1 - \beta_n (\gamma_0 - \beta - \alpha_n (1 - \overline{\gamma}\beta))] \|x_n - x_{n-1}\| \\ &\quad + M (\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\quad + (\beta_n \alpha_n + (1 - \beta_n \gamma_0)) \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\ &\leq \left[1 - \beta_n \left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)} (1 - \overline{\gamma}\beta)\right)\right] \\ &\quad \times \|x_n - x_{n-1}\| + M (\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\quad + 2 \|T_n x_{n-1} - T_{n-1} x_{n-1}\| \\ &= \left[1 - \frac{1}{2} \beta_n (\gamma_0 - \beta)\right] \|x_n - x_{n-1}\| \\ &\quad + M (\beta_n |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\quad + 2 \|T_n x_{n-1} - T_{n-1} x_{n-1}\|, \end{aligned} \tag{79}$$

where $\sup_{n\geq 0}\{\|T_nx_n - Af(x_n)\| + \|y_n - FT_nx_n\|\} \leq M$ for some $M > 0$ (it is easy to see that $\{y_n\}$ is bounded due to the boundedness of $\{x_n\}$). Utilizing Lemma 2, we conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ from conditions (i)-(ii) and the property imposed on $\{T_n\}$.

Next let us show that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{80}$$

Indeed, from (76), (77), and $\beta_n \to 0$, it follows that

$$\begin{aligned}\|x_n - T_nx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| \\ &= \|x_n - x_{n+1}\| + \beta_n \|y_n - FT_nx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{81}$$

That is,

$$\lim_{n\to\infty} \|x_n - T_nx_n\| = 0. \tag{82}$$

Also, it is clear that

$$\|x_n - Tx_n\| \leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\|. \tag{83}$$

By Lemma 12, we conclude from (82) and (83) that (80) holds. Let $x_t = (I - \theta_tF)Tx_t + \theta_t[f(x_t) - t(Af(x_t) - Tx_t)]$. According to Theorem 11, we know that $\{x_t\}$ converges strongly to $z \in \mathrm{Fix}(T) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) = \Omega$, which is the unique solution in $\Omega$ to the VIP:

$$\langle (F - f)z, J(z - p)\rangle \leq 0, \quad \forall p \in \Omega. \tag{84}$$

Further, let us show that

$$\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle \leq 0. \tag{85}$$

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle \\ &\quad = \lim_{i\to\infty} \left\langle (f - F)z, J\left(x_{n_i} - z\right)\right\rangle.\end{aligned} \tag{86}$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \tilde{x}$. Utilizing Lemma 5 we obtain from (80) that $\tilde{x} \in \mathrm{Fix}(T)$. Hence from (84) and (86) we get

$$\limsup_{n\to\infty} \langle (f - F)z, J(x_n - z)\rangle = \langle (f - F)z, J(\tilde{x} - z)\rangle \leq 0. \tag{87}$$

As required, let us show that $x_n \to z$ as $n \to \infty$.

As a matter of fact, we observe that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz \\
&\quad + \beta_n[(I - \alpha_nA)f(x_n) - (I - \alpha_nA)f(z) \\
&\qquad + \alpha_n(T_nx_n - z)] + \beta_n(f - F)z + \beta_n\alpha_n(I - Af)z\|^2 \\
&\leq \|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz \\
&\quad + \beta_n[(I - \alpha_nA)(f(x_n) - f(z)) + \alpha_n(T_nx_n - z)]\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\langle (I - Af)z, J(x_{n+1} - z)\rangle \\
&\leq [\|(I - \beta_nF)T_nx_n - (I - \beta_nF)T_nz\| \\
&\quad + \beta_n(\|I - \alpha_nA\|\|f(x_n) - f(z)\| + \alpha_n\|T_nx_n - z\|)]^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \{(1 - \beta_n\gamma_0)\|T_nx_n - T_nz\| \\
&\quad + \beta_n[(1 - \alpha_n\overline{\gamma})\beta\|x_n - z\| + \alpha_n\|T_nx_n - z\|]\}^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \{(1 - \beta_n\gamma_0)\|x_n - z\| \\
&\quad + \beta_n[(1 - \alpha_n\overline{\gamma})\beta\|x_n - z\| + \alpha_n\|x_n - z\|]\}^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&= [1 - \beta_n(\gamma_0 - \beta - \alpha_n(1 - \overline{\gamma}\beta))]^2\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq [1 - \beta_n(\gamma_0 - \beta - \alpha_n(1 - \overline{\gamma}\beta))]\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\| \\
&\leq \left[1 - \beta_n\left(\gamma_0 - \beta - \frac{\gamma_0 - \beta}{2(1 - \overline{\gamma}\beta)}(1 - \overline{\gamma}\beta)\right)\right]\|x_n - z\|^2 \\
&\quad + 2\beta_n\langle (f - F)z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n\|(I - Af)z\|\|x_{n+1} - z\|
\end{aligned}$$

$$
\begin{aligned}
&= \left[1 - \frac{1}{2}\beta_n(\gamma_0 - \beta)\right]\|x_n - z\|^2 \\
&\quad + 2\beta_n \langle (f - F) z, J(x_{n+1} - z)\rangle \\
&\quad + 2\beta_n\alpha_n \|(I - Af) z\| \|x_{n+1} - z\| \\
&= (1 - \mu_n)\|x_n - z\|^2 + \mu_n\nu_n,
\end{aligned} \tag{88}
$$

where $\mu_n = (1/2)\beta_n(\gamma_0 - \beta)$ and

$$
\nu_n = \frac{4\left(\langle (f - F) z, J(x_{n+1} - z)\rangle + \alpha_n \|(I - Af) z\| \|x_{n+1} - z\|\right)}{\gamma_0 - \beta}. \tag{89}
$$

It can be easily seen from (85) and conditions (i) and (ii) that

$$
\sum_{n=0}^{\infty}\mu_n = \infty, \qquad \limsup_{n\to\infty}\nu_n \le 0. \tag{90}
$$

In terms of Lemma 8, we infer that $x_n \to z$ as $n \to \infty$. □

Finally, we provide an example to illustrate Theorem 18.

*Example 19.* Let $X = \mathbf{R}^2$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ which are defined by

$$
\langle x, y\rangle = ac + bd, \qquad \|x\| = \sqrt{a^2 + b^2}, \tag{91}
$$

for all $x, y \in \mathbf{R}^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{(a, a) : a \in \mathbf{R}\}$. Clearly, $C$ is a nonempty closed convex subset of a uniformly smooth Banach space $X = \mathbf{R}^2$ such that $C \pm C \subset C$. Let $\{T_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive mappings from $C$ to itself such that $\Omega = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$, for instance, putting $T_n = (1 - 1/2^{n+1})T$ with $T = \left\{\begin{smallmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{smallmatrix}\right\}$. Then $\|T\| = 1$ and $\|T_n\| = 1 - 1/2^{n+1}, \forall n \ge 0$. It is clear that $T_n$ and $T$ are nonexpansive mappings with $\Omega = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n) = \{0\} \neq \emptyset$, and $\{T_n\}$ satisfies the assumption in Theorem 18. Let $F : C \to C$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$, and $f : C \to C$ be a fixed contractive mapping with contractive coefficient $\beta \in (0, \gamma_0), \gamma_0 = 1 - \sqrt{(1 - \alpha)/\lambda}$, for instance, putting $S = \left\{\begin{smallmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{smallmatrix}\right\}$, $F = (1/2)S$, and $f = \left\{\begin{smallmatrix} 3/25 & 2/25 \\ 2/25 & 3/25 \end{smallmatrix}\right\}$, we know that $\|F\| = (1/2)\|S\| = 1/2, \|f\| = 1/5$ and that $F$ is a $(1/2)$-strongly accretive and $(8/9)$-strictly pseudocontractive mapping and $f$ is a $(1/5)$-contraction with $(1/5) \in (0, \gamma_0)$ and $\gamma_0 = 1/4$. Let $A : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma}\beta < 1$; for instance, putting $A = (7/6)S$, we know that $A$ is a $(7/6)$-strongly positive linear bounded operator with $\overline{\gamma}\beta = (7/6) \times (1/5) < 1$. In this case, from iterative scheme (70) in Theorem 18, we obtain that for any given $x_0 \in C$,

$$
\begin{aligned}
x_1 &= (I - \beta_0 F) T_0 x_0 + \beta_0 \left[f(x_0) - \alpha_0 (Af(x_0) - T_0 x_0)\right] \\
&= \left(1 - \frac{1}{2}\beta_0\right)\left(1 - \frac{1}{2^{0+1}}\right)x_0 \\
&\quad + \beta_0\left[\frac{1}{5}x_0 - \alpha_0\left(\frac{7}{6}\cdot\frac{1}{5}x_0 - \left(1 - \frac{1}{2^{0+1}}\right)x_0\right)\right] \\
&= \left[\left(1 - \frac{1}{2}\beta_0\right)\left(1 - \frac{1}{2^{0+1}}\right) + \frac{1}{5}\beta_0 \right.\\
&\quad \left. - \alpha_0\beta_0\left(\frac{7}{30} - \left(1 - \frac{1}{2^{0+1}}\right)\right)\right] \quad x_0 \in C.
\end{aligned} \tag{92}
$$

It can be readily seen that

$$
\begin{aligned}
x_{n+1} = &\left[\left(1 - \frac{1}{2}\beta_n\right)\left(1 - \frac{1}{2^{n+1}}\right) + \frac{1}{5}\beta_n \right.\\
&\left. - \alpha_n\beta_n\left(\frac{7}{30} - \left(1 - \frac{1}{2^{n+1}}\right)\right)\right]x_n, \quad \forall n \ge 0.
\end{aligned} \tag{93}
$$

We claim that $x_n$ converges to the unique point 0 in $\Omega$ if $\alpha_n = (6/23)\beta_n$ and $\sum_{n=0}^{\infty}\beta_n = \infty$. Indeed, observe that

$$
\begin{aligned}
\|x_{n+1}\| &= \left[\left(1 - \frac{1}{2}\beta_n\right)\left(1 - \frac{1}{2^{n+1}}\right)\right.\\
&\quad \left. + \frac{1}{5}\beta_n - \alpha_n\beta_n\left(\frac{7}{30} - \left(1 - \frac{1}{2^{n+1}}\right)\right)\right]\|x_n\| \\
&\le \left[\left(1 - \frac{1}{2}\beta_n\right) + \frac{1}{5}\beta_n - \alpha_n\beta_n\left(\frac{7}{30} - 1\right)\right]\|x_n\| \\
&= \left(1 - \frac{3}{10}\beta_n + \frac{23}{30}\alpha_n\beta_n\right)\|x_n\| \\
&= \left(1 - \frac{3}{10}\beta_n + \frac{23}{30}\cdot\frac{6}{23}\beta_n\beta_n\right)\|x_n\| \\
&\le \left(1 - \frac{3}{10}\beta_n + \frac{1}{5}\beta_n\right)\|x_n\| \\
&= \left(1 - \frac{1}{10}\beta_n\right)\|x_n\| \le \prod_{i=0}^{n}\left(1 - \frac{1}{10}\beta_i\right)\|x_0\|.
\end{aligned} \tag{94}
$$

Thus, we conclude from $\sum_{n=0}^{\infty}\beta_n = \infty$ that $x_n$ converges to the unique point 0 in $\Omega$. It is clear that $z = 0$ is a unique solution in $\Omega$ for the following variational inequality problem (VIP):

$$
\langle (f - F) z, J(p - z)\rangle \le 0, \quad \forall p \in \Omega. \tag{95}
$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## References

[1] G. Cai and S. Bu, "Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 1998–2004, 2011.

[2] L. C. Ceng, S. Schaible, and J. C. Yao, "Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 139, no. 2, pp. 403–418, 2008.

[3] L. C. Ceng, P. Cubiotti, and J. C. Yao, "Strong convergence theorems for finitely many nonexpansive mappings and applications," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 5, pp. 1464–1473, 2007.

[4] L. C. Ceng and J. C. Yao, "Relaxed viscosity approximation methods for fixed point problems and variational inequality problems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 69, no. 10, pp. 3299–3309, 2008.

[5] L. C. Ceng and J. C. Yao, "Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 729–741, 2008.

[6] H. Zegeye, E. U. Ofoedu, and N. Shahzad, "Convergence theorems for equilibrium problem, variational inequality problem and countably infinite relatively quasi-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 216, no. 12, pp. 3439–3449, 2010.

[7] T. Suzuki, "Moudafi's viscosity approximations with Meir-Keeler contractions," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 342–352, 2007.

[8] Y. H. Wang and W. Xu, "Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities," *Fixed Point Theory and Applications*, vol. 2013, article 121, 9 pages, 2013.

[9] J. S. Jung, "Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 509–520, 2005.

[10] Y. Yao, Y. C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 12, pp. 3311–3317, 2007.

[11] A. Moudafi and P.-E. Maingé, "Towards viscosity approximations of hierarchical fixed-point problems," *Fixed Point Theory and Applications*, vol. 2006, Article ID 95453, 10 pages, 2006.

[12] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[13] M. Tian, "A general iterative algorithm for nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 73, no. 3, pp. 689–694, 2010.

[14] L. C. Ceng, S. M. Guu, and J. C. Yao, "A general composite iterative algorithm for nonexpansive mappings in Hilbert spaces," *Computers and Mathematics with Applications*, vol. 61, no. 9, pp. 2447–2455, 2011.

[15] L. C. Ceng, Q. H. Ansari, and J. C. Yao, "Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in banach spaces," *Numerical Functional Analysis and Optimization*, vol. 29, no. 9-10, pp. 987–1033, 2008.

[16] I. Yamada, "The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor, and S. Reich, Eds., pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[17] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[18] H. K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[19] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[20] K. Goebel and W. A. Kirk, "Topics in metric fixed point theory," in *Cambridge Studies in Advanced Mathematics*, vol. 28, Cambridge University Press, 1990.

[21] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, no. 4, pp. 365–374, 1974.

[22] Y. Song and R. Chen, "Convergence theorems of iterative algorithms for continuous pseudocontractive mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 2, pp. 486–497, 2007.

[23] G. Cai and C. S. Hu, "Strong convergence theorems of a general iterative process for a finite family of $\lambda_i$-strict pseudo-contractions in $q$-uniformly smooth Banach spaces," *Computers and Mathematics with Applications*, vol. 59, no. 1, pp. 149–160, 2010.

[24] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.