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Équations aux dérivées partielles/*Partial Differential Equations*

A nodal domain property for the p-Laplacian

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Abstract. We show a partial version of the Courant nodal domain theorem for the p-Laplacian: any eigenfunction associated to the second eigenvalue has exactly two nodal domains. A similar result is also proved for the Fučik spectrum. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une propriété nodale pour le p-Laplacien

Résumé. *Nous obtenons une extension partielle au p-Laplacien du théorème de Courant sur les domaines nodaux : toute fonction propre associée à la seconde valeur propre admet exactement deux domaines nodaux. Un résultat analogue est aussi démontré pour le spectre de Fučik.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots \to +\infty$ la suite de toutes les valeurs propres de $-\Delta$ sur $\mathrm{H}_0^1(\Omega)$, où Ω est un domaine borné de $\mathbb{R}^N$. Le théorème de Courant [7] affirme que si u est fonction propre associée à μ_k, alors u admet au plus k domaines nodaux.

Ce théorème a été partiellement étendu au p-Laplacien par Anane–Tsouli [4]. Soit $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to +\infty$ la suite des valeurs propres de $-\Delta_p$ sur $\mathrm{W}_0^{1,p}(\Omega)$ obtenue par la méthode de Ljusternik–Schnirelman (*cf*. [10,3], ...). Dans le cas linéaire $p = 2$, cette suite $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ fournit toutes les valeurs propres et coïncide avec la suite précédente $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$ (*cf*. [5]). Le résultat de [4] est le suivant. Soit λ une valeur propre de $-\Delta_p$ et supposons que pour un certain k, $\lambda < \lambda_k$. Alors le nombre de domaines nodaux d'une fonction propre associée à λ est $< k$. (Il est aussi démontré dans [4], comme application de cette propriété nodale, qu'il n'y a pas de valeur propre dans $]\lambda_1, \lambda_2[$, c'est-à-dire que λ_2 est vraiment la deuxième valeur propre de $-\Delta_p$ sur $\mathrm{W}_0^{1,p}(\Omega)$.)

Note présentée par Haïm BRÉZIS.

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Notre travail concerne cette seconde valeur propre λ_2. Dans le cas linéaire $p = 2$, le théorème de Courant implique que le nombre de domaines nodaux d'une fonction propre associée à λ_2 est exactement 2. Dans le cas non linéaire, le résultat précédent de [4] implique que le nombre de domaines nodaux d'une fonction propre associée à λ_2 est ($\geqslant 2$ et) $\leqslant 2 + m - 1$, où m est la multiplicité de Ljusternik–Schnirelman de λ_2 (c'est-à-dire $\lambda_1 < \lambda_2 = \lambda_3 = \cdots = \lambda_{2+m-1} < \lambda_{2+m}$). Nous démontrons que dans le cas non linéaire également, le nombre de domaines nodaux d'une fonction propre associée à λ_2 est exactement 2.

Plus généralement, nous considérons le spectre de Fučik Σ_p du p-Laplacien, qui est défini comme l'ensemble des $(\alpha, \beta) \in \mathbb{R}^2$ tels que (2.1) ci-dessous admet une solution non triviale u. Une première courbe dans Σ_p a été construite dans [6] en utilisant le théorème du col de la montagne sur une variété. Nous montrons ici que si u est une solution non triviale de (2.1) associée à un point (α, β) de cette première courbe, alors u admet exactement deux domaines nodaux.

La démonstration usuelle du théorème de Courant utilise la propriété de continuation unique (*cf.* [7,13]). La validité éventuelle de cette dernière propriété pour le p-Laplacien est une question largement ouverte (*cf.* [1,14,15]). Notre démonstration n'utilise pas de propriété de continuation unique. Elle est basée sur la caractérisation variationnelle de la première courbe du spectre de Fučik et utilise de façon essentielle le principe du maximum de Hopf pour le p-Laplacien (*cf.* [16]).

1. Introduction

Let us start by considering the sequence $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots \to +\infty$ of all eigenvalues of $-\Delta$ on $\mathrm{H}_0^1(\Omega)$, Ω being a bounded domain in $\mathbb{R}^N$, where each μ_k is repeated according to its multiplicity. A well-known theorem of Courant [7] states that if $u \in \mathrm{H}_0^1(\Omega)$ is an eigenfunction associated to μ_k, then u admits at most k nodal domains.

This theorem was partially extended to the p-Laplacian by Anane–Tsouli [4]. Let us denote by $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to +\infty$ the sequence of eigenvalues of $-\Delta_p$ on $\mathrm{W}_0^{1,p}(\Omega)$ obtained by the Ljusternik–Schnirelman method (*cf.* [10,3], ...). In the linear case $p = 2$, this sequence $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ yields all eigenvalues and coincides with the previous sequence $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$ (*cf.* [5]). The result of [4] is the following. Let λ be an eigenvalue of $-\Delta_p$ on $\mathrm{W}_0^{1,p}(\Omega)$ and suppose that, for some k, $\lambda < \lambda_k$. Then the number of nodal domains of an eigenfunction associated to λ is $< k$. (It is also proved in [4], as an application of this nodal property, that there is no eigenvalue in $]\lambda_1, \lambda_2[$, i.e., that λ_2 is really the second eigenvalue of $-\Delta_p$ on $\mathrm{W}_0^{1,p}(\Omega)$.)

We are interested in this paper in this second eigenvalue λ_2. In the linear case $p = 2$, the Courant theorem implies that the number of nodal domains of an eigenfunction associated to λ_2 is exactly 2. While in the nonlinear case, the above result of [4] implies that the number of nodal domains of an eigenfunction associated to λ_2 is ($\geqslant 2$ and) $\leqslant 2 + m - 1$, where m is the Ljusternik–Schnirelman multiplicity of λ_2 (i.e., $\lambda_1 < \lambda_2 = \lambda_3 = \cdots = \lambda_{2+m-1} < \lambda_{2+m}$).

It is our purpose in this paper to prove that in the nonlinear case too, the number of nodal domains of an eigenfunction associated to λ_2 is exactly 2. In fact we prove a more general result, which concerns the first curve of the Fučik spectrum of $-\Delta_p$ on $\mathrm{W}_0^{1,p}(\Omega)$. We show that any nontrivial solution of the Fučik equation (*cf.* (2.1) below) associated to a point (α, β) in that first curve admits exactly two nodal domains. In the linear case $p = 2$, this latter property was obtained in [8].

Our result is stated in detail in Section 2 and proved in Section 3. Some special care must be taken in the proof. Indeed, the usual argument which leads to the Courant theorem uses the unique continuation property (*cf.* [7,13]). This property is also used in [8]. On the other hand the unique continuation property is still largely an open question in the case of the p-Laplacian (*see* [1,14,15]). Our proof does not use any unique continuation property. It is based on a recent mountain-pass characterization of the first Fučik curve (*cf.* [6]). Another crucial ingredient is the Hopf maximum principle for the p-Laplacian (*cf.* [16]). We also use some ideas from [17].

2. Statement

The Fučik spectrum of the p-Laplacian on $W_0^{1,p}(\Omega)$ is defined as the set Σ_p of those $(\alpha,\beta)\in\mathbb{R}^2$ such that

$$\begin{cases} -\Delta_p u = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a nontrivial solution u. Here $1<p<\infty$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $u^\pm := \max(\pm u, 0)$, and Ω is a bounded domain in $\mathbb{R}^N$. The usual spectrum of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ corresponds to $\alpha=\beta$ in (2.1).

The first eigenvalue λ_1 is of special importance. It is defined as the minimum of $\int_\Omega |\nabla u|^p$ on the manifold $S := \{u\in W_0^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$. Here are some of its properties which will be of interest for us (*cf*. [3,12]): λ_1 is >0, simple, and admits an eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)\cap C^1(\Omega)$, with $\varphi_1(x)>0$ in Ω and $\int_\Omega \varphi_1^p = 1$; moreover, any eigenfunction associated to an eigenvalue different from λ_1 changes sign. This latter property immediately extends to any nontrivial solution of (2.1) with $(\alpha,\beta)\notin(\lambda_1\times\mathbb{R})\cup(\mathbb{R}\times\lambda_1)$. One also deduces directly from the definition of λ_1 that Σ_p is contained in $\{(\alpha,\beta)\in\mathbb{R}^2;\ \alpha \text{ and } \beta\geqslant\lambda_1\}$. Of course the lines $\lambda_1\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1$ are included in Σ_p.

A first curve in Σ_p was constructed in [6] in the following manner. Fix $s\geqslant 0$ and consider the functional

$$J_s(u) := \int_\Omega |\nabla u|^p - s\int_\Omega (u^+)^p$$

on $W_0^{1,p}(\Omega)$ as well as its restriction $\widetilde{J}_s$ to the manifold S. Denote by $c(s)$ the mountain-pass value:

$$c(s) := \inf_{\gamma\in\Gamma}\max_{u\in\gamma}\widetilde{J}_s(u), \tag{2.2}$$

where Γ represents the set of all continuous paths γ in S going from φ_1 to $-\varphi_1$. (Here and below, the topology on S is that induced by $W_0^{1,p}(\Omega)$.) It is proved in [6] that: (i) $c(s)>\lambda_1$, (ii) $(s+c(s),c(s))\in\Sigma_p$, (iii) $(s+c(s),c(s))$ is the first (nontrivial) point of Σ_p on the parallel to the diagonal passing through $(s,0)$. Letting s vary in $\mathbb{R}^+$ and taking into consideration the symmetric points with respect to the diagonal, we get in this way a first curve in Σ_p, which we denote by $\mathcal{C}$. Various properties of $\mathcal{C}$ are investigated in [6]: regularity, monotonicity, asymptotic behaviour.

We recall that a nodal domain for a function $u\in C(\Omega)$ is a maximal open connected subset of the set $\{x\in\Omega : u(x)\neq 0\}$. Note here that any solution $u\in W_0^{1,p}(\Omega)$ of (2.1) belongs to $L^\infty(\Omega)\cap C^1(\Omega)$. This follows by combining the L^∞ estimates of [3] with the local regularity results of [9].

THEOREM 2.1. – *Let u be a nontrivial solution of* (2.1) *with $(\alpha,\beta)\in\mathcal{C}$. Then u admits exactly two nodal domains.*

COROLLARY 2.2. – *An eigenfunction associated to λ_2 admits exactly two nodal domains.*

3. Proof of Theorem 2.1

Let u be a nontrivial solution of (2.1) with $(\alpha,\beta)\in\mathcal{C}$. Replacing u by $-u$ if necessary, we can assume $\alpha\geqslant\beta$. So, writing $s=\alpha-\beta$, we have $(\alpha,\beta)=(s+c(s),c(s))$ with $c(s)$ defined in (2.2).

As observed at the beginning of Section 2, u must change sign and consequently admits at least a positive nodal domain Ω_1 and a negative nodal domain Ω_2. Let us assume by contradiction the existence of a third nodal domain Ω_3 with, say, $u>0$ in Ω_3 (the argument would be similar if $u<0$ in Ω_3).

CLAIM 3.1. – *There exists $\widetilde{\Omega}_2\subset\Omega$, open connected with $\widetilde{\Omega}_2\supsetneqq\Omega_2$, such that $\widetilde{\Omega}_2$ is disjoint of Ω_1 or of Ω_3.*

Let us admit this claim for a moment and show how to derive a contradiction. The argument here is partly adapted from the proofs of Theorem 3.1 and Lemma 5.3 of [6] and we will only sketch it. We will assume below that the claim is verified with $\widetilde{\Omega}_2$ disjoint of Ω_1 (the argument would be similar in the other case).

The first part of the argument consists in showing the existence of a function $v \in \mathrm{W}_0^{1,p}(\Omega)$ which changes sign and satisfies

$$\int_\Omega |\nabla v^+|^p \Big/ \int_\Omega (v^+)^p < \alpha \quad \text{and} \quad \int_\Omega |\nabla v^-|^p \Big/ \int_\Omega (v^-)^p < \beta. \tag{3.1}$$

To do so we first observe that $\lambda_1(\Omega_1) = \alpha$ and $\lambda_1(\widetilde{\Omega}_2) < \lambda_1(\Omega_2) = \beta$, where $\lambda_1(\mathcal{O})$ denotes the first eigenvalue of $-\Delta_p$ on $\mathrm{W}_0^{1,p}(\mathcal{O})$. This follows from the fact that we deal with nodal domains and from the monotonicity dependence of $\lambda_1(\mathcal{O})$ with respect to $\mathcal{O}$. We then decrease a little bit $\widetilde{\Omega}_2$ and increase a little bit Ω_1 so as to get two new open sets in Ω, $\widetilde{\widetilde{\Omega}}_2$ and $\widetilde{\Omega}_1$, with empty intersection and such that $\lambda_1(\widetilde{\widetilde{\Omega}}_2) < \beta$ and $\lambda_1(\widetilde{\Omega}_1) < \alpha$. This construction involves the continuous dependence of $\lambda_1(\mathcal{O})$ with respect to $\mathcal{O}$ as well as the monotonicity dependence again. The desired function v in (3.1) is then obtained by putting $v = v_1 - v_2$, where v_1 (resp. v_2) is the extension by zero outside $\widetilde{\Omega}_1$ (resp. $\widetilde{\widetilde{\Omega}}_2$) of the positive eigenfunction associated to $\lambda_1(\widetilde{\Omega}_1)$ (resp. $\lambda_1(\widetilde{\widetilde{\Omega}}_2)$).

In the second part of the argument, we use this function v to construct a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma} \widetilde{J}_s(u) < \beta. \tag{3.2}$$

Since $\beta = c(s)$, this yields a contradiction with (2.2).

To construct this path, we start from $v/\|v\|_p$ and go to $v^+/\|v^+\|_p$ by convex combination on S, i.e., through the path

$$\frac{tv + (1-t)v^+}{\|tv + (1-t)v^+\|_p}, \quad t \in [0,1].$$

We then go on by convex combination on S from $v^+/\|v^+\|_p$ to $v^-/\|v^-\|_p$. Using (3.1), one verifies that the levels of $\widetilde{J}_s$ along these two paths remain $< \beta$; moreover, the level at $v^-/\|v^-\|_p$ is $< \beta - s$. One then uses Lemma 3.6 from [6] which says that any component of a set of the form $\{u \in S : \widetilde{J}_s(u) < r\}$ contains a critical point of $\widetilde{J}_s$. This allows us to go on from $v^-/\|v^-\|_p$ to φ_1 or $-\varphi_1$ (which are the only critical points of $\widetilde{J}_s$ at levels $< \beta - s$) by staying at levels $< \beta - s$. Let us call ν this last path and assume that it is, say, φ_1 which is reached in this way (the argument would be similar in the other case). One then considers the path $-\nu$ which goes from $-\varphi_1$ to $-v^-/\|v^-\|_p$. Finally by convex combination on S, one returns from $-v^-/\|v^-\|_p$ to the starting point $v/\|v\|_p$. It is easily seen that the path on S from φ_1 to $-\varphi_1$ constructed in this way satisfies (3.2).

It remains to give the:

Proof of claim 3.1. – We distinguish two cases: (i) $\partial\Omega_2 \cap \Omega$ not included in $\partial\Omega_1 \cap \Omega$, (ii) $\partial\Omega_2 \cap \Omega \subset \partial\Omega_1 \cap \Omega$. In case (i), take $x \in \partial\Omega_2 \cap \Omega$ with $x \notin \partial\Omega_1$. Thus, for some $\varepsilon > 0$, $\mathrm{B}(x,\varepsilon) \subset \Omega$ and $\mathrm{B}(x,\varepsilon) \cap \Omega_1 = \varnothing$. The set $\widetilde{\Omega}_2 = \Omega_2 \cup \mathrm{B}(x,\varepsilon)$ is then disjoint of Ω_1 and yields the conclusion of the claim. Let us now deal with case (ii). The function u on Ω_2 is C^1, < 0 and satisfies there $-\Delta_p u \leqslant 0$ in the weak sense. Let $z \in \partial\Omega_2 \cap \Omega$ with z satisfying the interior ball condition with respect to Ω_2. (It is easily verified that for any open set $\mathcal{O}$, the set of such points in $\partial\mathcal{O}$ is dense in $\partial\mathcal{O}$.) Since u is C^1 in a neighbourhood of z, one deduces from the Hopf maximum principle in [16] that $\partial u/\partial n(z) > 0$, where n is the exterior normal direction to the interior ball at z. So at least one partial derivative of u at z is nonzero. Assume for instance that it is $\partial u/\partial x_N$. Let us now consider the C^1 mapping $\Phi : \Omega \to \mathbb{R}^N : (x_1, \ldots, x_N) \to (y_1, \ldots, y_N)$ defined by

$y_i = x_i - z_i$ for $i = 1, \ldots, N-1$, $y_N = u(x_1, \ldots, x_N)$. By the inverse mapping theorem, there is an open neighbourhood U of z which is diffeomorphic through Φ to $V := \{y \in \mathbb{R}^N : |y| < \varepsilon\}$ for some $\varepsilon > 0$. Since $u(\Phi^{-1}(y)) = y_N$, we have that $u = 0$ on $\Phi^{-1}(V^0)$, $u > 0$ on $\Phi^{-1}(V^+)$ and $u < 0$ on $\Phi^{-1}(V^-)$, where V^0 (resp. V^+, V^-) $:= \{y \in V : y_N = 0$ (resp. $> 0, < 0$)$\}$. Moreover $U = \Phi^{-1}(V^0) \cup \Phi^{-1}(V^+) \cup \Phi^{-1}(V^-)$. Now $z \in \partial\Omega_1 \cap \Omega$ (because we are in case (ii)), $\Phi^{-1}(V^+)$ is open connected and Ω_1 is a positive nodal domain. This implies $\Omega_1 \supset \Phi^{-1}(V^+)$. Similarly, $\Omega_2 \supset \Phi^{-1}(V^-)$. Consequently, $z \notin \partial\Omega_3$. The conclusion of the claim can then be derived by arguing as in case (i) to get a set $\widetilde{\Omega}_2$ disjoint of Ω_3.

Remark 3.2. – The claim above can not be deduced from purely topological arguments. Indeed there exist disjoint bounded open connected sets Ω_1, Ω_2 and Ω_3 in $\mathbb{R}^2$ with $\partial\Omega_1 = \partial\Omega_2 = \partial\Omega_3$ (*cf*. [11]).

Remark 3.3. – The proof above is based on the mountain-pass formula (2.2). The usual Ljusternik–Schnirelman characterization of λ_2 however suffices if one is only interested in deriving Corollary 2.2. Indeed, the intersection with S of the vector space generated by v^+ and v^- (where v is as in (3.1) with $\alpha = \beta = \lambda_2$) yields a set of genus 2 on which J_0 is $< \lambda_2$.

Remark 3.4. – The present approach can be adapted to deal with a linear second order operator in divergence form with Hölder continuous coefficients. Unique continuation may fail in this situation. The nodal domain property derived in this way for such operators is however already known (*cf*. [2]).

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References

[1] Alessandrini G., Critical points of solutions to the p-Laplace equation in dimension two, Bull. Un. Mat. Ital. 7 (1987) 239–246.

[2] Alessandrini G., On Courant's nodal domain theorem, Forum Math. 10 (1998) 521–532.

[3] Anane A., Étude des valeurs propres et de la résonnance pour l'opérateur p-Laplacien, Thèse, Université libre de bruxelles, 1987; See also C. R. Acad. Sci. Paris, Série I 305 (1987) 725–728.

[4] Anane A., Tsouli N., On the second eigenvalue of the p-Laplacian, in: Nonlinear Partial Differential Equations, Benkirane A., Gossez J.-P. (Eds.), Pitman Res. Notes in Math. 343, 1996, pp. 1–9.

[5] Azizieh C., Méthodes variationnelles et spectre du p-Laplacien, Mém. Lic., Université libre de Bruxelles, 1997.

[6] Cuesta M., De Figueiredo D., Gossez J.-P., The beginning of the Fučik spectrum for the p-Laplacian, J. Differ. Eq. 159 (1999) 212–238.

[7] Courant R., Hilbert D., Methods of Mathematical Physics, Intersience, New York, 1943.

[8] De Figueiredo D., Gossez J.-P., On the first curve of the Fučik spectrum of an elliptic operator, Differ. Integ. Eq. 7 (1994) 1285–1302.

[9] Dibenedetto E., $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal., TMA 7 (1983) 827–850.

[10] Fučik S., Nečas J., Souček J., Souček V., Spectral Analysis of Nonlinear Operators, Lect. Notes Math. 346 Springer-Verlag, 1973.

[11] Gelbaum B., Olmsted J., Counterexamples in Analysis, Holden-Day, Oakland, 1964.

[12] Lindqvist P., On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990) 157–164; addendum, Proc. Amer. Math. Soc. 116 (1992) 583–584.

[13] Pleijel A., Remarks on Courant's nodal line theorem, Commun. Pure Appl. Math. 9 (1956) 543–550.

[14] Manfredi J., p-harmonic functions in the plane, Proc. Amer. Math. Soc. 103 (1988) 473–479.

[15] Martio O., Counterexamples for unique continuation, Manusc. Math. 60 (1988) 21–47.

[16] Vazquez J.L., A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.

[17] Véron L., Première valeur propre non nulle du p-Laplacien et équations quasi linéaires elliptiques sur une variété riemannienne compacte, C. R. Acad. Sci. Paris, Série I 314 (1992) 271–276.