Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication

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Summary: We study the difficulty of discriminating between an arbitrary quantum channel and a "replacer" channel that discards its input and replaces it with a fixed state.[1] The results obtained here generalize those known in the theory of quantum hypothesis testing for binary state discrimination. We show that, in this particular setting, the most general adaptive discrimination strategies provide no asymptotic advantage over non-adaptive tensor-power strategies. This conclusion follows by proving a quantum Stein's lemma for this channel discrimination setting, showing that a constant bound on the Type I error leads to the Type II error decreasing to zero exponentially quickly at a rate determined by the maximum relative entropy registered between the channels. The strong converse part of the lemma states that any attempt to make the Type II error decay to zero at a rate faster than the channel relative entropy implies that the Type I error necessarily converges to one. We then refine this latter result by identifying the optimal strong converse exponent for this task. As a consequence of these results, we can establish a strong converse theorem for the quantum-feedback-assisted capacity of a channel, sharpening a result due to Bowen. Furthermore, our channel discrimination result demonstrates the asymptotic optimality of a non-adaptive tensor-power strategy in the setting of quantum illumination, as was used in prior work on the topic. The sandwiched Rényi relative entropy, [16, 27], is a key tool in our analysis.

Background: Quantum channel discrimination is a natural extension of a basic problem in quantum hypothesis testing, that of distinguishing between the possible states of a quantum system. In an i.i.d. binary state discrimination problem, the discriminator is provided with n quantum systems in the state $\rho^{\otimes n}$ or $\sigma^{\otimes n}$, and the task is to apply a binary measurement $\{Q_n, I^{\otimes n} - Q_n\}$ to these n systems, with $0 \leq Q_n \leq I^{\otimes n}$. One is then concerned with two kinds of error probabilities: $\alpha_n(Q_n) \equiv \mathrm{Tr}\left\{(I^{\otimes n} - Q_n)\rho^{\otimes n}\right\}$, the probability of incorrectly rejecting the null hypothesis, the Type I error, and $\beta_n(Q_n) \equiv \mathrm{Tr}\left\{Q_n \sigma^{\otimes n}\right\}$, the probability of incorrectly rejecting the alternative hypothesis, the Type II error. One studies the asymptotic behaviour of α_n and β_n as $n \to \infty$, expecting there to be a trade-off between minimising α_n and minimising β_n.

In quantum channel discrimination, we have a quantum channel with input system A and output system B, and we are given that the channel is described by either the completely positive trace-

[1] A more detailed version of this work is available on the arXiv, [5]. We propose to the QIP program committee that it would be natural to merge our submission with the related work of Hayashi and Tomamichel, [10], because their results and ours can be combined to strengthen the results of both papers. In our work, we optimise over an arbitrary choice of input state and adaptive maps but the alternative hypothesis is restricted to a replacer channel with a memoryless, tensor-power structure. In the Hayashi-Tomamichel paper, the input state is restricted to be tensor-power, i.e., of the form $\psi_{RA}^{\otimes n}$, but they optimise over states of the more general form $\psi_R \otimes \sigma_{B^n}$ in the alternative hypothesis. One can combine our results with theirs to obtain a Stein's Lemma and strong converse exponent where one optimizes over both the choice of input, arbitrary adaptive maps, and replacer channels with general structure. The main conclusion of our work persists in this more general setting: the adaptive maps are not necessary and a tensor-power discrimination strategy suffices in these regimes.

preserving map $\mathcal{N}_1$ or $\mathcal{N}_2$. We assume that n uses of the channel are described by either $\mathcal{N}_1^{\otimes n}$ or $\mathcal{N}_2^{\otimes n}$. A non-adaptive discrimination strategy for n uses of the channel consists of feeding an input state $\psi_{R_n A^n}$ into the n-fold tensor-product channel, and then performing a binary measurement $\{Q_n, I - Q_n\}$ on the output, which is either $\mathcal{N}_1^{\otimes n}(\psi_{R_n A^n})$ or $\mathcal{N}_2^{\otimes n}(\psi_{R_n A^n})$. Here, R_n is an ancilla system on which the channel acts trivially. When an adaptive strategy is used, the output of the first k uses of the channel can be used to prepare the input for the $(k+1)$-th use.

Overview of Results: We specialise to the case of discriminating between an arbitrary quantum channel $\mathcal{N}$ and a replacer channel $\mathcal{R}$ that discards its input and replaces it with a fixed state σ. This scenario interpolates between the fully understood case of state discrimination and the still open problem of general quantum channel discrimination. Here we consider the setup in which the null hypothesis is the use of $\mathcal{N}$ and the alternative hypothesis is the use of the replacer channel.

In asymmetric hypothesis testing, one fixes a constraint on the Type I error, say, and then seeks to minimise the Type II error. The central result in the asymptotic setting for binary state discrimination is the quantum Stein's lemma, [11, 20]. The direct part of the lemma states that for any constant bound on the Type I error, there exists a sequence of measurements $\{Q_n, I^{\otimes n} - Q_n\}$ that meets this constraint and is such that the Type II error decreases to zero exponentially fast with a decay exponent given by the quantum relative entropy $D(\rho\|\sigma)$, defined as in [26, 11]. We also assume that for each k and $\psi_{R_k A^k}$, $\operatorname{supp}\left(\mathcal{N}_{A\to B}^{\otimes k}\left(\psi_{R_k A^k}\right)\right) \subseteq \operatorname{supp}(\psi_{R_k} \otimes \sigma_B^{\otimes k})$ in order to avoid trivial counterexamples to the strong converse property.

Theorem 1 *Let $\varepsilon \in (0,1)$ be a fixed constant. Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be an arbitrary quantum channel and let $\mathcal{R} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be the replacer quantum channel $\mathcal{R}(X_A) = \operatorname{Tr}\{X_A\}\sigma_B$, for some fixed density matrix σ_B. Let $\beta_\varepsilon^{\mathrm{ad}}\left(\mathcal{N}^{\otimes n}\|\mathcal{R}^{\otimes n}\right)$ denote the optimal Type II error for discriminating between $\mathcal{N}^{\otimes n}$ and $\mathcal{R}^{\otimes n}$, obtained by optimising over adaptive strategies for which the Type I error is less than ε. Then the channel version of Stein's lemma holds, i.e., for any $\varepsilon \in (0,1)$,*

$$\lim_{n\to\infty} -\frac{1}{n}\log \beta_\varepsilon^{\mathrm{ad}}\left(\mathcal{N}^{\otimes n}\middle\|\mathcal{R}^{\otimes n}\right) = \sup_{\psi_{RA}} D\left(\mathcal{N}_{A\to B}\left(\psi_{RA}\right)\|\psi_R \otimes \sigma_B\right). \tag{0.1}$$

This result has implications for the theory of quantum illumination. Building on prior work in [22, 23], Lloyd *et al.* show how the use of entangled photons can provide a significant improvement over unentangled light when detecting the presence of an object [14, 24]. The goal in quantum illumination is to determine whether a distant object is present or not by employing quantum light along with a quantum detection strategy. If the object is present (the alternative hypothesis), then the signal beam is reflected off the object and returns to the transmitter. The resulting state is described by $(\mathcal{N}_S \otimes \mathrm{id}_I)(\psi_{SI})$, where $\mathcal{N}_S$ describes the noise characteristics of the reflection channel. If the object is not present (the null hypothesis), then the signal mode is lost and is replaced by a thermal state θ_S, so that the joint state becomes $\theta_S \otimes \psi_I$. Clearly, this is an instance of the replacer channel. However, our results do not apply to this setting if one takes the null and alternative hypotheses in the natural way suggested above. Our results apply to the alternative scenario in which the transmitter and receiver are in different locations and the roles of the null and alternative hypotheses are switched.

Implicit in prior analyses on quantum illumination is the assumption that a tensor-power, non-adaptive strategy is optimal. Our results support this assumption (at least in the particular setting of asymmetric hypothesis testing described above) by showing that no asymptotic advantage is provided by instead using an adaptive strategy for quantum channel discrimination.[2]

[2]Strictly speaking, the results in our paper apply to finite-dimensional systems, whereas the quantum illumination

As a refinement of the quantum Stein's lemma, one can study the optimal Type I error given that the Type II error decays with a given exponential speed. In the case of binary state discrimination, one is then interested in the asymptotics of the optimal Type I error

$$\alpha_{n,r} \equiv \min\left\{\alpha_n(Q_n) : 0 \leq Q_n \leq I,\, \beta_n(Q_n) \leq 2^{-nr}\right\}, \tag{0.2}$$

with $r > 0$ a constant. In the "direct domain," when $r < D(\rho\|\sigma)$, $\alpha_{n,r}$ also decays with an exponential speed, as was shown in [19]. The exact decay rate is determined by the quantum Hoeffding bound theorem [9, 18, 1] as

$$\lim_{n\to\infty} -\frac{1}{n}\log\alpha_{n,r} = H_r(\rho\|\sigma) \equiv \sup_{0<\alpha<1}\frac{\alpha-1}{\alpha}(r - D_\alpha(\rho\|\sigma)), \tag{0.3}$$

where D_α is a quantum Rényi relative entropy and $H_r(\rho\|\sigma)$ is the (direct) Hoeffding divergence of ρ and σ. On the other hand, in the "strong converse domain," when $r > D(\rho\|\sigma)$, $\alpha_{n,r}$ goes to 1 exponentially fast [20, 17]. The rate of this convergence has been determined in [8, pages 80-81] in terms of the limit of post-measurement Rényi divergences. A "single-letter" expression has been obtained recently in [15] using the sandwiched quantum Rényi relative entropy, $\widetilde{D}_\alpha$, [16, 27]:

$$\lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}) = \sup_{\alpha>1}\frac{\alpha-1}{\alpha}(r - \widetilde{D}_\alpha(\rho\|\sigma)), \tag{0.4}$$

A corresponding result also holds for channel discrimination; we let $\alpha_{n,r}^{\mathrm{ad}}$ denote the optimal Type I error when constrained to use adaptive strategies for which the Type II error satisfies $\beta_n < 2^{-nr}$. The following theorem identifies the strong converse exponent, the optimal rate at which the success probability $1-\alpha_{n,r}^{ad}$ decreases to zero when $r > \sup_{\psi_{RA}} D\left(\mathcal{N}_{A\to B}\left(\psi_{RA}\right)\|\psi_R\otimes\sigma_B\right)$.

Theorem 2 *Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be an arbitrary quantum channel and let $\mathcal{R} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be the replacer quantum channel $\mathcal{R}(X) = \mathrm{Tr}\{X\}\sigma_B$, for some fixed density matrix σ_B. For any $r > 0$,*

$$\lim_{n\to+\infty} -\frac{1}{n}\log(1-\alpha_{n,r}^{\mathrm{ad}}) = \sup_{\alpha>1}\inf_{\psi_{RA}}\frac{\alpha-1}{\alpha}\left[r - \widetilde{D}_\alpha\left(\mathcal{N}_{A\to B}(\psi_{RA})\|\psi_R\otimes\sigma_B\right)\right] \tag{0.5}$$

$$= \inf_{\psi_{RA}}\sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left[r - \widetilde{D}_\alpha\left(\mathcal{N}_{A\to B}(\psi_{RA})\|\psi_R\otimes\sigma_B\right)\right]. \tag{0.6}$$

One consequence of Theorem 1 is a strong converse theorem for the quantum-feedback-assisted classical capacity of a quantum channel. Bowen, [4], proved that a noiseless quantum feedback channel does not increase the entanglement-assisted capacity of a noisy channel, proving that the quantum-feedback-assisted capacity of a channel $\mathcal{N}$ is equal to $I(\mathcal{N})$, its entanglement-assisted capacity, [2, 3, 12]. However, Bowen's result did not exclude the possibility of a trade-off between the communication rate and the error probability; our result sharpens Bowen's, strengthens the main result of [7], and generalizes [21, Theorem 7] to the quantum case. The approach taken is inspired by that used in [17] and [21] and later used to prove several strong converse theorems for quantum channels [13, 27, 7, 25]; our result relies upon the properties of the sandwiched Rényi relative entropy, [16, 27], and related completely bounded norms, [6].

Theorem 3 *For any sequence of quantum-feedback-assisted codes for a channel $\mathcal{N}$ with rate $C > I(\mathcal{N})$, the success probability decays exponentially to zero as $n\to\infty$.*

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protocols apply to infinite-dimensional, albeit finite-energy, systems. Given that our analysis never has any dimension dependence, it should be a straightforward exercise to extend our results to infinite-dimensional systems.

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