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Relating two semantic descriptions of functional logic programs

(Work in progress) [1]

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Abstract

A distinctive feature of modern functional logic languages like Toy or Curry is the possibility of programming non-strict and non-deterministic functions with call-time choice semantics. For almost ten years the CRWL framework [6,7] has been the only formal setting covering all these semantic aspects. But recently [1] an alternative proposal has appeared, focusing more on operational aspects. In this work we investigate the relation between both approaches, which is far from being obvious due to the wide gap between both descriptions, even at syntactical level.

Key words: Functional Logic Programming, Operational Semantics, Declarative Semantics

1 Introduction

In its origin functional logic programming (FLP) did not consider non-deterministic functions (see [8] for a survey of that era). Inspired in those ancestors and in Hussmann's work [12], the *CRWL* framework [6,7] was proposed in 1996 as a formal basis for *FLP* having as main notion that of non-strict non-deterministic function with call-time choice semantics. At the operational level, modern FLP has been mostly influenced by the notions of definitional trees [2] and needed narrowing [3].

Both approaches –*CRWL* and needed narrowing– coexist with success in the development of FLP (see [15,9] for recent respective surveys). It is tacitly accepted in the FLP community that they essentially speak of the same 'programming stuff', realized by systems like Curry [11] or Toy [14], but up to now they remain technically disconnected. One of the reasons has been that the formal setting for needed narrowing is classical rewriting, which is known to be unsound for call-time choice, which requires sharing.

But recently [1] a new operational formal description of FLP has been proposed, coping with narrowing, residuation, laziness, non-determinism and sharing, for a flat language, called here *FLC* for its proximity to *Flat Curry* [10].

[1] This work has been partially supported by the spanish projects TIN2005-09207-C03-03 and S-0505/TIC/0407.

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There is a long distance in the formal aspects of the two approaches, each having its own merit: *CRWL* provides a concise and clear way for giving logical semantics to programs, with a high level of abstraction and a syntax close to the user, while *FLC* and its semantics are closer to computations and concrete implementations with details about variable bindings representation.

The goal of our work is to relate both in a technically precise manner. In this way, some known or future results obtained for one of them could be applied to the other.

The rest of the paper is organized as follows. Sections 2 and 3 present the essentials of *CRWL* and *FLC* needed to relate them. Section 4 sets some restrictions assumed in our work and gives an overview of the structure of our results. Section 5 relates *CRWL* to $CRWL_{FLC}$, a new intermediate formal description. Section 6 is the main part of the work and studies the relation between $CRWL_{FLC}$ and *FLC*. Section 7 gives some conclusions. Proofs are mostly omitted and some of them are still under development [3].

2 The *CRWL* Framework: a Summary

We assume a signature $\Sigma = CS \cup FS$, where CS (FS) is a set of constructor symbols (defined function symbols) each of them with an associated arity; we sometimes write CS^n (FS^n resp.) to denote the set of constructor (function) symbols of arity n. As usual notations write $c, d \ldots$ for constructors, $f, g \ldots$ for functions and $x, y \ldots$ for variables taken from a numerable set $\mathcal{V}$.

The set of expressions Exp is defined as usual: $e ::= x \mid h(e_1, \ldots, e_n)$, where $h \in CS^n \cup FS^n$ and $e_1, \ldots, e_n \in Exp$. The set of *constructed* terms is defined analogously but with h restricted to CS, i.e., function symbols are not allowed. The intended meaning is that Exp stands for evaluable expressions while $CTerm$ are data terms. We will also use the extended signature $\Sigma_\perp = \Sigma \cup \{\perp\}$, where $\perp$ is a new constant (0-arity constructor) that stands for *undefined value*. Over this signature we build the sets $Exp_\perp$ and $Exp_\perp$ in the natural way. The set $CSubst$ ($CSubst_\perp$ resp.) stands for substitutions or mappings from $\mathcal{V}$ to $CTerm$ ($CTerm_\perp$ resp.). Both kind of substitutions will be written as $\theta, \sigma \ldots$. The notation $\sigma\theta$ denotes the composition of substitutions in the usual way. The notation $\overline{o}$ stands for tuples of any of the previous syntactic constructions.

$$\textbf{(B)}\ \frac{}{e \to \perp} \qquad \textbf{(RR)}\ \frac{}{x \to x}\quad x \in \mathcal{V}$$

$$\textbf{(DC)}\ \frac{e_1 \to t_1\ \ldots\ e_n \to t_n}{c(e_1, \ldots, e_n) \to c(t_1, \ldots, t_n)} \quad c \in CS^n,\ t_i \in CTerm_\perp$$

$$\textbf{(DF)}\ \frac{e_1 \to t_1\theta\ \ldots\ e_n \to t_n\theta \quad e\theta \to t}{f(e_1, \ldots, e_n) \to t} \quad \begin{array}{l}(f(t_1, \ldots, t_n) = e) \in \mathcal{P} \\ \theta \in CSubst_\perp\end{array}$$

Fig. 1. Rules of *CRWL*

The original *CRWL* logic introduces strict equality as a built-in constraint and program-rules optionally contain a sequence of equalities as condition. In the current work, as *FLC* does not consider built-in equality, we restrict the class of programs. Then a *CRWL*-program $\mathcal{P}$ is a

[3] See http://gpd.sip.ucm.es/juanrh/fullprole06.pdf for an extended version of this work.

set of rules of the form: $f(\bar{t}) = e$, where $f \in FS^n$, $\bar{t}$ is a linear (without multiple occurrences of the same variable) n-tuple of c-terms and $e \in Exp$.

Rules of *CRWL* (without equality) are presented in Figure 1. Rule (1) allows any expression to be undefined or not evaluated (non-strict semantics). Rule (4) is a proper reduction rule: for evaluating a function call it uses a compatible program-rule, makes the parameter passing (by means of a substitution θ) and then reduces the body. This logic proves *approximation* statements of the form $e \to t$, where $e \in Exp_\perp$ and $t \in CTerm_\perp$. Given a program $\mathcal{P}$, the denotation of an expression e with respect to *CRWL* is defined as $[\![e]\!]^{\mathcal{P}}_{CRWL} = \{t \mid e \to t\}$.

3 The *FLC* Language and its Natural Semantics

The language *FLC* considered in [1] is a convenient –although somehow low-level– format to which functional logic programs like those of Curry or Toy can be transformed (not in a unique manner). This transformation embeds important aspects of the operational procedure of FLP languages, like are definitional trees and inductive sequentiality.

The syntax of *FLC* is given in Fig. 2. Notice that each function symbol f has exactly one definition rule $f(x_1, \ldots, x_n) = e$ with distinct variables $x_1, \ldots, x_n$ as formal parameters. All non-determinism ix expressed by the use of *or* choices in right-hand sides and also all pattern matching has been moved to right-hand sides by means of nesting of *(f)case* expressions. *Let* bindings are a convenient way to achieve sharing.

$$
\begin{array}{ll}
\textit{Programs}:\ P \ ::=\ D_1 \ldots D_m & \\
\textit{Function definitions}:\ D \ ::=\ f(x_1, \ldots, x_n) = e & \\
\textit{Expressions} & \\
e \ ::=\ x & \text{(variable)} \\
\quad\mid\ c(e_1, \ldots, e_n) & \text{(constructor call)} \\
\quad\mid\ f(e_1, \ldots, e_n) & \text{(function call)} \\
\quad\mid\ case\ e\ of\ \{p_1 \to e_1; \ldots; p_n \to e_n\} & \text{(rigid case)} \\
\quad\mid\ fcase\ e\ of\ \{p_1 \to e_1; \ldots; p_n \to e_n\} & \text{(flexible case)} \\
\quad\mid\ e_1\ or\ e_2 & \text{(disjunction)} \\
\quad\mid\ let\ x_1 = e_1, \ldots, x_n = e_n\ in\ e & \text{(let binding)} \\
\textit{Patterns}:\ p \ ::=\ c(x_1, \ldots, x_n) = e &
\end{array}
$$

Fig. 2. Syntax for FLC programs

An additional *normalization step* over programs is assumed in [1]. In normalized expressions each constructor o function symbol appears applied only to distinct variables. This can be achieved via let-bindings. The normalization of e is written as e^*.

In [1] two operational semantics are given to *FLC*: a natural (*big-step*) semantics in the style of Launchbury's semantics [13] for lazy evaluation (with sharing) for functional programming, and a small step semantics. CRWL itself being a big-step semantics, it seems more adequate to compare it to the natural semantics of [1], which is shown [4] in Fig. 3. It consists of a set of rules for a relation $\Gamma : e \Downarrow \Delta : v$, indicating that one of the possible evaluations of e ends

[4] Some of the rules are skipped, because they are not needed here due to some restrictions to be imposed in the next section.

up with the head normal form (variable or constructor rooted) v. Γ, Δ are *heaps* consisting of bindings $x \mapsto e$ for variables. An initial configuration has the form $[] : e$.

Rule		Condition
(VarCons)	$\Gamma[x \mapsto t] : x \Downarrow \Gamma[x \mapsto t] : t$	t constructor-rooted
(VarExp)	$\dfrac{\Gamma[x \mapsto e] : e \Downarrow \Delta : v}{\Gamma[x \mapsto e] : x \Downarrow \Delta[x \mapsto v] : v}$	e not constructor-rooted, $e \neq x$
(Val)	$\Gamma : v \Downarrow \Gamma : v$	v constructor-rooted or variable with $\Gamma[v] = v$
(Fun)	$\dfrac{\Gamma : e\rho \Downarrow \Delta : v}{\Gamma : f(\overline{x_n}) \Downarrow \Delta : v}$	$f(\overline{x_n}) = e \ \in P$ and $\rho = \{\overline{y_n \mapsto x_n}\}$
(Let)	$\dfrac{\Gamma[\overline{y_k \mapsto e_k\rho}] : e \Downarrow \Delta : v}{\Gamma : let\ \{\overline{x_k = e_k}\}\ in\ e \Downarrow \Delta : v}$	$\rho = \{\overline{x_k \mapsto y_k}\}$ and $\overline{y_k}$ are fresh variables
(Or)	$\dfrac{\Gamma : e_i \Downarrow \Delta : v}{\Gamma : e_1 or\ e_2 \Downarrow \Delta : v}$	$i \in \{1, 2\}$
(Select)	$\dfrac{\Gamma : e \Downarrow \Delta : c(\overline{y_n}) \qquad \Delta : e_i\rho \Downarrow \Theta : v}{\Gamma : (f)case\ e\ of\ \{\overline{p_k \mapsto e_k}\}\ in\ e \Downarrow \Theta : v}$	$p_i = c(\overline{x_n})$ and $\rho = \{\overline{x_n \mapsto y_n}\}$

Fig. 3. Natural Semantics for *FLC*

4 *CRWL* vs. *FLC*: Working Plan

In order to establish the relation between *CRWL* and *FLC* (in Section 6) firstly we adapt *CRWL* to the syntax of *FLC*. For this purpose we introduce the rewriting logic $CRWL_{FLC}$ as a variant of *CRWL* with specific rules for managing *let*, *or* and *case* expressions.

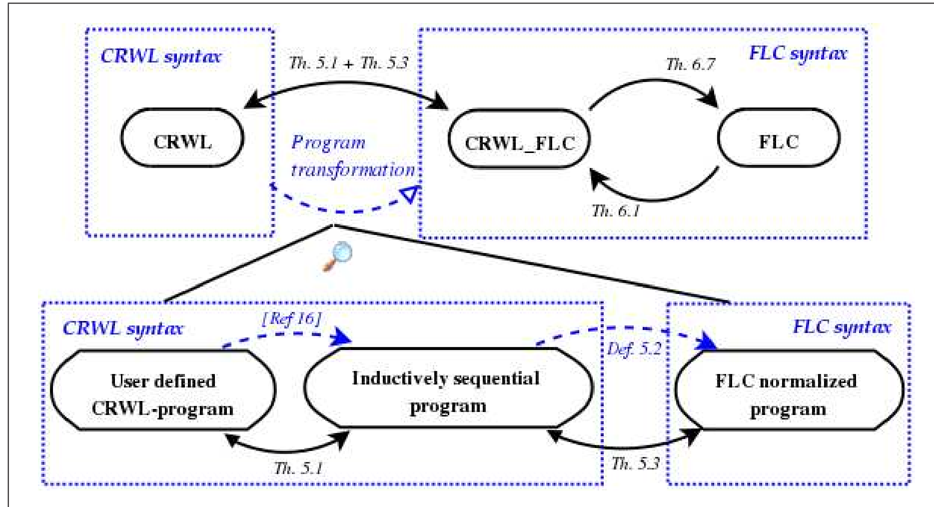

Fig. 4. Proof's plan

The relation between *CRWL* and *FLC* is established through this intermediate logic. The working plan is sketched in Figure 4. Given a pair program/expression in *CRWL* we transform them into *FLC*-syntax and study the semantic equivalence of both versions of *CRWL* (Theorems 5.1 and 5.3). Then we focus on the equivalence of *FLC* with respect to $CRWL_{FLC}$ in a common

syntax context (Theorems 6.7 and 6.1). *FLC* and *CRWL* are very different frameworks from the syntactical and the semantical points of view. The advantage of splitting the problem is that on one hand both versions of *CRWL* are very close from the point of view of semantics; on the other hand $CRWL_{FLC}$ and *FLC* share the same syntax. The syntactic transformation and its correctness will be explained in Sect. 5.1.

There are important differences between *FLC* and $CRWL_{FLC}$ that makes not easy its relation. The heaps used in *FLC* for storing variable bindings have not any (explicit) correspondence in *CRWL*. Another important difference is that the first obtains *head normal forms* for expressions, while the second is able to obtain any value of the denotation of an expression (in particular a normal form if it exists).

Differences do not end here. There are still two important points that enforces us to take some decisions: (1) *FLC* performs narrowing while *CRWL* is a pure rewriting relation. In this paper we address this inconvenience by considering only the rewriting fragment of *FLC*. Narrowing acts in *FLC* either due to the presence of logical variables in expressions to evaluate or because of the use of extra variables in program rules (those not appearing in left-hand sides). So we can isolate the rewriting fragment by excluding this kind of variables throughout this work. (2) The other difference is due to the fact that *FLC* allows recursive *let* constructions. There is not a general consensus about the semantics of such constructions in a non-deterministic context (it is not clear if *sharing* must be done or not). Due to the divergence of opinions on the matter and for the sake of simplicity, we exclude recursive *let*'s from the language in this work. Once this decision is taken it is not difficult to see that a *let* with multiple variable bindings may be expressed as a sequence of nested *let*'s, each with a unique binding. For simplicity and without loss of generality we will consider only this kind of *let*'s.

We assume from now on that programs and expressions fulfil the conditions imposed in (1) and (2).

5 The proof calculus $CRWL_{FLC}$

The rewriting logic $CRWL_{FLC}$ preserves the main features of *CRWL* from a semantical point of view, but it uses the *FLC*-syntax for expressions and programs. In particular it allows *let*, *case* and *or* constructs, but like *CRWL* it proves statements of the form $e \to t$ where $t \in CTerm_\perp$.

Rules of $CRWL_{FLC}$ are presented in Figure 5. The first three ones **(B)**, **(RR)** and **(DC)** are directly incorporated from *CRWL*. Rules **(CASE)**, **(OR)** and **(LET)**) has also a clear reading. Finally, rule **(DF)** is a simplified version of the corresponding in *CRWL*, as now we can guarantee that any function call in a derivation can only use c-terms as arguments. This is easy to check: the initial expression to reduce is in normalized form (arguments are all variables) and the substitutions applied by the calculus (in rules **(DF)**, **(CASE)** and **(LET)**) can only introduce c-terms. Given a program $\mathcal{P}$ the denotation of an expression e with respect to $CRWL_{FLC}$ is defined as $[\![e]\!]^{\mathcal{P}}_{CRWL_{FLC}} = \{t \mid e \to t\}$.

5.1 Relation between $CRWL_{FLC}$ and *CRWL*

We obtain here an equivalence result for $CRWL_{FLC}$ and *CRWL*. A skeleton of the proof is given in the zoomed part of Fig 4. It is based on a program transformation from *CRWL*-syntax (user syntax) to *FLC*-syntax. This translation is assumed but not made explicit in [1]. But we need here to make it more precise, since otherwise *CRWL* and $CRWL_{FLC}$ will remain technically disconnected. For technical convenience we split the transformation in two parts: first, and still

$$\textbf{(B)}\ \frac{}{e \rightarrow \bot} \qquad \textbf{(RR)}\ \frac{}{x \rightarrow x} \quad x \in \mathcal{V}$$

$$\textbf{(DC)}\ \frac{e_1 \rightarrow t_1 \ \ldots \ e_n \rightarrow t_n}{c(e_1, \ldots, e_n) \rightarrow c(t_1, \ldots, t_n)} \quad c \in CS^n,\ t_i \in CTerm_\bot$$

$$\textbf{(DF)}\ \frac{e\theta \rightarrow t}{f(\overline{t}) \rightarrow t} \quad (f(\overline{y}) = e) \in \mathcal{P},\ \theta = [\overline{y/t}]$$

$$\textbf{(CASE)}\ \frac{e \rightarrow c(\overline{t}) \qquad e_i\theta \rightarrow t}{case\ e\ of\ \{\overline{p_i \rightarrow e_i}\} \rightarrow t} \quad \begin{array}{l} p_i = c(\overline{x}) \text{ for some } i \\ \theta = [\overline{y/t}] \end{array}$$

$$\textbf{(OR)}\ \frac{e_i \rightarrow t}{e_1\ or\ e_2 \rightarrow t} \quad \text{for some } i \in \{1, 2\}$$

$$\textbf{(LET)}\ \frac{e' \rightarrow t' \quad e[x/t'] \rightarrow t}{let\ \{x = e'\}\ in\ e \rightarrow t}$$

Fig. 5. Rules of $CRWL_{FLC}$

within *CRWL*-syntax, we transform P into another program P' which is *inductively sequential* ([2,9]), except for a function *or* [5], defined by the two rules $X\ or\ Y = X$ and $X\ or\ Y = Y$. The function *or* concentrates all the non-sequentiality (hence, all the indeterminism) of functions in right-hand sides. We speak of 'inductively sequential with *or* (IS_{or}) programs'. Alternatively, programs can be transformed into overlapping inductively sequential format (see [9]), where a function might have several rules with the same left-hand side (as happens with the rules of *or*). Both formats are easily interchangeable. Such kind of transformations are well-known in functional logic programming. In the *CRWL* setting, a particular transformation has been proposed in [16], where it is proved the following result:

Theorem 5.1 *Let P be a CRWL-program and e an expression.*
Then $[\![e]\!]^P_{CRWL} = [\![e]\!]^{P'}_{CRWL}$ where P' is the IS_{or} transformed program of P.

Now, to transform IS_{or} programs into (normalized) *FLC*-syntax is not difficult, by simply mimicking the inductive structure of function definitions by means of (possibly nested) *case* expressions. The following algorithm performs it.

Definition 5.2 [***FLC*-transformation**] Let P be an IS_{or} *CRWL*-program.

A) **Transformation of sets of rules.** Let $\mathcal{Q} = \{(f(\overline{t}_1) \rightarrow e_1), \ldots, (f(\overline{t}_n) \rightarrow e_n)\}$ be a set of rules for a function f in P ($\mathcal{Q} \subseteq \mathcal{P}_f$) and $f(\overline{s})$ a pattern compatible with $\mathcal{Q}$ (i.e., it subsumes the left-hand side of all the rules in $\mathcal{Q}$). The expression $\Delta(\mathcal{Q}, f(\overline{s}))$ is defined according to the following (exhaustive, due to inductive sequentiality) possibilities:

(i) **There is an inductive position** (if several, choose any) in $f(\overline{s})$ wrt $\mathcal{Q}$, i.e., a position u occupied by a variable X in ($f(\overline{s})$) and by constructor symbols $c_1, \ldots, c_k$ in the left-hand sides of rules of $\mathcal{Q}$. For each $i \in \{1, \ldots, k\}$ we write $\mathcal{Q}_{c_i}$ for the set of rules in $\mathcal{Q}$ having the constructor c_i at position u, and $\overline{s}_{c_i}$ for $\overline{s}[X/c_i(\overline{Y})]$, where $\overline{Y}$ are fresh variables. Then

[5] Not to be confused with boolean disjunction; *or* is written as `//` in Toy and `?` in Curry.

$\Delta(\mathcal{Q}, f(\overline{s})) = case\ X\ of\ \{c_1 \rightarrow \Delta(\mathcal{Q}_{c_1}, f(\overline{s}_{c_1})); \ldots ; c_k \rightarrow \Delta(\mathcal{Q}_{c_k}, f(\overline{s}_{c_k}))\}$

(ii) There is **no inductive position** in $f(\overline{s})$ wrt $\mathcal{Q}$. It should be the case that $\mathcal{Q} = \{f(\overline{s}) = e\}$. Then: $\Delta(\mathcal{Q}, f(\overline{s})) = e^*$, where e^* is the normalization of e (see sect. 3).

B) Transformation of whole programs. The (normalized) *FLC*-transformation of P is

$$\hat{P} = \bigcup_{f \in FS} \{f(\overline{X}) = \Delta(\mathcal{P}_f, f(\overline{X}))\}$$

An example of the two program transformation steps (first to IS_{or}, then to *FLC*) is given in Fig. 6. Notice that the final *FLC*-program does not contain rules for *or*, since it is included in the syntax of *FLC*, and there is a specific rule governing its semantics in the $CRWL_{FLC}$-calculus.

Constructor symbols: $0 \in CS^0$, $s \in CS^1$

Source CRWL-program

```
f(0,Y) = s(Y)
f(X,0) = X
f(s(X),s(Y)) = s(f(X,Y))
```

Transformed IS_{or} CRWL-program

```
f(X,Y) = f1(X,Y) or f2(X,Y)
f1(0,Y) = s(Y)
f1(s(X),s(Y)) = s(f(X,Y))
f2(X,0) = X
X or Y = X      X or Y = Y
```

Transformed normalized FLC-program

```
f(X,Y) = f1(X,Y) or f2(X,Y)
f1(X,Y) = case X of {  0 → s(Y);
                      s(X1) → case Y of { s(Y1) → let U=f(X1,Y1)
                                                   in s(U)} }
f2(X,Y) = case Y of {0 → X}
```

Fig. 6. Transformation from *CRWL* to *FLC* syntax

The following equivalence result states the correctness of the transformation.

Theorem 5.3 *Let P be an IS CRWL-program and, e an CRWL-expression, and $\hat{P}, \hat{e}$ their FLC-transformations. Then $[\![e]\!]^P_{CRWL} = [\![\hat{e}]\!]^{\hat{P}}_{CRWL_{FLC}}$.*

6 Relation between $CRWL_{FLC}$ and *FLC*

We need some more technical preliminaries and notations:

- $dom(\Gamma)$: The set of variables bound in the heap Γ.
- $var(e)$: The set of free variables in the expression e.
- *Valid heap*: A heap Γ is valid if $[] : e \Downarrow \Gamma : v$ for some e, v, i.e., Γ is reachable in a computation.
- $ligs(\Gamma, e)$: The bindings of a valid heap Γ can be ordered in a way such that $\Gamma = [x_1 \mapsto e_1, \ldots, x_n \mapsto e_n]$ where each e_i does not depend on x_j iff $j >= i$. That is because recursive bindings are forbidden. Then we define $ligs([x_1 \mapsto e_1, \ldots, x_n \mapsto e_n], e) =_{def} let\ \{x_1 = e_1\}\ in \ldots let\ \{x_n = e_n\}\ in\ e$.
- $[\![\Gamma, e]\!]$: Expresses the set of terms we can reach in $CRWL_{FLC}$, applying a heap to an expression. Formally, $[\![\Gamma, e]\!] =_{def} [\![ligs(\Gamma, e)]\!]_{CRWL_{FLC}} = \{t \mid ligs(\Gamma, e) \rightarrow t\}$.
- $norm2(e)$: If $e^* = let\ \{x_1 = e_1\}\ in \ldots\ in\ let\ \{x_n = e_n\}\ in\ e'$, then $norm2(e) = ([x_1 \mapsto e_1, \ldots, x_n \mapsto e_n], e')$. It is a kind of reverse of *ligs*.

6.1 Completeness of $CRWL_{FLC}$ wrt FLC

Our **main result** concerning the completeness of $CRWL_{FLC}$ with respect to *FLC* is:

Theorem 6.1 (From *FLC* to $CRWL_{FLC}$) *If* $\Gamma : e \Downarrow \Delta : v$, *then* $[\![\Delta, v]\!] \subseteq [\![\Gamma, e]\!]$.

Before proving it we must formulate some auxiliary results.

Lemma 6.2 *If* $[\![\Delta, x]\!] \subseteq [\![\Gamma, x]\!]$, *for all* $x \in var(e)$, *then* $[\![\Delta, e]\!] \subseteq [\![\Gamma, e]\!]$.

Theorem 6.3 *If* $\Gamma : e \Downarrow \Delta : v$, *then:*

(H) $[\![\Delta, x]\!] \subseteq [\![\Gamma, x]\!]$, *for all* $x \in dom(\Gamma)$.

(R) $[\![\Delta, v]\!] \subseteq [\![\Delta, e]\!]$.

The property *(H)* tells us what happens with heaps, while *(R)* relates the results of the computation. The following Corollary is an immediate consequence of Lemma 6.2 and *(H)*.

Corollary 6.4 (H') *If* $\Gamma : e \Downarrow \Delta : v$, *then* $[\![\Delta, e]\!] \subseteq [\![\Gamma, e]\!]$, *for all* e *with* $var(e) \subseteq dom(\Gamma)$.

Now the proof of Theorem 6.1 becomes easy:

Proof. *(Theorem 6.1)*
Assume $\Gamma : e \Downarrow \Delta : v$. Then, by property *(R)* of Theorem 6.3 we have $[\![\Delta, v]\!] \subseteq [\![\Delta, e]\!]$, and by Corollary 6.4 (H') we have $[\![\Delta, e]\!] \subseteq [\![\Gamma, e]\!]$, because it must happen $var(e) \subseteq dom(\Gamma)$, because the FLC-derivation has succeeded. But then $[\![\Delta, v]\!] \subseteq [\![\Gamma, e]\!]$. □

Some additional conclusions can be extracted from Theorem 6.1, but to explain them we must introduce the concept of **shell** of an expression in a heap, which is defined in Figure 7.

$$|\Gamma : x| = \begin{cases} x & \text{if } \Gamma[x] = x \\ c(|\Gamma : x_1|, \ldots, |\Gamma : x_n|) & \text{if } \Gamma[x] = c(x_1, \ldots, x_n) \\ \bot & \text{in other case} \end{cases}$$

$$|\Gamma : c(x_1, \ldots, x_n)| = c(|\Gamma : x_1|, \ldots, |\Gamma : x_n|) \qquad |\Gamma : e| = \bot \text{ in other case}$$

Fig. 7. Shell of an expression in a heap

We can prove the following results involving shells:

Lemma 6.5 *If* $\Gamma : e \Downarrow \Delta : v$, *then* $|\Delta : v| \in [\![\Delta, v]\!]$.

Corollary 6.6 *If* $\Gamma : e \Downarrow \Delta : v$, *then* $|\Delta : v| \in [\![\Gamma, e]\!]$.

Proof. Assume $\Gamma : e \Downarrow \Delta : v$. Then by Lemma 6.5 we have $|\Delta : v| \in [\![\Delta, v]\!]$ and by Theorem 6.1 we have $[\![\Delta, v]\!] \subseteq [\![\Gamma, e]\!]$, so $|\Delta : v| \in [\![\Gamma, e]\!]$. □

6.2 Completeness of FLC wrt $CRWL_{FLC}$

Our **main result** concerning the completeness of *FLC* with respect to $CRWL_{FLC}$ is:

Theorem 6.7 (From $CRWL_{FLC}$ to *FLC*) *If* $e \to c(t_1, \ldots, t_n)$ *and* $(\Gamma, e') = norm2(e)$, *then* $\Gamma : e' \Downarrow \Delta : c(x_1, \ldots, x_n)$, *for some* $x_1, \ldots, x_n$ *verifying* $ligs(\Delta, x_i) \to t_i$ *for each* $i \in \{1, \ldots, n\}$.

$$\textbf{(Var*)}\ \frac{\Gamma : e \Downarrow \Delta : x}{\Gamma : e \Downarrow^* \Delta : x} \qquad \textbf{(RR*)}\ \frac{}{\Gamma : x \Downarrow^* \Gamma : x}$$

$$\textbf{(Cons*)}\ \frac{\Gamma : e \Downarrow \Delta : c(x_1, \ldots, x_n) \quad \Delta_i : x_i \Downarrow^* \Delta_{i+1} : t_i}{\Gamma : e \Downarrow^* \Delta_{n+1} : c(t_1, \ldots, t_n)} \qquad \begin{array}{l} i \in \{1, \ldots, n\} \\ \Delta_1 = \Delta \end{array}$$

Fig. 8. Rules of $\Downarrow^*$

On the other hand we could follow an alternative approach consisting on defining a new relation based on the $\Downarrow$ relation of *FLC*, that evaluates expressions beyond head normal forms. We call this relation $\Downarrow^*$ and define it in Figure 8. This new relation has several interesting properties:

- $\Downarrow \subseteq \Downarrow^*$, because for any FLC-derivation of the form $\Gamma : e \Downarrow \Delta : v$ if v is a variable then $\Gamma : e \Downarrow^* \Delta : v$ by the rule Var*, and if $v = c(x_1, \ldots, x_n)$ then $\Gamma : e \Downarrow^* \Delta : c(x_1, \ldots, x_n)$ applying Cons* and RR* for the premises.
- $\forall\, \Gamma, t$ such that t is a normalized *Cterm* and Γ is a valid heap, it happens $\Gamma : t \Downarrow^* \Gamma : t$. We can get this applying Cons* and RR* for the premises.

Now we can formulate an alternative theorem using this new relation:

Theorem 6.8 (From $CRWL_{FLC}$ to FLC, alternative version) *If* $e \to c(t_1, \ldots, t_n)$, *then* $\Gamma : e' \Downarrow^* \Delta : c(t_1, \ldots, t_n)$, *where* $(\Gamma, e') = norm2(e)$

The proofs for these theorems are still under development.

7 Conclusions and Future Work

In this paper we study the relationship between *CRWL* [6,7] and *FLC* [1], two formal semantical descriptions of first order functional logic programming with call-time choice semantics for non-deterministic functions. The long distance between these two settings, even at syntactical level, discourages any direct proof of equivalence. Instead, we have chosen *FLC* as common language, to which *CRWL* can be adapted by means of a program transformation and a new $CRWL_{FLC}$ proof calculus for the resulting *FLC*-programs. The program transformation itself is not very novel, although its formulation here is original, but the $CRWL_{FLC}$ calculus and its relation to the original are indeed novel and could be useful for future works.

The most important and involved part of the paper establishes the relation between the $CRWL_{FLC}$ logic and the natural semantics given to *FLC* in [1]. We give an equivalence result for ground expressions and for the class of *FLC*-programs not having recursive *let* bindings nor extra variables. This is not so restrictive as it could seem: it has been proved [5,4] that extra variables can be eliminated from programs, and recursive *let*'s do not appear in the translation to *FLC*-syntax of *CRWL*-programs. Still, dropping such restrictions is desirable, and we hope to do it in the next future.

We did not expect proofs to be easy. Despite of that, we are a bit surprised by the great difficulties we have encountered, even with the imposed restrictions over expressions and programs. This suggest to look for new insights, not only at the level of the proofs but also in the sense of finding new alternative semantical descriptions of functional logic programs.

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8 Appendix A: Proofs

In order to clarify the proofs, some extra **notation** is introducted:

$deps(\Gamma, e)$: This is the sets of variables in $dom(\Gamma)$ such that e depends on, directly or indirectly. It can be defined as $deps(\Gamma, e) = var(e) \cup \{x \mid y \in deps(\Gamma, e) \wedge x \in deps(\Gamma, \Gamma[y])\}$.

$subs(\Gamma)$: Given a heap Γ, $subs(\Gamma)$ is the set of all substitutions under the variables in $dom(\Gamma)$, then came evaluating this heap. If we order the bings in Γ in a way such that $\Gamma = [x_1 \mapsto e_1, \ldots, x_n \mapsto e_n]$ and each e_i could depend on x_j iff $j < i$, then we define $subs([x_1 \mapsto e_1, \ldots, x_n \mapsto e_n]) =_{def} \{[x_i/t_i, \ldots, x_n/t_n] \mid ligs([x_1 \mapsto e_1, \ldots, x_n \mapsto e_n], (x_1, \ldots, x_n)) \rightarrow (t_1, \ldots, t_n)\}$.
Note that $\forall\ \Gamma.\ subs(\Gamma) \subseteq CSubst_\perp$, because every t_i is in the right side of a $CRWL_{LET}$-derivation.

$norm2(e)$: As we said in Section 6, this is a pair corresponding to the result of normalizing the expression e in a similar way as Albert et al., but instead of using let expressions to store arguments, storing them in the heap. Formally we can define this operation as follows:

$$norm2(x) = ([], x)$$

$$norm2(\varphi(x_1, \ldots, x_n)) = ([], \varphi(x_1, \ldots, x_n))$$

$$norm2(\varphi(x_1, \ldots, x_{i-1}, e_i, e_{i+1}, \ldots, e_n)) = ([x_i \mapsto e'_i] \uplus \Gamma \uplus \Delta, e')$$

$$\text{where } e_i \text{ is not a variable and } (\Gamma, e'_i) = norm2(e_i)$$

$$, (\Delta, e') = norm2(\varphi(x_1, \ldots, x_{i-1}, x_i, e_{i+1}, \ldots, e_n)) \text{ and } x_i \text{ is fresh}$$

$$norm2(let\ \{x = e_1\}\ in\ e) = (\Gamma \uplus \Delta, let\ \{x = e'_1\}\ in\ e')$$

$$\text{where } (\Gamma, e'_1) = norm2(e_1) \text{ and } (\Delta, e') = norm2(e)$$

$$norm2(e_1\ or\ e_2) = (\Gamma_1 \uplus \Gamma_2, e'_1\ or\ e'_2)$$

$$\text{where } (\Gamma_1, e'_1) = norm2(e_1) \text{ and } (\Gamma_2, e'_2) = norm2(e_2)$$

$$norm2((f)case\ e\ of\ \{\overline{p_k \rightarrow e_k}\}) = (\Gamma \uplus (\biguplus \Delta_k), (f)case\ e'\ of\ \{\overline{p_k \rightarrow e'_k}\})$$

$$\text{where } (\Gamma, e') = norm2(e) \text{ and } \forall x_k\ (\Delta_k, e'_k) = norm2(e_k)$$

We can use disjoint union ($\uplus$) because each variable introducted in the heap is fresh.

Additionally, in the following we will suppose we are working with FLC-programs and FLC-expressions subject to the following additional transformation,

$$case\ e\ of\ \{\overline{p_k \mapsto e_k}\} \hookrightarrow let\ \{x = e\}\ in\ case\ x\ of\ \{\overline{p_k \mapsto e_k}\}$$

being x a fresh variable and e not a variable (in case e is a variable the transformation leaves the expression untouched). Once this transformation has been applied, as all the substitutions made in FLC are from variables to variables, we can state that this transformation lasts in the calculus. Furthermore, if the calculus succeeds for a case expression like that, we can state that x is defined in the heap, becase x is always demanded to compute the case expression.

Proof. [For Theorem 5.3](Sketch) We prove the inclusion $[\![e]\!]^{P}_{CRWL} \subseteq [\![\hat{e}]\!]^{\hat{P}}_{CRWL_{FLC}}$ (resp. $[\![e]\!]^{P}_{CRWL} \supseteq [\![\hat{e}]\!]^{\hat{P}}_{CRWL_{FLC}}$) by induction over the depth of the *CRWL*-derivation (resp. $CRWL_{FLC}$-derivation) of an arbitrary arrow $e \to t$ (resp. $\hat{e} \to t$), with $t \in [\![e]\!]^{P}_{CRWL}$ (resp. $t \in [\![\hat{e}]\!]^{\hat{P}}_{CRWL_{FLC}}$). □

Before proving Theorem 6.3 some auxiliary lemmas are nedeed:

Lemma 8.1 (Ligs) $ligs(\Gamma, e) \to t$ *iff* $\exists \sigma \in subs(\Gamma)$ *such that* $\sigma e \to t$. *In other words,* $t \in [\![\Gamma, e]\!]$ *iff* $\exists \sigma \in sus(\Gamma)$ *such that* $\sigma e \to t$.

Lemma 8.2 $\forall\, \Gamma, x.\ [\![\Gamma[x \mapsto e], e]\!] = [\![\Gamma[x \mapsto e], x]\!]$

Lemma 8.3 *For any FLC-derivation of the form* $\Gamma : e \Downarrow \Delta : v$ *it happens that* $dom(\Gamma) \subseteq dom(\Delta)$. *Heaps are always growing during the computation.*

Lemma 8.4 $\forall\, \Gamma, x$ *such that* $\Gamma[x] = c(\overline{y})$ *then for all FLC-derivation of the form* $\Gamma : e \Downarrow \Delta : v$ *it happens that* $\Delta[x] = c(\overline{y})$. *Bindings to returning values remain in all the heaps that follow in the computation.*

Lemma 8.5 $\forall\, \Gamma, x, e, e_1$ *such that* $x \notin deps(\Gamma, e)$, *then* $[\![\Gamma[x \mapsto e_1], e]\!] = [\![\Gamma, e]\!]$.

Lemma 8.6 $\forall\, \Gamma, x, e_1, e_2, e$ *such that* Γ *is a valid heap,* $x \notin dom(\Gamma)$ *and* $x \notin var(e_1) \cup var(e_2)$, *then* $([\![\Gamma, e_1]\!] \subseteq [\![\Gamma, e_2]\!]) \Longrightarrow ([\![\Gamma[x \mapsto e_1], y]\!] \subseteq [\![\Gamma[x \mapsto e_2], y]\!])$, $\forall y \in dom(\Gamma) \cup \{x\}$.

Lemma 8.7 *For every valid heap* Γ *and every case-expression of the form case* $c(\overline{y_n})$ *of* $\{\overline{p_k \mapsto e_k}\}$ *such that* $p_i = c(\overline{x_n})$, *if we define the substitution* $\rho = [\overline{x_n / y_n}]$ *then* $[\![\Gamma, e_i\rho]\!] \subseteq [\![\Gamma, case\ c(\overline{y_n})\ of\ \{\overline{p_k \mapsto e_k}\}]\!]$

Lemma 8.8 $\forall\, \Gamma, \Delta, x, v$ *such that* $\Gamma : x \Downarrow \Delta : v$ *it happens that* $\Delta[x] = v$.

These are the proofs for those lemmas:

Proof. [For Lemma 8.4](Sketch) Using Lemma 8.3 we know that there must be a binding for x, all that's left is ensuring that this binding never changes. The only way a binding for a variable changes is through the rule VarExp, but this rule cannot be applied if e is constructor-rooted, and that's the case because $e = c(\overline{y})$, so the binding for x remains the same. □

Proof. [For Lemma 8.5](Sketch) To prove this statement we use Lemma 8.1 (Ligs) and realise that the substitution σ can give $\perp$ for x, and for every variable y such that $y \notin deps(\Gamma, e)$, so we can get the same result in $[\![\Gamma[x \mapsto e_1], e]\!]$ as in $[\![\Gamma, e]\!]$. □

Proof. [For Lemma 8.6] There are two possibilities:

- $y \in dom(\Gamma)$: Then $x \notin deps(\Gamma, y)$ because Γ is a valid heap and so no binding in Γ can depend on x, since this variable isn't in the heap and free variables are forbidden. So $[\![\Gamma[x \mapsto e_1], y]\!] =_{Lemma 8.5} [\![\Gamma, y]\!] =_{Lemma 8.5} [\![\Gamma[x \mapsto e_2], y]\!]$
- $y = x$: Then $x \notin deps(\Gamma, e_1) \cup deps(\Gamma, e_2)$ because $x \notin var(e_1) \cup var(e_2)$ and $x \notin dom(\Gamma)$. So $[\![\Gamma[x \mapsto e_1], x]\!] =_{Lemma 8.2} [\![\Gamma[x \mapsto e_1], e_1]\!] =_{Lemma 8.5} [\![\Gamma, e_1]\!] \subseteq_{hypothesis} [\![\Gamma, e_2]\!] =_{Lemma 8.5} [\![\Gamma[x \mapsto e_2], e_2]\!] =_{Lemma 8.2} [\![\Gamma[x \mapsto e_2], x]\!]$

□

Proof. [For Lemma 8.7] If we have that $ligs(\Gamma, e_i\rho) \to t$ then $\exists \sigma \in sus(\Gamma)$ such that $e_i\rho\sigma \to t$. Now let's see the derivation for the case:

$$\frac{(c(\overline{y_n}))\sigma \rightarrow c(\overline{y_n}\sigma) \qquad e_i\sigma_{|(Var\setminus\{\overline{x_n}\})}\ \gamma \rightarrow t}{(case\ c(\overline{y_n})\ of\ \{\overline{p_k \mapsto e_k}\})\sigma \rightarrow t}\ \text{CASE}$$

where $\gamma = [\overline{x_n \mapsto y_n\sigma}]$. We can now prove that $\forall z \in vars(e_i).\ z\rho\sigma = z\sigma_{|(Var\setminus\{\overline{x_n}\})}\gamma$, doing a case distinction:

$z \in \{\overline{x_n}\}$: For example $z = x_i$. Then $x_i\rho\sigma = y_i\sigma$, and $x_i\sigma_{|(Var\setminus\{\overline{x_n}\})}\gamma = x_i\gamma = y_i\sigma$.

$z \notin \{\overline{x_n}\}$: Then $z\rho\sigma = z\sigma$, and $z\sigma_{|(Var\setminus\{\overline{x_n}\})}\gamma = z\sigma\gamma = z\sigma$, because all the variables that could come from a substitution in $sus(\Gamma)$ are variables from Γ, and $dom(\Gamma) \cap \{\overline{x_n}\} = \emptyset$. We can state this intersection is empty because all the variables in Γ are introducted by the Let rule of FLC and so are fresh, and no substitution over a case expression can change the variables in its patterns, because those are bounded variables.

□

Proof. [For Lemma 8.8] We can check this very easily looking at the rules VarCons and VarExp of FLC: these are the only rules applicable for that derivation and they keep this property. □

Now we are ready to prove Theorem 6.3:

Proof. [For Theorem 6.3] By induction of the structure of FLC-derivations:

Notation: IH_{H_i} means apply the induction hypothesis for the property H over the *i-th* premise of the rule Select. IH_{R_i} means the same but for the property R.

(i) Base:
- **VarCons**
  H: It follows trivially because we have only one heap.
  R: $[\![\Gamma[x \mapsto t], t]\!] =_{Lemma\ 8.2} [\![\Gamma[x \mapsto t], x]\!]$, so this condition is fulfilled also.
- **Val**
  H: It follows trivially because we have only one heap.
  R: It follows trivially because we have only one heap and one expression to reduce.

(ii) Inductive step:
- **VarExp**
  R: $[\![\Delta[x \mapsto v], v]\!] =_{Lemma\ 8.2} [\![\Delta[x \mapsto v], x]\!]$, so this condition is fulfilled.
  H: Heap Δ in the premise must fulfil $\Delta[x] = e$, because Δ is one of the results obtained during the calculation of e. If x had changed in any heap during this calculation, that should be because x had been consulted to calculate e and so e depends on x. That should mean we have a recursive binding and this is forbidden, hence $\Delta \equiv \Delta'[x \mapsto e]$, in other worrds, $\Delta[x] = e$

  Now we want to prove this property: $(P1) \equiv \forall y \in dom(\Delta[x \mapsto v]),\ [\![\Delta[x \mapsto v], y]\!] \subseteq [\![\Delta[x \mapsto e], y]\!]$. Applying Lemma 8.5 all we have to prove to have (P1) is that $[\![\Delta', v]\!] \subseteq [\![\Delta', e]\!]$, and we can prove that in the following way:

$$\begin{array}{ccc} [\![\Delta', v]\!] & \subseteq_? & [\![\Delta', e]\!] \\ \|_{Lemma\ 8.6} & & \|_{Lemma\ 8.6} \\ [\![\Delta, v]\!] & \subseteq_{IH_R} & [\![\Delta, e]\!] \end{array}$$

where IH_R denotes the R part of the induction hypothesis. We are sure we can apply Lemma 8.6 because of the absence of recursive bindings that forces e and v to be independent from x.

So we have (P1), and we have also $dom(\Gamma[x \mapsto e]) \subseteq dom(\Delta[x \mapsto v])$ by Lemma 8.3, hence we have $\forall y \in dom(\Gamma[x \mapsto e])$, $[\![\Delta[x \mapsto v], y]\!] \subseteq [\![\Delta, y]\!]$ (because $\Delta \equiv \Delta[x \mapsto e]$). Applying the H part of the induction hypothesis we have $\forall y \in dom(\Gamma[x \mapsto e])$, $[\![\Delta, y]\!] \subseteq [\![\Gamma[x \mapsto e], y]\!]$, so H follows by transitivity of subsets.

- **Select**

  H: $\forall x \in dom(\Gamma)$, $[\![\Theta, x]\!] \subseteq_{IH_{H_2} + \, dom(\Gamma) \subseteq dom(\Delta), \, Lemma\ 8.3} [\![\Delta, x]\!] \subseteq_{IH_{H_1}} [\![\Gamma, x]\!]$, so the property holds.

  R: Following the assumptions in Section 4 we can suppose that e is a variable, x for example. So we want to prove $[\![\Theta, v]\!] \subseteq [\![\Theta, case\ x\ of\ \{\overline{p_k \mapsto e_k}\}]\!]$

  $[\![\Theta, v]\!] \subseteq_{IH_{R_2}} [\![\Theta, \rho(e_i)]\!] \subseteq_{Lemma\ 8.7} [\![\Theta, case\ c(\overline{y_n})\ of\ \{\overline{p_k \mapsto e_k}\}]\!]$

  So all we need to prove is that $[\![\Theta, c(\overline{y_n})]\!] \subseteq [\![\Theta, x]\!]$. To do that we apply Lemma 8.8 to the first premise to obtain that $\Delta[x] = c(\overline{y_n})$, and with this and Lemma 8.4 we get that $\Theta[x] = c(\overline{y_n})$. So $\Theta \equiv \Theta[x \mapsto c(\overline{y_n})]$, hence by Lemma 8.2 $[\![\Theta, c(\overline{y_n})]\!] = [\![\Theta, x]\!]$: the property holds.

- **Fun**

  H: It follows by induction hypothesis.

  R: We want to prove that $[\![\Delta, v]\!] \subseteq [\![(\Delta, f(\overline{x_n})]\!]$, that is,$(ligs(\Delta, v) \to t) \Longrightarrow (ligs(\Delta, f(\overline{x_n})) \to t)$.

  $ligs(\Delta, v) \to t \Longrightarrow_{HI_R} ligs(\Delta, \rho(e)) \to t \Longrightarrow_{Lema\ 8.1} \exists \sigma \in sus(\Delta)$ such that $\sigma(\rho(e)) \to t$. If for this substitution σ we could prove that $\sigma(f(\overline{x_n})) \to t$ then, using Lema 8.1 we would prove that $ligs(\Delta, f(\overline{x_n})) \to t$ and so R would be proved. That proof should look like this:

  $$\frac{\theta(e) \to t}{\sigma(f(\overline{x_n})) \equiv f(\sigma(\overline{x_n})) \to t}\ \text{DF}$$

  , with $\theta = [\overline{y_n \mapsto \sigma(x_n)}]$, because $f(\overline{y_n}) = e \in P$

  So, as free vars in e must be in $\{\overline{y_n}\}$, $\theta(e) \equiv \sigma(\rho(e))$ if $\forall y_i \in \overline{y_n}$, $\theta(y_i) = \sigma(\rho(y_i))$, and that happens because $\theta(y_i) =_{def\ of\ \theta} \sigma(x_i) =_{def\ of\ \rho} \sigma(\rho(y_i))$. So que $\theta(e) \equiv \sigma(\rho(e))$ and as $\sigma(\rho(e)) \to t$ then $\theta(e) \to t$ thusR holds.

- **Or**

  H: It follows by induction hypothesis.

  R: We want to prove that $[\![\Delta, v]\!] \subseteq [\![(\Delta, e_1\ or\ e_2)]\!]$, that is, $(ligs(\Delta, v) \to t) \Longrightarrow (ligs(\Delta, e_1\ or\ e_2) \to t)$

  $ligs(\Delta, v) \to t \Longrightarrow_{IH_R}$ for $e_j \in \{e_1, e_2\}$ used in the premise we get $ligs(\Delta, e_j) \to t \Longrightarrow_{Lema\ 4.5} \exists \sigma \in sus(\Delta)$ such that $\sigma(e_j) \to t$. So:

$$\cfrac{\cfrac{\sigma(e_j) \rightarrow t}{\sigma(e_1 \; or \; e_2) \equiv \sigma(e_1) \; or \; \sigma(e_2) \rightarrow t} \; OR}{ligs(\Delta, e_1 \; or \; e_2) \rightarrow t} \; Lemma 8.1$$

using Lemma 8.1 in the first step.

- **Let**

  H: It follows by induction hypothesis, because $dom(\Gamma[\overline{y_k \mapsto \rho(e_k)}]) \supset dom(\Gamma)$, since we get $\Gamma[\overline{y_k \mapsto \rho(e_k)}]$ adding bindings for fresh variables to Γ, and because $\forall x \in dom(\Gamma)$, $\Gamma[\overline{y_k \mapsto \rho(e_k)}][x] = \Gamma[x]$. So we get R applying Lemma 8.5 to every variable in $dom(\Gamma)$.

  R: It follows by induction hypothesis since except for renaming:

$$ligs(\Gamma[\overline{y_k \mapsto \rho(e_k)}], \rho(e)) \equiv_{renombramiento} ligs(\Gamma, let \; \{\overline{x_k = e_k}\} \; in \; e)$$

□

Proof. [For Lemma 6.5] As v is a result of a FLC-derivation it can only be $v = c(x_1, \ldots, x_m)$. So what we want to prove that $ligs(\Delta, c(x_1, \ldots, x_m)) \rightarrow |\Delta : c(x_1, \ldots, x_m)|$. To do that we order the bindings in Δ in the same way as in $Ligs$, that is, in a way such that $\Gamma = [z_1 \mapsto e_1, \ldots, z_m \mapsto e_m]$ and each e_i could depend on z_j iff $j < i$. Now we define $\sigma_i =_{def} [z_i \mapsto |\Delta : z_i|]$, and $\sigma^i =_{def} \sigma_i \circ \sigma_{i-1} \circ \ldots \circ \sigma_1$. We affirm that $(\forall z_i \in dom(\Delta), \; (\Delta[z_i])\sigma^{i-1} \rightarrow |\Delta : z_i|) \equiv (L)$ holds. We can prove distinguishing cases over $\Delta[z_i]$:

a) $\Delta[z_i] = c(x_1, \ldots, x_n)$, then $|\Delta : z_i| = c(|\Delta : x_1|, \ldots, |\Delta : x_n)$, so

$$\cfrac{x_1\sigma^{i-1} \rightarrow |\Delta : x_1| \quad \ldots \quad x_n\sigma^{i-1} \rightarrow |\Delta : x_n|}{(\Delta[z_i])\sigma^{i-1} \equiv c(x_1\sigma^{i-1}, \ldots, x_n\sigma^{i-1}) \rightarrow c(|\Delta : x_1|, \ldots, |\Delta : x_n)} \; \mathrm{DC}$$

so we have to prove that $\forall x_k \in \{x_1, \ldots, x_n\}, \; x_k\sigma^{i-1} \rightarrow |\Delta : x_k|$. For the ordering of dependences in Γ we have done before, we know that each x_k must belong to $\{z_1, \ldots, z_{i-1}\}$. Let's suppose $x_k = z_j$ such that $j < i$, then:

$$z_j\sigma^{i-1} =_1 z_j\sigma_1 \ldots \sigma_j\sigma_{j+1} \ldots \sigma_{i-1} =_2 z_j\sigma_j\sigma_{j+1} \ldots \sigma_{i-1} =_3 |\Delta : z_j|\sigma_{j+1} \ldots \sigma_{i-1} =_4 |\Delta : z_j|$$

We must justify each one of these steps:

1. By definition of σ^{i-1}
2. Because the domain of each σ_l is just $\{z_l\}$ and z_j is distinct from each element in $\{z_1, \ldots, z_{j-1}\}$
3. By definition of σ_j
4. Because under our working context $|\Delta : z_j|$ must be ground (it has no variables), because logical variables are forbidden.

So, as $|\Delta : z_j$ is a ground $Cterm_\perp$, using the rule DC many times we can get $x_k\sigma^{i-1} \equiv |\Delta : x_k| \rightarrow |\Delta : x_k|$.

b) As we have forbidden logical variables, the only possibility is that $|\Delta : z_i| = \perp$, so using the rule B we are done.

Now when we derive $ligs(\Delta, c(x_1, \ldots, x_m))$ we can use L with the fists binding to see that $\Delta[z_1] \rightarrow |\Delta : z_1|$, doing that with the rest of the bindings, following the LET rule (note that $ligs(\Delta, c(x_1, \ldots, x_m))$ is a let expression), we go constructing σ^1, σ^2, ..., until σ^m. So all we have to prove is that $c(x_1, \ldots, x_m)\sigma^m \rightarrow |\Delta : c(x_1, \ldots, x_m)|$, and this can we proven using rule DC and similar techniques as the used in the a case of the proof for (L). □

9 Appendix B: Examples

9.1 An example for Section 6.1

Maybe an example could come in useful to assimilate the results above. Given the following CRWL-program (where the pair constructor '()' is a constructor symbol):

$$
\begin{array}{ll}
f(x) = (x, plus(x)) & f(x) = (x, minus(x)) \\
plus(0) = 1 & plus(1) = 2 \\
minus(0) = -1 & minus(1) = 0 \\
coin = 0 & coin = 1
\end{array}
$$

and $f(coin)$ as the CRWL-expression to evaluate, we can translate it to normalized FLC syntax getting the FLC-program:

$$
\begin{array}{l}
f(x) = (let\ \{p = plus(x)\}\ in\ (x,p))\ or\ (let\ \{m = minus(x)\}\ in\ (x,p)) \\
plus(x) = case\ x\ of\ \{0 \to 1; 1 \to 2\} \\
minus(x) = case\ x\ of\ \{0 \to -1; 1 \to 0\} \\
coin = 0\ or\ 1
\end{array}
$$

and $let\ \{c = coin\}\ in\ f(c)$ as the FLC-expression to evaluate. Now let's see how Theorem 6.1 and Corollary 6.6 apply to this setting. Firstly we must construct a FLC-derivation for that expression:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1) \Downarrow [c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)}\ Val}{[c1 \mapsto coin] : let\ \{p = plus(c1)\}\ in\ (c1,p) \Downarrow [c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)}\ Let}{[c1 \mapsto coin] : (let\ \{p = plus(c1)\}\ in\ (c1,p))\ or \ldots \Downarrow [c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)}\ Or}{[c1 \mapsto coin] : f(c1) \Downarrow [c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)}\ Fun}{[] : let\ \{c = coin\}\ in\ f(c) \Downarrow [c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)}\ Let
$$

By Theorem 6.1, $[\![[c1 \mapsto coin, p1 \mapsto plus(c1)], (c1,p1)]\!] \subseteq [\![[], let\ \{c = coin\}\ in\ f(c)]\!]$. We can check this by constructing each CRWL-derivation from $ligs([c1 \mapsto coin, p1 \mapsto plus(c1)], (c1,p1))$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{0 \to 0}\ DC}{0\ or\ 1 \to 0}\ OR}{coin \to 0}\ DF \quad \dfrac{\dfrac{\dfrac{\dfrac{}{0 \to 0}\ DC \quad \dfrac{}{1 \to 1}\ DC}{case\ 0\ of\ \{0 \to 1; 1 \to 2\} \to 1}\ CASE}{plus(0) \to 1}\ DF \quad \dfrac{\dfrac{}{0 \to 0}\ DC \quad \dfrac{}{1 \to 1}\ DC}{(0,1) \to (0,1)}\ DC}{let\ \{p1 = plus(0)\}\ in\ (0,p1) \to (0,1)}\ LET}{let\ \{c1 = coin\}\ in\ let\ \{p1 = plus(c1)\}\ in\ (c1,p1) \to (0,1)}\ LET
$$

The other possible CRWL-derivations for this expression are similar and lead us to the following conclusion: $[\![[c1 \mapsto coin, p1 \mapsto plus(c1)], (c1,p1)]\!] = \{(0,1), (1,2), (0,\perp), (1,\perp), (\perp,1), (\perp,2), (\perp,\perp), \perp\}$. By Theorem 6.1, it must happen that $(0,1) \in [\![[], let\ \{c = coin\}\ in\ f(c)]\!]$, let's

check this deriving $ligs([], let\ \{c = coin\}\ in\ f(c))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{0 \to 0}\,DC}{0\ or\ 1 \to 0}\,OR}{coin \to 0}\,DF \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{0 \to 0}\,DC \quad \dfrac{}{1 \to 1}\,DC}{case\ 0\ of\ \{0 \to 1; 1 \to 2\} \to 1}\,CASE}{plus(0) \to 1}\,DF}{let\ \{p = plus(0)\}\ in\ (0,p)) \to (0,1)}\,LET \quad \dfrac{\dfrac{}{0 \to 0}\,DC \quad \dfrac{}{1 \to 1}\,DC}{(0,1) \to (0,1)}\,DC}{(let\ \{p = plus(0)\}\ in\ (0,p))\ or \ldots \to (0,1)}\,OR}{f(0) \to (0,1)}\,DF}{let\ \{c = coin\}\ in\ f(c) \to (0,1)}\,LET$$

Doing the other CRWL-derivations for this expression we get $[\![[], let\ \{c = coin\}\ in\ f(c)]\!] = \{(0,1),(1,2),(0,\perp),(1,\perp),(\perp,1),(\perp,2),(\perp,\perp),\perp,(0,-1),(1,0),(\perp,-1),(\perp,0)\}$. Therefore Theorem 6.1 holds and so does Corollary 6.6 as $|[c1 \mapsto coin, p1 \mapsto plus(c1)] : (c1,p1)| = (\perp,\perp)$.