Asymptotic Behavior of HKM Paths in Interior Point Method for Monotone Semidefinite Linear Complementarity Problem: General Theory

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Abstract

An interior point method (IPM) defines a search direction at an interior point of the feasible region. These search directions form a direction field which in turn defines a system of ordinary differential equations (ODEs). Thus, it is natural to define the underlying paths of the IPM as the solutions of the systems of ODEs. In [8], these off-central paths are shown to be well-defined analytic curves and any of their accumulation points is a solution to the given monotone semidefinite linear complementarity problem (SDLCP). The off-central path of a simple example is also studied in [8] whose asymptotic behavior near the solution of the example is analyzed. In this paper, which is an extension of [8], we study the asymptotic behavior of off-central path for general SDLCP (using the dual HKM direction), instead of for a given example. We give a necessary and sufficient condition for when an off-central path is analytic as a function of $\sqrt{\mu}$ at the solution of the SDLCP. Then we show that if the given SDLCP has a unique solution, the first derivative of its off-central path, as a function of $\sqrt{\mu}$, is bounded. We work under the assumption that the given SDLCP satisfies strict complementarity condition.

1   Introduction.

The notion of central path was introduced by Sonnevend [9] in 1985 to interior point method (IPM). Since then, people realize that IPM is actually a homotopy method following underlying paths (central and off-central paths) and that many remarkable properties of IPM are attributed to the nice geometry of the underlying paths. Readers who are interested in basic geometry of the underlying paths may refer to [1].

In [10, 11, 20, 19, 22] it was found that, for solving a linear program (LP) or a linear complementarity problem (LCP), the number of iterations needed by an interior point algorithm to reduce the duality gap μ from μ_0 to $\epsilon > 0$ is equivalent to the integral of the curvature of the central path from μ_0 to ϵ. This equivalence relates a discrete analysis (complexity analysis) to a continuous analysis (curvature of path) and thus opens a new way to estimate upper and lower bounds of the complexity of IPM. In [15] (Mizuno, Meggido, Tsuchiya and Monteiro also have papers on this topic), the authors showed that the complexity depends only on the

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constraint matrix, by observing those regions where the central path is straight or crossing over.

Another important role the underlying paths play in the study of IPM is to show fast local convergence. Classical proof of the local convergence of an iterative method, such as the Newton's method, for finding the solution of a system of equations relies on the nonsingularity of the Jacobian matrix. However, the Jacobian matrix of the equation system defining the central path in IPM may be singular at the optimal solution. Thus traditional approach of local convergence analysis does not work for IPM. Fast local convergence of IPM has instead been successfully proved by relating it to the boundedness of derivatives of the underlying paths in [6, 16, 17, 13].

The study of fast local convergence is particularly important for semidefinite linear complementarity problem (SDLCP), with semidefinite program (SDP) as a special case, because, in contrast to LCP, the exact solution of a SDLCP cannot be obtained from an approximate solution by determining a complementary basis.

There are various ways in which the underlying paths, using different search directions, for SDLCP are defined in the literature [4, 5, 7, 8]. In [8], a new definition of the underlying paths of IPM for SDLCP, using ordinary differential equations (ODEs), is proposed. The motivation for defining paths in this way is to relate paths to the vector field of search directions of the IPM (see more details in [8]). In this paper, we use this definition of paths for SDLCP to study the asymptotic behavior of the paths for general SDLCP. As mentioned in earlier paragraphs, studying the asymptotic behavior of paths is important in the investigation of local convergence of IPM for SDLCP.

Throughout what follows, we restrict ourselves to the dual HKM direction and assume that SDLCP satisfies strict complementarity. The HKM direction and its dual are among the most used directions in designing interior point algorithms, besides, the AHO and NT directions. The asymptotic analyticity behavior of off-central paths for SDLCP using the AHO direction has been studied in [5, 7]. In [4], the asymptotic analyticity behavior of off-central paths for SDP using the HKM direction was studied in general. The authors in [4] show that the off-central paths are analytic as a function of $\sqrt{\mu}$ in the limit. The off-central path that the authors in [4, 5, 7] are defined by algebraic equations and are not directly related to search directions of the IPM , while in [8] and this paper, it is defined using ODEs obtained from search directions. In [8], the asymptotic analyticity behavior of off-central paths, using the dual HKM direction, is investigated for a simple example. In this paper, we attempt to investigate the asymptotic behavior of off-central paths, using the dual HKM direction, for general SDLCP.

In [8], it is shown, through an example, that there are two sets of off-central paths: paths in one set are analytic at $\mu = 0$ and those in the other set are not. For that example, the authors found a condition which characterizes analytic and non-analytic paths. For general problems, similar conditions have not been found. In this paper, we show that a path $(X(\mu), Y(\mu))$ is analytic with respect to $\sqrt{\mu}$ if and only if an off-diagonal submatrix of Y (or X) is analytic with respect to $\sqrt{\mu}$ and the submatrix is equal to $O(\mu)$ as $\mu \to 0$. This result is interesting on its own.

Another phenomenon observed in [8], again by an example, is that the first derivative of an

off-central path with respect to μ is unbounded as $\mu \to 0$. A natural question is whether the first order derivative of an off-central path with respect to $\sqrt{\mu}$ is bounded as $\mu \to 0$. One may guess that the first order derivatives of those non-analytic paths are likely to be unbounded near $\mu = 0$ even as a function of $\sqrt{\mu}$. Our study in this paper shows a fact contrary to this intuition.

In Section 2, we first define SDLCP and off-central paths for SDLCP. We also describe in detail a reformulation of the ODE system that described an off-central path for SDLCP. The main result in Section 3 is a necessary and sufficient condition for when an off-central path, as a function of $\sqrt{\mu}$ (where μ is the parameter of the path, and proportional to the duality gap between the primal and dual variables), is analytic at the solution of SDLCP. This condition is not intuitively obvious and may provide some insight into the study of asymptotic analyticity behavior of off-central paths. We also derive in this section a weak sufficient condition for convergence of off-central path. In Section 4, we show that if the given SDLCP has a unique solution, then the first derivative of any off-central path, as a function of $\sqrt{\mu}$, is bounded. Finally, we give some concluding remarks and future directions in Section 5.

1.1 Notation.

The space of symmetric $n \times n$ matrices is denoted by S^n. Given matrices X and Y in $\Re^{p \times q}$, the standard inner product is defined by $X \bullet Y \equiv \mathrm{Tr}(X^T Y)$, where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. If $X \in S^n$ is positive semidefinite (resp., positive definite), we write $X \succeq 0$ (resp., $X \succ 0$). The cone of positive semidefinite (resp., positive definite) symmetric matrices is denoted by S^n_+ (resp., S^n_{++}). Either the identity matrix or operator will be denoted by I.
$\| \cdot \|$ for a vector in $\Re^n$ refers the Euclidean norm and for a matrix in $\Re^{p \times q}$, it refers to the Frobenius norm.
For a matrix $X \in \Re^{p \times q}$, we denote its component at the ith row and j^{th} column by X_{ij}. Also, $X_{i.}$ denotes the ith row of X and $X_{.j}$ the jth column of X. In case X is partitioned into blocks of submatrices, then X_{ij} refers to the submatrix in the corresponding (i, j) position.
Given square matrices $A_i \in \Re^{n_i \times n_i}$, $i = 1, \ldots, m$, $\mathrm{diag}(A_1, \ldots, A_m)$ is a square matrix with A_i as its diagonal blocks arranged in accordance to the way they are lined up in $\mathrm{diag}(A_1, \ldots, A_m)$. All the other entries in $\mathrm{diag}(A_1, \ldots, A_m)$ are taken to be zero.

Given functions $f : \Omega \longrightarrow E$ and $g : \Omega \longrightarrow \Re_{++}$, where Ω is an arbitrary set and E is a normed vector space, and a subset $\widetilde{\Omega} \subseteq \Omega$. We write $f(w) = O(g(w))$ for all $w \in \widetilde{\Omega}$ to mean that $\|f(w)\| \leq Mg(w)$ for all $w \in \widetilde{\Omega}$ and a constant $M > 0$; Moreover, for a function $U : \Omega \longrightarrow S^n_{++}$, we write $U(w) = \Theta(g(w))$ for all $w \in \widetilde{\Omega}$ if $U(w) = O(g(w))$ and $U(w)^{-1} = O(g(w))$ for all $w \in \widetilde{\Omega}$. The latter condition is equivalent to the existence of a constant $M > 0$ such that

$$\frac{1}{M} I \preceq \frac{1}{g(w)} U(w) \preceq MI \quad \forall \, w \in \widetilde{\Omega}.$$

The subset $\widetilde{\Omega}$ should be clear from the context whenever it is used. Usually, $\widetilde{\Omega} = (0, \bar{w})$ for a small $\bar{w} > 0$.
If $\{u(\nu) \ : \ \nu > 0\}$ and $\{v(\nu) \ : \ \nu > 0\}$ are real sequences with $v(\nu) > 0$, then $u(\nu) = o(v(\nu))$ means that $\lim_{\nu \to 0} \frac{u(\nu)}{v(\nu)} = 0$. If $u(\nu)$ is a matrix or a vector, then $u(\nu) = o(v(\nu))$ means that $\lim_{\nu \to 0} \frac{\|u(\nu)\|}{v(\nu)} = 0$.

A function $f = (f_1, \ldots, f_m)$ from an open subset $\mathcal{O}$ of $\Re^k$ to $\Re^m$ is analytic at a point $x = (x_1, \ldots, x_k) \in \mathcal{O}$ if each f_i, $i = 1, \ldots, m$, can be written as a convergent power series expansion about $(x_1, \ldots, x_k)$ in open neighborhood of x. Furthermore, if $x^0 \in \Re^k$ is on the boundary of $\mathcal{O}$, we say f is analytic at x^0 (or can be extended analytically to x^0), and we let $f(x^0) = \lim_{x \to x^0} f(x)$, if there exists an analytic function g which is analytic at x^0 and coincides with f wherever both are defined.

Note that the above also applies if an argument of f is a symmetric matrix, in which case, we consider the variable to lie in an Euclidean space of appropriate dimension. If the range of f is in the space of matrices, we also consider it to be in an appropriate Euclidean space when considering analyticity, so that the above applies.

2 Formulation and Reformulation of ODEs for Off-Central Path.

Let us consider the following SDLCP:

$$
\begin{aligned}
XY &= 0 \\
A(X) + B(Y) &= q \\
X, Y &\in \mathcal{S}^n_+
\end{aligned}
\tag{1}
$$

where $A, B : \mathcal{S}^n \longrightarrow \Re^{\tilde{n}}$ are linear operators mapping $\mathcal{S}^n$ to the space $\Re^{\tilde{n}}$, where $\tilde{n} := n(n+1)/2$. Hence A and B have the form $A(X) = (A_1 \bullet X, \ldots, A_{\tilde{n}} \bullet X)^T$, resp. $B(Y) = (B_1 \bullet Y, \ldots, B_{\tilde{n}} \bullet Y)^T$, where $A_i, B_i \in \mathcal{S}^n$ for all $i = 1, \ldots, \tilde{n}$.

We have the following assumption on SDLCP throughout the paper:

Assumption 2.1

(a) *SDLCP is monotone, i.e.* $A(X) + B(Y) = 0$ *for* $X, Y \in \mathcal{S}^n \Rightarrow X \bullet Y \geq 0$.

(b) *There exists* $X^1, Y^1 \succ 0$ *such that* $A(X^1) + B(Y^1) = q$.

(c) $\{A(X) + B(Y) : X, Y \in S^n\} = \Re^{\tilde{n}}$.

The above assumptions are basic assumptions used in the literature when SDLCP is studied in the context of IPM. Besides Assumption 2.1, we also need another assumption in this paper, given below:

Assumption 2.2 *There exists a strictly complementary solution,* (X^*, Y^*), *to SDLCP (1).*

The analysis of the asymptotic behavior of an off-central path for a general SDLCP is considered to be difficult without this assumption (Assumption 2.2). However, we note that there have been some work done in this area for special classes of SDLCP without the assumption. See for example [3].

Let us now define the off-central path for SDLCP passing through a point (X^0, Y^0), $X^0, Y^0 \succ 0$, satisfying $A(X) + B(Y) = q$.

Definition 2.1 *The solution* $(X(\mu), Y(\mu))$, $\mu > 0$, *to*

$$
H_P(XY' + X'Y) = \frac{1}{\mu} H_P(XY),
\tag{2}
$$

$$
A(X') + B(Y') = 0,
\tag{3}
$$

with the initial condition $(X(1), Y(1)) = (X^0, Y^0)$, $X^0, Y^0 \succ 0$, *is the* **off-central path** *for SDLCP, corresponding to P, passing through* (X^0, Y^0).
Here $H_P(U) := \frac{1}{2}(PUP^{-1} + (PUP^{-1})^T)$, *and* $P \in \Re^{n \times n}$ *is an invertible matrix.*

Assuming P is an analytic function of X, Y and $PXYP^{-1}$ is always symmetric (such P include the well-known directions like the HKM (and its dual) and NT directions), it is proved in [8] that the above definition is well-defined, and $(X(\mu), Y(\mu))$, $X(\mu), Y(\mu) \succ 0$, is unique, analytic and exists over $\mu \in (0, \infty)$. The motivation for defining an off-central path as in Definition 2.1 is also given in [8].
The following theorem, which we quote from [8], will be referred to in this paper. We state it here for easy reference.

Theorem 2.1 *Let* $(X(\mu), Y(\mu))$, $\mu > 0$, *be the off-central path for SDLCP passing through* (X^0, Y^0), *then* $\lambda_{min}(X(\mu)Y(\mu)) = \lambda_{min}(X^0Y^0)\mu$ *and* $\lambda_{max}(X(\mu)Y(\mu)) = \lambda_{max}(X^0Y^0)\mu$.
Here $\lambda_{min}(\cdot)$ *and* $\lambda_{max}(\cdot)$ *are the minimum and maximum eigenvalue of the given matrix, respectively.*

Remark 2.1 *The central path* $(X_c(\mu), Y_c(\mu))$ *for SDLCP, which satisfies* $X_c(\mu)Y_c(\mu) = \mu I$, *is a special example of off-central path for SDLCP. When* $\mu = 1$, *it satisfies* $Tr(X_c(1)Y_c(1)) = n$. *Therefore, we also require the initial data* (X^0, Y^0) *when* $\mu = 1$ *in (2)-(3) to satisfy* $Tr(X^0Y^0) = n$. *In this case, it is easy to see, using (2), that the parameter* μ *in the ODE system (2)-(3) actually represents the duality gap,* $X(\mu) \bullet Y(\mu)$, *at the point* $(X(\mu), Y(\mu))$ *on the path.*

Using the operation $\otimes_s$ and the map *svec* (with inverse *smat*), whose properties are given on pp. 775-776 and the appendix of [14], we can rewrite (2)-(3) as

$$\begin{pmatrix} svec(A_1)^T & svec(B_1)^T \\ \vdots & \vdots \\ svec(A_{\tilde{n}})^T & svec(B_{\tilde{n}})^T \\ P \otimes_s P^{-T}Y & PX \otimes_s P^{-T} \end{pmatrix} \begin{pmatrix} svec(X') \\ svec(Y') \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} 0 \\ svec(H_P(XY)) \end{pmatrix}, \quad (4)$$

where $\tilde{n} = n(n+1)/2$.

As was mentioned in the Introduction section, we consider only the dual HKM direction in this paper. This corresponds to $P = Y^{1/2}$[18]. Therefore, (4) becomes

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & X \otimes_s Y^{-1} \end{pmatrix} \begin{pmatrix} svec(X') \\ svec(Y') \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} 0 \\ svec(X) \end{pmatrix}, \quad (5)$$

where $\mathcal{A} = \begin{pmatrix} svec(A_1)^T \\ \vdots \\ svec(A_{\tilde{n}})^T \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} svec(B_1)^T \\ \vdots \\ svec(B_{\tilde{n}})^T \end{pmatrix}$.

As $\mu \to 0$, $(X(\mu), Y(\mu))$ will tend to the boundary of the feasible region, thus, they are expected to be singular at the limit. Therefore, the left-hand matrix in (5) is not invertible, and may not be defined, in the limit as $\mu \to 0$ on an off-central path for SDLCP. Hence using (5) is not likely to yield results on the asymptotic behavior of off-central paths for SDLCP. To overcome this, we will make a transformation to (5). We wish that in the transformed system the coefficient

matrix on the left-hand side will be invertible at $\mu = 0$, and the original and new systems have the same solution for $\mu > 0$. If such a new system can be formulated and its solution can be shown to be analytic (with respect to μ or $\sqrt{\mu}$) at the $\mu = 0$, then the solution of the original system can be analytically extended to $\mu = 0$. Therefore, the system of ODEs obtained after the transformation will provide us an appropriate platform to answer the question when an off-central path $(X(\mu), Y(\mu))$ converges and is analytic at its limit point.

We only attempt to study the analyticity of the off-central path at its limit point with respect to $\sqrt{\mu}$ instead of μ in this paper because $\sqrt{\mu}$ naturally appears in the off diagonal entries of $X(\mu), Y(\mu)$, as shown in (6) and (7) below. This leads us to naturally investigate asymptotic behavior of $X(\mu), Y(\mu)$ w.r.t $\sqrt{\mu}$ first.

In what follows, we occasionally suppress the dependence of a vector or matrix on its parameters for the sake of clarity, and whether these matrices or vectors are dependent on a parameter and the parameter involved should be clear from the context.

Let (X^*, Y^*) be a strictly complementary solution to SDLCP (1), which exists by Assumption 2.2.
Since X^* and Y^* commutes, they are jointly diagonalizable by some orthogonal matrix. So, using a suitable orthogonal similarity transformation of the matrices in SDLCP (1), we may assume, without loss of generality, that

$$X^* = \left(\begin{array}{cc} \Lambda_{11}^* & 0 \\ 0 & 0 \end{array} \right), Y^* = \left(\begin{array}{cc} 0 & 0 \\ 0 & \Lambda_{22}^* \end{array} \right),$$

where $\Lambda_{11}^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_m^*) \succ 0$ and $\Lambda_{22}^* = \mathrm{diag}(\lambda_{m+1}^*, \ldots, \lambda_n^*) \succ 0$. Here $\lambda_1^*, \ldots, \lambda_n^*$ are real numbers greater than zero.

Hereafter, whenever we partitioned a matrix $S \in S^n$, we do it in a similar way, i.e., S is always partitioned as $\left(\begin{array}{cc} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{array} \right)$, where $S_{11} \in S^m, S_{22} \in S^{n-m}$ and $S_{12} \in \Re^{m \times (n-m)}$.

In order to transform the ODE system (5) into a more "manageable" system of ODEs, we will perform a variable transformation. For this purpose, we first prove a few lemmas below. These lemmas are adapted from [7].

Lemma 2.1 *On an off-central path, $X(\mu)$, $Y(\mu)$ are bounded near $\mu = 0$.*

Proof. This can be easily seen using $X(\mu) \bullet Y(\mu) = (X^0 \bullet Y^0)\mu = n\mu$ (where the second equality follows from Remark 2.1) and from $(X(\mu) - X^1) \bullet (Y(\mu) - Y^1) \geq 0$, which follows from Assumption 2.1(a) and (b). **QED**

Lemma 2.2 *([7] Lemma 3.10) $Y_{11}(\mu)$ and $X_{22}(\mu)$ are equal to $\mathcal{O}(\mu)$ and $\|X_{12}(\mu)\|$ and $\|Y_{12}(\mu)\|$ are equal to $\mathcal{O}(\sqrt{\mu})$.*

Proof. Now, $A(X(\mu) - X^*) + B(Y(\mu) - Y^*) = 0$ implies, by Assumption 2.1(a), that $(X(\mu) - X^*) \bullet (Y(\mu) - Y^*) \geq 0$. Hence $X(\mu) \bullet Y^* + X^* \bullet Y(\mu) \leq X(\mu) \bullet Y(\mu) = \mathrm{Tr}(X(\mu)Y(\mu))$.

Note that by (2), we can see easily that $\mathrm{Tr}(X(\mu)Y(\mu)) = \mathrm{Tr}(X^0Y^0)\mu = n\mu$. Hence, $X(\mu) \bullet Y^* + X^* \bullet Y(\mu) = \mathcal{O}(\mu)$. That is, $\sum_{i=m+1}^n \lambda_i^* x_{ii}(\mu) + \sum_{i=1}^m \lambda_i^* y_{ii}(\mu) = \mathcal{O}(\mu)$, where $x_{ii}(\mu), y_{ii}(\mu)$

6

are the diagonal elements of $X(\mu)$ and $Y(\mu)$ respectively. This implies that $X_{22}(\mu) = \mathcal{O}(\mu)$ and $Y_{11}(\mu) = \mathcal{O}(\mu)$.

Also, we have $\|X_{12}(\mu)\|^2 \leq \text{Tr}(X_{11}(\mu))\text{Tr}(X_{22}(\mu))$ (by Lemma 2.2 of [7]), together with the fact that $X(\mu)$ is bounded near μ equal to zero (by Lemma 2.1) and $X_{22}(\mu) = \mathcal{O}(\mu)$, implies that $\|X_{12}(\mu)\| = \mathcal{O}(\sqrt{\mu})$.

Similarly, we can show that $\|Y_{12}(\mu)\| = \mathcal{O}(\sqrt{\mu})$. **QED**

Lemma 2.3 *([7] Lemma 3.11) $X_{11}(\mu)$ and $Y_{22}(\mu)$ are equal to $\Theta(1)$, and $X_{22}(\mu)$ and $Y_{11}(\mu)$ are equal to $\Theta(\mu)$.*

Proof. Now, $\det\left(\frac{X(\mu)Y(\mu)}{\mu}\right) = \prod_{i=1}^n \frac{\lambda_i(X(\mu)Y(\mu))}{\mu} \geq \lambda_{min}(X^0Y^0)^n$, where the inequality follows from Theorem 2.1.

On the other hand, $\det\left(\frac{X(\mu)Y(\mu)}{\mu}\right) = \frac{1}{\mu^n}\det(X(\mu))\det(Y(\mu)) \leq \det(X_{11}(\mu)) \ \det\left(\frac{X_{22}(\mu)}{\mu}\right) \times$ $\det(Y_{22}(\mu)) \ \det\left(\frac{Y_{11}(\mu)}{\mu}\right)$ (where the inequality follows from Theorem 2.4 in [7]). Therefore, we have $\lambda_{min}(X^0Y^0)^n \leq \det(X_{11}(\mu))\det\left(\frac{X_{22}(\mu)}{\mu}\right)\det(Y_{22}(\mu))\det\left(\frac{Y_{11}(\mu)}{\mu}\right)$. Taking log on both sides of the inequality, we have

$$
\begin{aligned}
n\log\lambda_{min}(X^0Y^0) \leq & \ \log\det(X_{11}(\mu)) + \log\det\left(\frac{X_{22}(\mu)}{\mu}\right) + \log\det(Y_{22}(\mu)) \\
& + \log\det\left(\frac{Y_{11}(\mu)}{\mu}\right).
\end{aligned}
$$

Since $X_{22}(\mu)$ and $Y_{11}(\mu)$ are equal to $\mathcal{O}(\mu)$ (by the previous lemma) and $X(\mu), Y(\mu)$ are bounded (by Lemma 2.1), we must have, from the above logarithmic inequality, that $X_{11}(\mu)$ and $Y_{22}(\mu)$ are equal to $\Theta(1)$ and $X_{22}(\mu)$ and $Y_{11}(\mu)$ are equal to $\Theta(\mu)$. **QED**

The above lemmas show that for $(X(\mu), Y(\mu))$ on an off-central path, we have

$$
X(\mu) = \begin{pmatrix} X_{11} & \sqrt{\mu}\widetilde{X}_{12} \\ \sqrt{\mu}\widetilde{X}_{12}^T & \mu\widetilde{X}_{22} \end{pmatrix} \tag{6}
$$

and

$$
Y(\mu) = \begin{pmatrix} \mu\widetilde{Y}_{11} & \sqrt{\mu}\widetilde{Y}_{12} \\ \sqrt{\mu}\widetilde{Y}_{12}^T & Y_{22} \end{pmatrix}, \tag{7}
$$

where $X_{11}, Y_{22}, \widetilde{X}_{22}$ and $\widetilde{Y}_{11}$ are equal to $\Theta(1)$, and $\|\widetilde{X}_{12}(\mu)\|, \|\widetilde{Y}_{12}(\mu)\|$ are equal to $\mathcal{O}(1)$.

Letting $\widetilde{X}(\mu) = \begin{pmatrix} X_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{pmatrix}$ and $\widetilde{Y}(\mu) = \begin{pmatrix} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ \widetilde{Y}_{12}^T & Y_{22} \end{pmatrix}$, we can then write

$$
X(\mu) = \begin{pmatrix} I & 0 \\ 0 & \sqrt{\mu}I \end{pmatrix} \widetilde{X}(\mu) \begin{pmatrix} I & 0 \\ 0 & \sqrt{\mu}I \end{pmatrix}
$$

and

$$
Y(\mu) = \begin{pmatrix} \sqrt{\mu}I & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}(\mu) \begin{pmatrix} \sqrt{\mu}I & 0 \\ 0 & I \end{pmatrix}.
$$

Lemma 2.4 $\widetilde{X}(\mu)$ and $\widetilde{Y}(\mu)$ are positive definite for all $\mu > 0$, and any of their accumulation points are also positive definite.

Proof. Now, from the above relations between $\widetilde{X}$, X and $\widetilde{Y}$, Y, it is clear that $\widetilde{X}(\mu)$, $\widetilde{Y}(\mu)$ are positive definite for all $\mu > 0$.
It can be seen easily that

$$X(\mu)Y(\mu) = \mu \begin{pmatrix} I & 0 \\ 0 & \sqrt{\mu}I \end{pmatrix} \widetilde{X}(\mu)\widetilde{Y}(\mu) \begin{pmatrix} I & 0 \\ 0 & \sqrt{\mu}I \end{pmatrix}^{-1}.$$

That is, $X(\mu)Y(\mu)$ is similar to $\mu\widetilde{X}(\mu)\widetilde{Y}(\mu)$. Thus, by Thereom 2.1, $\lambda_{min}(\widetilde{X}(\mu)\widetilde{Y}(\mu)) = \lambda_{min}(X^0Y^0)$ and $\lambda_{max}(\widetilde{X}(\mu)\widetilde{Y}(\mu)) = \lambda_{max}(X^0Y^0)$. With $\widetilde{X}(\mu)$, $\widetilde{Y}(\mu)$ bounded and positive definite for μ in $(0, \mu_0]$ for any $\mu_0 > 0$, we deduce that any of their accumulation points are also positive definite. **QED**

Let $X_1(t) = X(t^2)$, $Y_1(t) = Y(t^2)$. Similarly, let $\widetilde{X}_1(t) = \widetilde{X}(t^2)$ and $\widetilde{Y}_1(t) = \widetilde{Y}(t^2)$. Then $X_1, \widetilde{X}_1$ and $Y_1, \widetilde{Y}_1$ are related by

$$X_1(t) = \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1(t) \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \tag{8}$$

and

$$Y_1(t) = \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \widetilde{Y}_1(t) \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}. \tag{9}$$

To study the analyticity of $(X(\mu), Y(\mu))$ w.r.t $\sqrt{\mu}$ at $\mu = 0$, it is the same as studying the analyticity of $(X_1(t), Y_1(t))$ when $t = 0$. The following proposition shows that it suffices to do this by studying the analyticity of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ at $t = 0$.

Proposition 2.1 $X_1(t)$ is analytic at $t = 0$ if and only if $\widetilde{X}_1(t)$ is analytic at $t = 0$. Similarly, $Y_1(t)$ is analytic at $t = 0$ if and only if $\widetilde{Y}_1(t)$ is analytic at $t = 0$.

Proof. By (8), we have $(X_1)_{11}(t) = (\widetilde{X}_1)_{11}(t)$, $(X_1)_{12}(t) = t(\widetilde{X}_1)_{12}(t)$, $(X_1)_{22}(t) = t^2(\widetilde{X}_1)_{22}(t)$. If $\widetilde{X}_1$ is analytic, it is obvious that X_1 is also analytic. Now we consider submatrix $(X_1)_{22}$ as example. If X_1 is analytic at $t = 0$, then $(X_1)_{12}$ can be represented as a power series near $t = 0$, say $(X_1)_{12}(t) = U_0 + \sum_{i=1}^{\infty} U_i t^i$. Now, using the fact that $\widetilde{X}_1$ is bounded near $t = 0$, which follows from Lemma 2.2, we have $(X_1)_{12}(0) = 0$. Thus, $U_0 = 0$. Then, $(\widetilde{X}_1)_{12}(t) = \sum_{i=1}^{\infty} U_i t^{i-1}$, which is analytic near $t = 0$. The other parts can be shown similarly. **QED**

Therefore, by this proposition, we need only study the analyticity of $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ at $t = 0$ to conclude the property for $X_1(t)$ and $Y_1(t)$. An advantage for using $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ rather than $X_1(t)$ and $Y_1(t)$ is because their accumulation points are positive definite, by Lemma 2.4 - which is a desirable property, unlike that of $X_1(t)$ and $Y_1(t)$.

Hence, we are going to express the system of ODEs (5) in terms of $\widetilde{X}_1$ and $\widetilde{Y}_1$.

In terms of X_1 and Y_1, (5) becomes

$$\frac{1}{2} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & X_1 \otimes_s Y_1^{-1} \end{pmatrix} \begin{pmatrix} svec(X_1') \\ svec(Y_1') \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 0 \\ svec(X_1) \end{pmatrix}. \tag{10}$$

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Let us reiterate again that if we consider X_1 and Y_1 on an off-central path, then the matrix on the extreme left in (10) is not invertible and may not even be defined as t tends to zero (since Y_1^{-1} does not exist in the limit) and hence it is not possible to analyze the asymptotic behavior of $X_1(t)$ and $Y_1(t)$ if we just use (10). This provides the motivation for us to express (10) in terms of $\widetilde{X}_1$ and $\widetilde{Y}_1$, after which we will see that further analysis is possible.

From here till Proposition 2.3, we describe how we obtain from (10) an ODE system in terms of $\widetilde{X}_1$ and $\widetilde{Y}_1$.

Note that since

$$X_1(t) = \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1(t) \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix},$$

we have

$$X_1'(t) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \widetilde{X}_1(t) \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1'(t) \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} +$$
$$\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \widetilde{X}_1(t) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Therefore,

$$svec(X_1'(t)) = 2 \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) svec(\widetilde{X}_1(t)) +$$
$$\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) svec(\widetilde{X}_1'(t)). \qquad (11)$$

Similarly,

$$svec(Y_1'(t)) = 2 \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) svec(\widetilde{Y}_1(t)) +$$
$$\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) svec(\widetilde{Y}_1'(t)). \qquad (12)$$

We first consider the second equation in (10):

$$\frac{1}{2} \left(svec(X_1') + (X_1 \otimes_s Y_1^{-1}) svec(Y_1') \right) = \frac{1}{t} svec(X_1). \qquad (13)$$

By (8), (9) and using the properties of $\otimes_s$, we have

$$(X_1 \otimes_s Y_1^{-1}) \left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) = \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1}). \qquad (14)$$

Substituting the expressions of (X_1', Y_1'), (11)-(12), and (14) into (13), we have

$$\frac{1}{2} \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) \left(svec(\widetilde{X}_1') + (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1}) svec(\widetilde{Y}_1') \right)$$
$$= \frac{1}{t} \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) svec(\widetilde{X}_1) - \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) svec(\widetilde{X}_1)$$
$$- \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1}) \left(\begin{pmatrix} \frac{1}{t}I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I \right) svec(\widetilde{Y}_1).$$

9

Taking the inverse of the first matrix, we obtain

$$\frac{1}{2}\left(svec(\widetilde{X}_1') + (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})svec(\widetilde{Y}_1')\right)$$

$$= \frac{1}{t}\left[\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right)svec(\widetilde{X}_1) - (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right)svec(\widetilde{Y}_1)\right]. \quad (15)$$

For any $t > 0$, the equation (13) is equivalent to

$$\frac{1}{2}\left((X_1 \otimes_s Y_1^{-1})^{-1}svec(X_1') + svec(Y_1')\right) = \frac{1}{t}svec(Y_1).$$

Similar to the above manipulation, we can derive

$$\frac{1}{2}\left((\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})^{-1}svec(\widetilde{X}_1') + svec(\widetilde{Y}_1')\right)$$

$$= \frac{1}{t}\left[-(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})^{-1}\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s I\right)svec(\widetilde{X}_1) + \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s I\right)svec(\widetilde{Y}_1)\right].$$

Multiplying $(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})$ on both sides, and then comparing with (15), we obtain

$$\left(svec(\widetilde{X}_1') + (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})svec(\widetilde{Y}_1')\right)$$

$$= \frac{1}{t}\left[\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right)svec(\widetilde{X}_1) - (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right)svec(\widetilde{Y}_1)\right]. \quad (16)$$

This equation is equivalent to the second equation in (10) for all $t > 0$. The nice thing in this equation is that all matrices in it are nonsingular at $t = 0$. Thus the only singularity in the equation is explicitly shown up in $1/t$.

Now, we turn to consider the first equation in (10):

$$\mathcal{A}svec(X_1') + \mathcal{B}svec(Y_1') = 0. \quad (17)$$

Substituting (11) and (12) into it, we obtain

$$\mathcal{A}\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}\right)svec(\widetilde{X}_1') + \mathcal{B}\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}\right)svec(\widetilde{Y}_1')$$

$$= -2\left[\mathcal{A}\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}\right)svec(\widetilde{X}_1) + \mathcal{B}\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}\right)svec(\widetilde{Y}_1)\right] \quad (18)$$

Now the coefficients of $svec(\widetilde{X}_1')$ and $svec(\widetilde{Y}_1')$ together may not be of full row rank at $t = 0$. Thus, we shall "eliminate" some t on both sides so that the equivalent (when $t > 0$) new equation will have coefficients of $svec(\widetilde{X}_1')$ and $svec(\widetilde{Y}_1')$ together of full row rank at $t = 0$.

To match the block structures of X and Y, it is convenient to have special form of $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$(\mathcal{A} \quad \mathcal{B}) = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \\ \mathcal{W} \end{pmatrix}, \quad (19)$$

where $\mathcal{U} \in \Re^{i_1 \times 2\tilde{n}}$, $\mathcal{V} \in \Re^{i_2 \times 2\tilde{n}}$ and $\mathcal{W} \in \Re^{i_3 \times 2\tilde{n}}$ are such that

$$\mathcal{U}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} (A_k)_{11} & (A_k)_{12} \\ (A_k)_{12}^T & (A_k)_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & (B_k)_{22} \end{array} \right) \right)^T \right), \quad 1 \le k \le i_1,$$

$$\mathcal{V}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} 0 & (A_k)_{12} \\ (A_k)_{12}^T & (A_k)_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & 0 \end{array} \right) \right)^T \right), \quad i_1 + 1 \le k \le i_1 + i_2,$$

$$\mathcal{W}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} 0 & 0 \\ 0 & (A_k)_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (B_k)_{11} & 0 \\ 0 & 0 \end{array} \right) \right)^T \right), \quad i_1 + i_2 + 1 \le k \le \tilde{n}.$$

for some i_1, i_2 and i_3.

Lemma 2.5 below (the lemma is inspired by a similar result in [7], see also [4]) shows that this special form of $(\mathcal{A} \quad \mathcal{B})$ is valid without loss of generality. Thus, the use of this special form does not sacrifice generality.

Lemma 2.5 *There exists an invertible matrix T such that*

$$T(\mathcal{A} \quad \mathcal{B}) =$$

$$T \left(\begin{array}{cc} svec(A_1)^T & svec(B_1)^T \\ \vdots & \vdots \\ svec(A_{\tilde{n}})^T & svec(B_{\tilde{n}})^T \end{array} \right) =$$

$$\left(\begin{array}{c} \widetilde{\mathcal{U}} \\ \widetilde{\mathcal{V}} \\ \widetilde{\mathcal{W}} \end{array} \right), \tag{20}$$

where $\widetilde{\mathcal{U}} \in \Re^{i_1 \times 2\tilde{n}}$, $\widetilde{\mathcal{V}} \in \Re^{i_2 \times 2\tilde{n}}$ and $\widetilde{\mathcal{W}} \in \Re^{i_3 \times 2\tilde{n}}$ are such that

$$\widetilde{\mathcal{U}}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} (\widetilde{A_k})_{11} & (\widetilde{A_k})_{12} \\ (\widetilde{A_k})_{12}^T & (\widetilde{A_k})_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (\widetilde{B_k})_{11} & (\widetilde{B_k})_{12} \\ (\widetilde{B_k})_{12}^T & (\widetilde{B_k})_{22} \end{array} \right) \right)^T \right), \quad 1 \le k \le i_1,$$

$$\widetilde{\mathcal{V}}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} 0 & (\widetilde{A_k})_{12} \\ (\widetilde{A_k})_{12}^T & (\widetilde{A_k})_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (\widetilde{B_k})_{11} & (\widetilde{B_k})_{12} \\ (\widetilde{B_k})_{12}^T & 0 \end{array} \right) \right)^T \right), \quad i_1 + 1 \le k \le i_1 + i_2,$$

$$\widetilde{\mathcal{W}}_{k\cdot} = \left(\left(svec \left(\begin{array}{cc} 0 & 0 \\ 0 & (\widetilde{A_k})_{22} \end{array} \right) \right)^T \quad \left(svec \left(\begin{array}{cc} (\widetilde{B_k})_{11} & 0 \\ 0 & 0 \end{array} \right) \right)^T \right), \quad i_1 + i_2 + 1 \le k \le \tilde{n}.$$

Here $0 \le i_1, i_2, i_3 \le \tilde{n}$ - how i_1, i_2 and i_3 are defined is clear from the proof of the lemma.

Proof. For the sake of proving the lemma, assume without loss of generality that the entries in $svec(A_i)^T$ are rearranged resulting in the row vector $((\widehat{A_i})_{11} \quad (\widehat{A_i})_{12} \quad (\widehat{A_i})_{22})$, where $(\widehat{A_i})_{11}$ comprises of the entries in $svec(A_i)^T$ corresponding to the upper left hand block $(A_i)_{11}$ of A_i, $(\widehat{A_i})_{12}$ entries of $svec(A_i)^T$ corresponding to the upper right hand block $(A_i)_{12}$ of A_i and $(\widehat{A_i})_{22}$ entries of $svec(A_i)^T$ corresponding to the lower right hand block $(A_i)_{22}$ of A_i. Similarly, assume that $svec(B_i)^T$ is written as $((\widehat{B_i})_{11} \quad (\widehat{B_i})_{12} \quad (\widehat{B_i})_{22})$.

Let

$$
i_1 = \text{rank} \begin{pmatrix} (\widehat{A_1})_{11} & (\widehat{B_1})_{22} \\ \vdots & \vdots \\ (\widehat{A_{\tilde{n}}})_{11} & (\widehat{B_{\tilde{n}}})_{22} \end{pmatrix},
$$

$$
i_2 = \text{rank} \begin{pmatrix} (\widehat{A_1})_{11} & (\widehat{A_1})_{12} & (\widehat{B_1})_{12} & (\widehat{B_1})_{22} \\ \vdots & \vdots & \vdots & \vdots \\ (\widehat{A_{\tilde{n}}})_{11} & (\widehat{A_{\tilde{n}}})_{12} & (\widehat{B_{\tilde{n}}})_{12} & (\widehat{B_{\tilde{n}}})_{22} \end{pmatrix} - i_1,
$$

$$
i_3 = \text{rank} \begin{pmatrix} svec(A_1)^T & svec(B_1)^T \\ \vdots & \vdots \\ svec(A_{\tilde{n}})^T & svec(B_{\tilde{n}})^T \end{pmatrix} - (i_1 + i_2),
$$

where $i_1 + i_2 + i_3$ is equal to $\tilde{n}$, by Assumption 2.1(c). Then the lemma holds by applying the technique of block Gaussian elimination method to

$$
\begin{pmatrix} svec(A_1)^T & svec(B_1)^T \\ \vdots & \vdots \\ svec(A_{\tilde{n}})^T & svec(B_{\tilde{n}})^T \end{pmatrix}.
$$

Namely, we first eliminate to zero, $(\tilde{n} - i_1)$ rows in

$$
\begin{pmatrix} (\widehat{A_1})_{11} & (\widehat{B_1})_{22} \\ \vdots & \vdots \\ (\widehat{A_{\tilde{n}}})_{11} & (\widehat{B_{\tilde{n}}})_{22} \end{pmatrix}.
$$

Then, we eliminate to zero, i_3 rows in the corresponding $(\tilde{n} - i_1)$ rows in

$$
\begin{pmatrix} (\widehat{A_1})_{12} & (\widehat{B_1})_{12} \\ \vdots & \vdots \\ (\widehat{A_{\tilde{n}}})_{12} & (\widehat{B_{\tilde{n}}})_{12} \end{pmatrix},
$$

to obtain the required result. **QED**

Remark 2.2 *It should be noted, by construction, that in the above*

$$
i_1 = rank(\widetilde{\mathcal{U}}_1),
$$

where $\widetilde{\mathcal{U}}_1 \in \Re^{i_1 \times 2\tilde{n}}$ is defined by

$$
(\widetilde{\mathcal{U}}_1)_{k\cdot} = \left(\left(svec \begin{pmatrix} (\widetilde{A_k})_{11} & 0 \\ 0 & 0 \end{pmatrix} \right)^T \quad \left(svec \begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{B_k})_{22} \end{pmatrix} \right)^T \right), 1 \leq k \leq i_1.
$$

$$
i_2 = rank(\widetilde{\mathcal{V}}_1),
$$

where $\widetilde{\mathcal{V}}_1 \in \Re^{i_2 \times 2\tilde{n}}$ is defined by

$$(\widetilde{\mathcal{V}}_1)_{k\cdot} = \left(\left(svec\left(\begin{array}{cc} 0 & (\widetilde{A}_k)_{12} \\ (\widetilde{A}_k)_{12}^T & 0 \end{array} \right) \right)^T \quad \left(svec\left(\begin{array}{cc} 0 & (\widetilde{B}_k)_{12} \\ (\widetilde{B}_k)_{12}^T & 0 \end{array} \right) \right)^T \right), \, i_1 + 1 \leq k \leq i_1 + i_2.$$

$$i_3 = rank(\widetilde{\mathcal{W}}_1),$$

where $\widetilde{\mathcal{W}}_1 \in \Re^{i_3 \times 2\tilde{n}}$ is defined by

$$(\widetilde{\mathcal{W}}_1)_{k\cdot} = \left(\left(svec\left(\begin{array}{cc} 0 & 0 \\ 0 & (\widetilde{A}_k)_{22} \end{array} \right) \right)^T \quad \left(svec\left(\begin{array}{cc} (\widetilde{B}_k)_{11} & 0 \\ 0 & 0 \end{array} \right) \right)^T \right), \, i_1 + i_2 + 1 \leq k \leq \tilde{n}.$$

Here $i_1 + i_2 + i_3 = \tilde{n}$.

From now onwards, we can assume, without loss of generality, that $\mathcal{A} = \left(\begin{array}{c} svec(A_1)^T \\ \vdots \\ svec(A_{\tilde{n}})^T \end{array} \right)$ and

$\mathcal{B} = \left(\begin{array}{c} svec(B_1)^T \\ \vdots \\ svec(B_{\tilde{n}})^T \end{array} \right)$ are given by (19). In these forms, again, $(\mathcal{A} \ \mathcal{B})$ have full row rank and

$$\mathcal{A}u + \mathcal{B}v = 0 \quad \Rightarrow \quad u^T v \geq 0. \tag{21}$$

Let us make use of these new structures of $\mathcal{A}$ and $\mathcal{B}$.

Now, for each $i = 1, \ldots, \tilde{n}$, observe that

$$svec(A_i)^T \left(\left(\begin{array}{cc} I & 0 \\ 0 & tI \end{array} \right) \otimes_s \left(\begin{array}{cc} I & 0 \\ 0 & tI \end{array} \right) \right) = \left(svec\left(\begin{array}{cc} (A_i)_{11} & t(A_i)_{12} \\ t(A_i)_{12}^T & t^2(A_i)_{22} \end{array} \right) \right)^T.$$

Together with form (19) for $\mathcal{A}$, we can see easily that

$$\mathcal{A}\left(\left(\begin{array}{cc} I & 0 \\ 0 & tI \end{array} \right) \otimes_s \left(\begin{array}{cc} I & 0 \\ 0 & tI \end{array} \right) \right) = \text{diag}(I, tI, t^2 I)\mathcal{A}(t),$$

where

$$\mathcal{A}(t)_{k\cdot} = \left\{ \begin{array}{ll} \left(svec\left(\begin{array}{cc} (A_k)_{11} & t(A_k)_{12} \\ t(A_k)_{12}^T & t^2(A_k)_{22} \end{array} \right) \right)^T & \text{for } 1 \leq k \leq i_1 \\[3mm] \left(svec\left(\begin{array}{cc} 0 & (A_k)_{12} \\ (A_k)_{12}^T & t(A_k)_{22} \end{array} \right) \right)^T & \text{for } i_1 + 1 \leq k \leq i_1 + i_2 \\[3mm] \left(svec\left(\begin{array}{cc} 0 & 0 \\ 0 & (A_k)_{22} \end{array} \right) \right)^T & \text{for } i_1 + i_2 + 1 \leq k \leq \tilde{n} \end{array} \right. \tag{22}$$

Remark 2.3 *Note that in this and the next section, diag$(I, tI, t^2 I)$ or diag$(I, tI, t^2 I, C)$, where C is a matrix, whenever it appears, has its first diagonal block the identity matrix of dimension i_1, its second diagonal block a multiple of the identity matrix of dimension i_2, and its third diagonal block a multiple of the identity matrix of dimension $\widetilde{n} - i_1 - i_2 = i_3$.*

In a similar fashion, we have

$$\mathcal{B}\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) = \mathrm{diag}(I, tI, t^2 I)\mathcal{B}(t),$$

where

$$\mathcal{B}(t)_{k\cdot} = \begin{cases} \left(svec \begin{pmatrix} t^2(B_k)_{11} & t(B_k)_{12} \\ t(B_k)_{12}^T & (B_k)_{22} \end{pmatrix} \right)^T & \text{for } 1 \le k \le i_1 \\[12pt] \left(svec \begin{pmatrix} t(B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & 0 \end{pmatrix} \right)^T & \text{for } i_1 + 1 \le k \le i_1 + i_2 \\[12pt] \left(svec \begin{pmatrix} (B_k)_{11} & 0 \\ 0 & 0 \end{pmatrix} \right)^T & \text{for } i_1 + i_2 + 1 \le k \le \tilde{n} \end{cases} . \tag{23}$$

Therefore, we have the following lemma:

Lemma 2.6 *With the matrices $\mathcal{A}$ and $\mathcal{B}$ in the form of (19), we have*

$$\mathcal{A}\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) = diag(I, tI, t^2 I)\mathcal{A}(t)$$

and

$$\mathcal{B}\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) = diag(I, tI, t^2 I)\mathcal{B}(t),$$

where $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined in (22) and (23) respectively.

Proof. As above. **QED**

From Lemma 2.6, we see that using the new structures of $\mathcal{A}$ and $\mathcal{B}$ in (19), we are able to "factor" out t and t^2 from various blocks in $\mathcal{A}\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right)$ and $\mathcal{B}\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right)$, while maintaining the "monotonicity" and "full row rank" properties of $(\mathcal{A} \ \mathcal{B})$ in $(\mathcal{A}(t) \ \mathcal{B}(t))$. This proves to be important in rewriting the ODE system (10) into a better form.

Lemma 2.6 can also be used to reformulate the right hand side of (18).

$$\mathcal{A}\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) =$$

$$\mathrm{diag}(I, tI, t^2 I)\mathcal{A}(t)\left(\begin{pmatrix} I & 0 \\ 0 & \frac{1}{t}I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & \frac{1}{t}I \end{pmatrix} \right)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) =$$

$$\mathrm{diag}(I, tI, t^2 I)\mathcal{A}(t)\left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{t}I \end{pmatrix} \otimes_s I \right).$$

Also,

$$\mathcal{B}\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}\right) = \mathrm{diag}(I, tI, t^2 I)\mathcal{B}(t)\left(\begin{pmatrix} \frac{1}{t}I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right).$$

Therefore, for any $t > 0$, the equation (18) is equivalent to

$$\frac{1}{2}\mathcal{A}(t)svec(\widetilde{X}_1') + \frac{1}{2}\mathcal{B}(t)svec(\widetilde{Y}_1')$$
$$= -\frac{1}{t}\mathcal{A}(t)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s I\right)svec(\widetilde{X}_1) - \frac{1}{t}\mathcal{B}(t)\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right)svec(\widetilde{Y}_1). \qquad (24)$$

As we assume the special structure of $\mathcal{A}$ and $\mathcal{B}$ in the form of (19), the reformulated equation of $\mathcal{A}X + \mathcal{B}Y = q$ have the following special form.

Proposition 2.2 *With $\mathcal{A}$ and $\mathcal{B}$ in the form of (19) and $\mathcal{A}(t)$ and $\mathcal{B}(t)$ defined by (22) and (23) respectively, the linear equation in the SDLCP (1) is equivalent to*

$$\mathcal{A}(t)svec(\widetilde{X}_1(t)) + \mathcal{B}(t)svec(\widetilde{Y}_1(t)) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} \qquad (25)$$

and we let $q = (q_1^T, 0, 0)^T$ from now onwards.

Proof. Since $\mathcal{A}svec(X_1(t)) + \mathcal{B}svec(Y_1(t)) = q$, we have

$$\mathcal{A}\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}\right)svec(\widetilde{X}_1(t)) + \mathcal{B}\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix}\right)svec(\widetilde{Y}_1(t)) = q.$$

Hence, by Lemma 2.6,

$$\mathrm{diag}(I, tI, t^2 I)\left(\mathcal{A}(t)svec(\widetilde{X}_1(t)) + \mathcal{B}(t)svec(\widetilde{Y}_1(t))\right) = q.$$

This implies that q is equal to $(q_1^T, 0, 0)^T$ where $q_1 \in \Re^{i_1}$, which can be seen by letting t tends to zero in above. Therefore,

$$\mathcal{A}(t)svec(\widetilde{X}_1(t)) + \mathcal{B}(t)svec(\widetilde{Y}_1(t)) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}. \qquad (26)$$

QED

We shall further simplify the right-hand side of (24). By the structure of $\mathcal{A}(t)$ in (22), we have the following:

$$\left(\mathcal{A}(t)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s I\right)\right)_{k\cdot} = \begin{cases} t\left(svec\begin{pmatrix} 0 & \frac{1}{2}(A_k)_{12} \\ \frac{1}{2}(A_k)_{12}^T & t(A_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \le k \le i_1 \\ \frac{1}{2}\mathcal{A}(t)_{k\cdot} + t\left(svec\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(A_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1 + 1 \le k \le i_1 + i_2 \\ \mathcal{A}(t)_{k\cdot} & \text{for } i_1 + i_2 + 1 \le k \le \tilde{n} \end{cases}.$$

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By the structure of $\mathcal{B}(t)$ in (23), an analogous structure for $\mathcal{B}(t)\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right)$ also holds.

Now, by Proposition 2.2, we have, for $k = i_1 + 1, \ldots, \tilde{n}$,

$$\mathcal{A}(t)_{k\cdot} svec(\widetilde{X}_1(t)) + \mathcal{B}(t)_{k\cdot} svec(\widetilde{Y}_1(t)) = 0.$$

Therefore,

$$\frac{2}{t}\mathcal{A}(t)\left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \otimes_s I\right) svec(\widetilde{X}_1) + \frac{2}{t}\mathcal{B}(t)\left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \otimes_s I\right) svec(\widetilde{Y}_1)$$

from the right-hand side of (24) equals

$$\mathcal{G}(t)svec(\widetilde{X}_1(t)) + \mathcal{H}(t)svec(\widetilde{Y}_1(t)),$$

where

$$\mathcal{G}(t)_{k\cdot} := \begin{cases} \left(svec\begin{pmatrix} 0 & (A_k)_{12} \\ (A_k)_{12}^T & 2t(A_k)_{22} \end{pmatrix}\right)^T & \text{for } 1 \le k \le i_1 \\ \left(svec\begin{pmatrix} 0 & 0 \\ 0 & (A_k)_{22} \end{pmatrix}\right)^T & \text{for } i_1 + 1 \le k \le i_1 + i_2 \\ 0 & \text{for } i_1 + i_2 + 1 \le k \le \tilde{n} \end{cases} \qquad (27)$$

and

$$\mathcal{H}(t)_{k\cdot} := \begin{cases} \left(svec\begin{pmatrix} 2t(B_k)_{11} & (B_k)_{12} \\ (B_k)_{12}^T & 0 \end{pmatrix}\right)^T & \text{for } 1 \le k \le i_1 \\ \left(svec\begin{pmatrix} (B_k)_{11} & 0 \\ 0 & 0 \end{pmatrix}\right)^T & \text{for } i_1 + 1 \le k \le i_1 + i_2 \\ 0 & \text{for } i_1 + i_2 + 1 \le k \le \tilde{n} \end{cases} \qquad (28)$$

Now we can present the main result of this section: a reformulated ODE system for the off-central paths.

Proposition 2.3 *The off-central path for SDLCP, $(X(\mu), Y(\mu))$, $\mu > 0$, is the solution of the system of ODEs (5) with $(X(1), Y(1)) = (X^0, Y^0)$, if and only if $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$, $t > 0$, is the solution to the following system of ODEs*

$$\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix} \begin{pmatrix} svec(\widetilde{X}_1') \\ svec(\widetilde{Y}_1') \end{pmatrix} = \qquad (29)$$

$$\begin{pmatrix} -\mathcal{G}(t) & -\mathcal{H}(t) \\ \frac{1}{t}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I & -\frac{1}{t}(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I\right) \end{pmatrix} \begin{pmatrix} svec(\widetilde{X}_1) \\ svec(\widetilde{Y}_1) \end{pmatrix},$$

with $(\widetilde{X}_1(1), \widetilde{Y}_1(1)) = (X^0, Y^0)$.
Here $X(\mu)(= X_1(t))$, $\widetilde{X}_1(t)$ and $Y(\mu)(= Y_1(t))$, $\widetilde{Y}_1(t)$ are related by (8) and (9) respectively, where $\mu = t^2$.

Proof. The ODE system (29) follows from (24) and (16) which are equivalent to the ODE system (10) and thus equivalent to (5) for $\mu > 0$ $(t > 0)$. **QED**

The importance of this proposition is that the coefficient matrix on the left-hand side is non-singular for all $t \geq 0$ (even at $t = 0$), as will be shown in Proposition 2.4 below. This enable us to investigate the asymptotic behavior of off-central paths as $t \to 0$ (or $\mu \to 0$).

We observe, in the following proposition, an important property of the matrix $\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix}$ on the left-hand side of the system of equations (29).

Proposition 2.4 $\begin{pmatrix} \beta\mathcal{A}(t) & \beta\mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix}$, *where $\beta \neq 0$, $\beta \in \Re$, is invertible for all $t \geq 0$ and* $\widetilde{X}_1$, $\widetilde{Y}_1$ *positive definite.*

Proof. To prove the proposition, it suffices to show that

$$\begin{pmatrix} \beta\mathcal{A}(t) & \beta\mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \quad \Rightarrow \quad u = v = 0,$$

for $t \geq 0$ and $\widetilde{X}_1$, $\widetilde{Y}_1$ positive definite.

A sufficient condition for the above to hold is to show that

$$\mathcal{A}(t)u + \mathcal{B}(t)v = 0 \quad \Rightarrow \quad u^T v \geq 0 \tag{30}$$

Now, for $t > 0$, (30) is true by Lemma 2.6 and since (21) holds.

Therefore, we need only show (30) for the case $t = 0$.

Suppose $\mathcal{A}(0)u + \mathcal{B}(0)v = 0$. We want to show that $u^T v \geq 0$. (The idea to prove this follows the proof of Theorem 3.13 in [7].)

Let $u = svec \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{pmatrix}$ and $v = svec \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}$.

We have $\mathcal{A} \, svec \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{B} \, svec \begin{pmatrix} 0 & 0 \\ 0 & V_{22} \end{pmatrix} = 0$ since $\mathcal{A}(0)u + \mathcal{B}(0)v = 0$.

Also, $\mathcal{A} \, svec \begin{pmatrix} W_1 & Z_1 \\ Z_1^T & U_{22} \end{pmatrix} + \mathcal{B} \, svec \begin{pmatrix} V_{11} & Z_2 \\ Z_2^T & W_2 \end{pmatrix} = 0$ for some $W_1 \in S^m$, $W_2 \in S^{n-m}$ and $Z_1, Z_2 \in \Re^{m \times (n-m)}$. This is possible because $\mathcal{A}(0)u + \mathcal{B}(0)v = 0$ and by Remark 2.2.

Letting $X(s) = \begin{pmatrix} W_1 & Z_1 \\ Z_1^T & U_{22} \end{pmatrix} + s \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and $Y(s) = \begin{pmatrix} V_{11} & Z_2 \\ Z_2^T & W_2 \end{pmatrix} + s \begin{pmatrix} 0 & 0 \\ 0 & V_{22} \end{pmatrix}$, we have $\mathcal{A} \, svec(X(s)) + \mathcal{B} \, svec(Y(s)) = 0$ for all $s \in \Re$. Therefore, by (21), $X(s) \bullet Y(s) \geq 0$ for all $s \in \Re$. Expanding $X(s) \bullet Y(s)$, we have $W_1 \bullet V_{11} + U_{22} \bullet W_2 + 2Z_1 \bullet Z_2 + s(U_{11} \bullet V_{11} + U_{22} \bullet V_{22}) \geq 0$ for all $s \in \Re$. This must imply that $U_{11} \bullet V_{11} + U_{22} \bullet V_{22} = 0$.

We are done if we can show that $U_{12} \bullet V_{12} \geq 0$. This is true since there exist $W_3 \in S^m$ and $W_4 \in S^{n-m}$ such that $\mathcal{A} \, svec \begin{pmatrix} W_3 & U_{12} \\ U_{12}^T & 0 \end{pmatrix} + \mathcal{B} \, svec \begin{pmatrix} 0 & V_{12} \\ V_{12}^T & W_4 \end{pmatrix} = 0$ (the reason for this is because $\mathcal{A}(0)u + \mathcal{B}(0)v = 0$ and by Remark 2.2) and then by (21).

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Therefore, we have $u^T v = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{pmatrix} \bullet \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix} \geq 0$. **QED**

Note that the matrix $\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix}$ in (29) is invertible at any accumulation point of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$. (This follows from Proposition 2.4 since any accumulation point of $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ is positive definite, by Lemma 2.4.) This fact implies that the matrix is still well-defined and invertible at the limit as t tends to zero and this enables us to study the asymptotic behavior of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$.

Using (29), we can give a necessary and sufficient condition for $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ of an off-central path to be analytic at $t = 0$. This will be studied in the next section.

3 Asymptotic Analyticity Behavior of Off-Central Path.

First, we have the following technical proposition:

Proposition 3.1 *Let $(\widetilde{X}_1^*, \widetilde{Y}_1^*)$ be an accumulation point of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ of an off-central path as t approaches zero. Then*

$$(\widetilde{Y}_1^*)_{12} = 0 \iff$$
$$(\widetilde{Y}_1^*)^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \widetilde{Y}_1^* \widetilde{X}_1^* + \widetilde{X}_1^* \widetilde{Y}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (\widetilde{Y}_1^*)^{-1} = \begin{pmatrix} 2(\widetilde{X}_1^*)_{11} & 0 \\ 0 & -2(\widetilde{X}_1^*)_{22} \end{pmatrix}$$

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose

$$(\widetilde{Y}_1^*)^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \widetilde{Y}_1^* \widetilde{X}_1^* + \widetilde{X}_1^* \widetilde{Y}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (\widetilde{Y}_1^*)^{-1} = \begin{pmatrix} 2(\widetilde{X}_1^*)_{11} & 0 \\ 0 & -2(\widetilde{X}_1^*)_{22} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \widetilde{Y}_1^* \widetilde{X}_1^* \widetilde{Y}_1^* + \widetilde{Y}_1^* \widetilde{X}_1^* \widetilde{Y}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 2\widetilde{Y}_1^* \begin{pmatrix} (\widetilde{X}_1^*)_{11} & 0 \\ 0 & -(\widetilde{X}_1^*)_{22} \end{pmatrix} \widetilde{Y}_1^*.$$

Now,

$$\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \widetilde{Y}_1^* \widetilde{X}_1^* \widetilde{Y}_1^* + \widetilde{Y}_1^* \widetilde{X}_1^* \widetilde{Y}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right)_{11} =$$
$$2\left((\widetilde{Y}_1^*)_{11}(\widetilde{X}_1^*)_{11}(\widetilde{Y}_1^*)_{11} + (\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11} + (\widetilde{Y}_1^*)_{11}(\widetilde{X}_1^*)_{12}(\widetilde{Y}_1^*)_{12}^T + (\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T \right)$$

and

$$2\left(\widetilde{Y}_1^* \begin{pmatrix} (\widetilde{X}_1^*)_{11} & 0 \\ 0 & -(\widetilde{X}_1^*)_{22} \end{pmatrix} \widetilde{Y}_1^* \right)_{11} = 2(\widetilde{Y}_1^*)_{11}(\widetilde{X}_1^*)_{11}(\widetilde{Y}_1^*)_{11} - 2(\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T.$$

Equating them together, we have

$$(\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11} + (\widetilde{Y}_1^*)_{11}(\widetilde{X}_1^*)_{12}(\widetilde{Y}_1^*)_{12}^T + 2(\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T = 0.$$

Hence,

$$(\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{12}^T + (\widetilde{Y}_1^*)_{11}(\widetilde{X}_1^*)_{12}(\widetilde{Y}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11}^{-1} = -2(\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11}^{-1}.$$

Therefore, $(\widetilde{Y}_1^*)_{12} \bullet (\widetilde{X}_1^*)_{12} = -\operatorname{Tr}((\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11}^{-1}) \leq 0$.

On the other hand, consider $X_1(t)$ and $Y_1(t)$ where $X_1(t), \widetilde{X}_1(t)$ and $Y_1(t), \widetilde{Y}_1(t)$ are related by (8) and (9) respectively. Let $\{t_k\}$ be a sequence tending to zero such that $(X_1(t_k), Y_1(t_k))$ approaches (X^*, Y^*) and $(\widetilde{X}_1(t_k), \widetilde{Y}_1(t_k))$ approaches $(\widetilde{X}_1^*, \widetilde{Y}_1^*)$. Note that (X^*, Y^*) is a solution to SDLCP (1). (Hence $X^* \bullet Y^* = 0$.) Also, $(X^*)_{11} = (\widetilde{X}_1^*)_{11}$ and $(Y^*)_{22} = (\widetilde{Y}_1^*)_{22}$.
Note also that since $(X_1(t_k), Y_1(t_k))$ and (X^*, Y^*) satisfy $A(X) + B(Y) = q$, we have, by Assumption 2.1(a) (or (21)), $(X_1(t_k) - X^*) \bullet (Y_1(t_k) - Y^*) \geq 0$.
Therefore, $X_1(t_k) \bullet Y_1(t_k) \geq X_1(t_k) \bullet Y^* + X^* \bullet Y_1(t_k)$, where we have used $X^* \bullet Y^* = 0$.
Note that $X_1(t_k) \bullet Y_1(t_k) = t_k^2 \widetilde{X}_1(t_k) \bullet \widetilde{Y}_1(t_k)$, $X_1(t_k) \bullet Y^* = t_k^2(\widetilde{X}_1(t_k))_{22} \bullet (\widetilde{Y}_1^*)_{22}$ and $X^* \bullet Y_1(t_k) = t_k^2(\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1(t_k))_{11}$, by (8), (9) and $(Y^*)_{22} = (\widetilde{Y}_1^*)_{22}$, $(X^*)_{11} = (\widetilde{X}_1^*)_{22}$. Hence $\widetilde{X}_1(t_k) \bullet \widetilde{Y}_1(t_k) \geq (\widetilde{X}_1(t_k))_{22} \bullet (\widetilde{Y}_1^*)_{22} + (\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1(t_k))_{11}$.
Letting t_k tends to zero, we have $\widetilde{X}_1^* \bullet \widetilde{Y}_1^* \geq (\widetilde{X}_1^*)_{22} \bullet (\widetilde{Y}_1^*)_{22} + (\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1^*)_{11}$.
Since $\widetilde{X}_1^* \bullet \widetilde{Y}_1^* = (\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1^*)_{11} + 2(\widetilde{X}_1^*)_{12} \bullet (\widetilde{Y}_1^*)_{12} + (\widetilde{X}_1^*)_{22} \bullet (\widetilde{Y}_1^*)_{22}$, we have $(\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1^*)_{11} + 2(\widetilde{X}_1^*)_{12} \bullet (\widetilde{Y}_1^*)_{12} + (\widetilde{X}_1^*)_{22} \bullet (\widetilde{Y}_1^*)_{22} \geq (\widetilde{X}_1^*)_{22} \bullet (\widetilde{Y}_1^*)_{22} + (\widetilde{X}_1^*)_{11} \bullet (\widetilde{Y}_1^*)_{11}$. This implies that $(\widetilde{X}_1^*)_{12} \bullet (\widetilde{Y}_1^*)_{12} \geq 0$.

Combining with $(\widetilde{Y}_1^*)_{12} \bullet (\widetilde{X}_1^*)_{12} \leq 0$ obtained earlier, we have $\operatorname{Tr}((\widetilde{Y}_1^*)_{12}(\widetilde{X}_1^*)_{22}(\widetilde{Y}_1^*)_{12}^T(\widetilde{Y}_1^*)_{11}^{-1}) = -(\widetilde{Y}_1^*)_{12} \bullet (\widetilde{X}_1^*)_{12} = 0$ which means that $(\widetilde{Y}_1^*)_{12} = 0$, since $(\widetilde{X}_1^*)_{22}$, $(\widetilde{Y}_1^*)_{11}$ are symmetric, positive definite. Hence we are done. **QED**

With this technical proposition, the following proposition follows almost immediately.

Proposition 3.2 *Let $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ be a solution to the system of ODEs (29) for $t > 0$. Suppose $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ converges as $t \longrightarrow 0$. Then $\lim_{t \to 0}(\widetilde{Y}_1)_{12}(t) = 0$.*

Proof. Suppose $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ converge as $t \longrightarrow 0$.
Let $\widetilde{X}_1(t) \longrightarrow \widetilde{X}_1^* \succ 0, \widetilde{Y}_1(t) \longrightarrow \widetilde{Y}_1^* \succ 0$ as $t \longrightarrow 0$. We must have

$$\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) svec(\widetilde{X}_1^*) - (\widetilde{X}_1^* \otimes_s (\widetilde{Y}_1^*)^{-1}) \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) svec(\widetilde{Y}_1^*)$$

is equal to zero. (If not, then from (29), we see that there exists at least one element of $(svec(\widetilde{X}_1')^T, svec(\widetilde{Y}_1')^T)^T$ that behaves like $1/t$ for all t close to zero. This implies that $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ is unbounded as $t \to 0$, which contradicts its convergence.)
Therefore,

$$\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) svec(\widetilde{X}_1^*) - (\widetilde{X}_1^* \otimes_s (\widetilde{Y}_1^*)^{-1}) \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) svec(\widetilde{Y}_1^*) = 0.$$

Using the properties of $\otimes_s$, we have

$$svec \begin{pmatrix} (\widetilde{X}_1^*)_{11} & 0 \\ 0 & -(\widetilde{X}_1^*)_{22} \end{pmatrix} - \tfrac{1}{2}\left(\left(\widetilde{X}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right) \otimes_s (\widetilde{Y}_1^*)^{-1} + \right.$$
$$\left. \left((\widetilde{Y}_1^*)^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right) \otimes_s \widetilde{X}_1^* \right) svec(\widetilde{Y}_1^*) = 0,$$

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which implies that

$$(\widetilde{Y}_1^*)^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \widetilde{Y}_1^* \widetilde{X}_1^* + \widetilde{X}_1^* \widetilde{Y}_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (\widetilde{Y}_1^*)^{-1} = \begin{pmatrix} 2(\widetilde{X}_1^*)_{11} & 0 \\ 0 & -2(\widetilde{X}_1^*)_{22} \end{pmatrix}.$$

Hence $(\widetilde{Y}_1^*)_{12} = 0$, by Proposition 3.1. Therefore, $\lim_{t \to 0} (\widetilde{Y}_1)_{12}(t) = 0$. **QED**

We are now ready to state a necessary and sufficient condition for $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ to be analytic at $t = 0$. We have the following theorem:

Theorem 3.1 *Let $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ be a solution to the system of ODEs (29) for $t > 0$. Then $\widetilde{X}_1(t)$, $\widetilde{Y}_1(t)$ converge as $t \to 0$ and are analytic at $t = 0$ if and only if $(\widetilde{Y}_1)_{12}(t)$ converges to zero as $t \to 0$ and is analytic at $t = 0$.*

Proof. ($\Rightarrow$) Suppose $\widetilde{X}_1(t)$ and $\widetilde{Y}_1(t)$ converge as $t \to 0$ and are analytic at $t = 0$. Therefore, by Proposition 3.2, $\lim_{t \to 0} (\widetilde{Y}_1)_{12}(t) = 0$. This, together with the analyticity of $(\widetilde{Y}_1)_{12}(t)$ at $t = 0$, implies our required result.

($\Leftarrow$) Suppose $(\widetilde{Y}_1)_{12}(t) = tW_1(t)$ for t (> 0) near 0, where $W_1(t)$ is analytic at $t = 0$.

From (29), we have

$$\begin{pmatrix} svec(\widetilde{X}_1') \\ svec(\widetilde{Y}_1') \end{pmatrix} = \frac{\mathcal{F}_1(t, \widetilde{X}_1, \widetilde{Y}_1)}{t},$$

where

$$\mathcal{F}_1(t, \widetilde{X}_1, \widetilde{Y}_1) = \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1} \end{pmatrix}^{-1} \times$$

$$\left(\begin{pmatrix} -t\mathcal{G}(t) \\ \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \end{pmatrix} \begin{pmatrix} -t\mathcal{H}(t) \\ -(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1}) \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) \end{pmatrix} \right) \begin{pmatrix} svec(\widetilde{X}_1) \\ svec(\widetilde{Y}_1) \end{pmatrix}.$$

We want to show that $\mathcal{F}_1(t, \widetilde{X}_1(t), \widetilde{Y}_1(t))$ can be written as $t\widetilde{a}_0(t, \widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$, where $\widetilde{a}_0$ as a function of $(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ is analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ with $\widetilde{X}_1$, $(\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$.

Now, it is clear that

$$\widetilde{B}_0(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22}) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ I & \widetilde{X}_1 \otimes_s \begin{pmatrix} (\widetilde{Y}_1)_{11} & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & (\widetilde{Y}_1)_{22} \end{pmatrix}^{-1} \end{pmatrix}^{-1}$$

is defined and analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$, since $(\widetilde{Y}_1)_{12}(t)$ is analytic at $t = 0$ and $(\widetilde{Y}_1)_{12}(0) = 0$, and by Proposition 2.4.

Next, let us consider

$$\left(\begin{pmatrix} -t\mathcal{G}(t) \\ \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \end{pmatrix} \begin{pmatrix} -t\mathcal{H}(t) \\ -(\widetilde{X}_1(t) \otimes_s \widetilde{Y}_1^{-1}(t)) \left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes_s I \right) \end{pmatrix} \right) \begin{pmatrix} svec(\widetilde{X}_1(t)) \\ svec(\widetilde{Y}_1(t)) \end{pmatrix}.$$

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Clearly, $-t\mathcal{G}(t)svec(\widetilde{X}_1(t))-t\mathcal{H}(t)svec(\widetilde{Y}_1(t))$ is equal to $t\widetilde{c}(t,\widetilde{X}_1(t),(\widetilde{Y}_1)_{11}(t),(\widetilde{Y}_1)_{22}(t))$, where

$$\widetilde{c}(t,\widetilde{X}_1,(\widetilde{Y}_1)_{11},(\widetilde{Y}_1)_{22}) := -\mathcal{G}(t)svec(\widetilde{X}_1) - \mathcal{H}(t)svec\begin{pmatrix} (\widetilde{Y}_1)_{11} & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & (\widetilde{Y}_1)_{22} \end{pmatrix}$$

is analytic at $(0,\widetilde{X}_1,(\widetilde{Y}_1)_{11},(\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1,(\widetilde{Y}_1)_{11},(\widetilde{Y}_1)_{22}\succ 0$.

Consider

$$smat\left(\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\otimes_s I\right)svec(\widetilde{X}_1(t)) - (\widetilde{X}_1(t)\otimes_s\widetilde{Y}_1^{-1}(t))\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\otimes_s I\right)svec(\widetilde{Y}_1(t))\right)$$

which is equal to

$$\frac{1}{4}\left[2\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix} - \widetilde{Y}_1^{-1}(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widetilde{Y}_1(t)\widetilde{X}_1(t) - \widetilde{X}_1(t)\widetilde{Y}_1(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widetilde{Y}_1^{-1}(t)\right].$$

Let

$$D(t) := \widetilde{Y}_1^{-1}(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widetilde{Y}_1(t)\widetilde{X}_1(t) + \widetilde{X}_1(t)\widetilde{Y}_1(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widetilde{Y}_1^{-1}(t) -$$

$$2\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}.$$

We have

$$\widetilde{Y}_1(t)D(t)\widetilde{Y}_1(t) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widetilde{Y}_1(t)\widetilde{X}_1(t)\widetilde{Y}_1(t)+$$

$$\widetilde{Y}_1(t)\widetilde{X}_1(t)\widetilde{Y}_1(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - 2\widetilde{Y}_1(t)\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}\widetilde{Y}_1(t). \tag{31}$$

Let $\widetilde{Y}_1(t) := \widehat{Y}_1(t) + \bar{Y}_1(t)$, where

$$\widehat{Y}_1(t) := \begin{pmatrix} (\widetilde{Y}_1)_{11}(t) & 0 \\ 0 & (\widetilde{Y}_1)_{22}(t) \end{pmatrix}, \quad \bar{Y}_1(t) := \begin{pmatrix} 0 & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & 0 \end{pmatrix}.$$

Then, noting that

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\widehat{Y}_1(t)\widetilde{X}_1(t)\widehat{Y}_1(t) + \widehat{Y}_1(t)\widetilde{X}_1(t)\widehat{Y}_1(t)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} =$$

$$2\widehat{Y}_1(t)\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}\widehat{Y}_1(t),$$

we observe, by writing $\widetilde{Y}_1 = \widehat{Y}_1 + \bar{Y}_1$ in the above expression (31) for $\widetilde{Y}_1(t)D(t)\widetilde{Y}_1(t)$ and simplifying, that

$$\widetilde{Y}_1(t)D(t)\widetilde{Y}_1(t) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\left[\bar{Y}_1(t)\widetilde{X}_1(t)\bar{Y}_1(t) + \bar{Y}_1(t)\widetilde{X}_1(t)\widehat{Y}_1(t) + \widehat{Y}_1(t)\widetilde{X}_1(t)\bar{Y}_1(t)\right]+$$

$$\left[\bar{Y}_1(t)\widetilde{X}_1(t)\bar{Y}_1(t) + \bar{Y}_1(t)\widetilde{X}_1(t)\widehat{Y}_1(t) + \widehat{Y}_1(t)\widetilde{X}_1(t)\bar{Y}_1(t)\right]\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} -$$

$$2\bar{Y}_1(t)\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}\bar{Y}_1(t) - 2\bar{Y}_1(t)\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}\widehat{Y}_1(t)- \tag{32}$$

$$2\widehat{Y}_1(t)\begin{pmatrix} (\widetilde{X}_1)_{11}(t) & 0 \\ 0 & -(\widetilde{X}_1)_{22}(t) \end{pmatrix}\bar{Y}_1(t).$$

Note that written this way, every term in the expression for $\widetilde{Y}_1(t)D(t)\widetilde{Y}_1(t)$ has at least a $\bar{Y}_1(t)$. Therefore, with $(\widetilde{Y}_1)_{12}(t) = tW_1(t)$ for $t\ (>0)$ near 0 and $W_1(t)$ analytic at $t = 0$, we have $\widetilde{Y}_1(t)D(t)\widetilde{Y}_1(t) = tD_0(t, \widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$, where D_0 written as a function of $(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ is analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$.

Now since $\begin{pmatrix} (\widetilde{Y}_1)_{11} & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & (\widetilde{Y}_1)_{22} \end{pmatrix}^{-1}$ exists and is analytic as a function of $(t, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ at $(0, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ with $(\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$, $D(t)$ can also be written as $t\widetilde{D}_0(t, \widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$, where

$$\widetilde{D}_0(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22}) :=$$
$$\begin{pmatrix} (\widetilde{Y}_1)_{11} & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & (\widetilde{Y}_1)_{22} \end{pmatrix}^{-1} D_0(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22}) \begin{pmatrix} (\widetilde{Y}_1)_{11} & (\widetilde{Y}_1)_{12}(t) \\ (\widetilde{Y}_1)_{12}^T(t) & (\widetilde{Y}_1)_{22} \end{pmatrix}^{-1}$$

is analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$.

Hence, $\mathcal{F}_1(t, \widetilde{X}_1(t), \widetilde{Y}_1(t)) = t\, \widetilde{a}_0(t, \widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$, where $\widetilde{a}_0$ as a function of $(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$ is analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$, is true.

Therefore, we have $(t, \widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$, for $t\ (>0)$ near 0, satisfies the following system of ODEs,

$$\begin{pmatrix} svec(\widetilde{X}_1') \\ svec(\widetilde{Y}_1') \end{pmatrix} = \widetilde{a}_0(t, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22}),$$

where its right-hand side is analytic at $(0, \widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22})$, with $\widetilde{X}_1, (\widetilde{Y}_1)_{11}, (\widetilde{Y}_1)_{22} \succ 0$.

Therefore, from Theorem 4.1 of [2], pp. 15 and Theorem 2.1 of [8], we have $(\widetilde{X}_1(t), (\widetilde{Y}_1)_{11}(t), (\widetilde{Y}_1)_{22}(t))$ can be extended and is analytic at $t = 0$, which together with the analyticity of $(\widetilde{Y}_1)_{12}(t)$ at $t = 0$, implies our required result. **QED**

From the sufficiency proof of Theorem 3.1, we observe that a sufficient condition for $\widetilde{X}_1(t)$, $\widetilde{Y}_1(t)$, and hence an off-central path, $(X(\mu), Y(\mu))$, to converge as t (or μ) tends to zero is $(\widetilde{Y}_1)_{12}(t) = O(t^\alpha)$, that is, $Y_{12}(\mu) = O(\mu^{0.5(1+\alpha)})$, for any $\alpha > 0$. Therefore, we have the following corollary:

Corollary 3.1 *Let $(X(\mu), Y(\mu))$ be an off-central path for SDLCP (1), $\mu > 0$, under Assumptions 2.1 and 2.2. Suppose $Y_{12}(\mu) = O(\mu^{0.5(1+\alpha)})$ for some $\alpha > 0$, then $(X(\mu), Y(\mu))$ converges as $\mu \to 0$.*

Proof. Suppose $Y_{12}(\mu) = O(\mu^{0.5(1+\alpha)})$ for some $\alpha > 0$.
Then $(\widetilde{Y}_1)_{12}(t) = O(t^\alpha)$. Therefore, from the sufficiency proof of Theorem 3.1, the key being expression (32), we see that $\begin{pmatrix} svec(\widetilde{X}_1') \\ svec(\widetilde{Y}_1') \end{pmatrix} = O(t^{\alpha-1})$, where $\alpha - 1 > -1$. This implies that $\widetilde{X}_1(t)$, $\widetilde{Y}_1(t)$ must converge as $t \to 0$. Hence, $X(\mu), Y(\mu)$ converge as $\mu \to 0$. **QED**

Using Theorem 3.1, we have the main theorem for the section:

Theorem 3.2 *Let $(X(\mu), Y(\mu))$ be an off-central path for SDLCP (1), $\mu > 0$, under Assumptions 2.1 and 2.2. Then $X(\mu), Y(\mu)$ converge as μ tends to zero and are analytic as a function of $t = \sqrt{\mu}$ at $t = 0$ if and only if $\lim_{\mu \to 0} Y_{12}(\mu)/\mu$ exists and the analyticity of $Y_{12}(\mu)/\mu$ as a function of $t = \sqrt{\mu}$ can be extended to $t = 0$.*

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Proof. ($\Rightarrow$) Suppose $X(\mu), Y(\mu)$ converge as μ tends to zero and are analytic as a function of $t = \sqrt{\mu}$ at $t = 0$. The result follows from Proposition 2.1, Theorem 3.1 and the fact that $(Y_1)_{12}(t) = t(\widetilde{Y}_1)_{12}(t)$, where $(Y_1)_{12}(t) = Y_{12}(t^2)$.

($\Leftarrow$) Suppose $\lim_{\mu \to 0} Y_{12}(\mu)/\mu$ exists and the analyticity of $Y_{12}(\mu)/\mu$ as a function of $t = \sqrt{\mu}$ can be extended to $t = 0$. Then $(\widetilde{Y}_1)_{12}(t)$ is analytic at $t = 0$ and $(\widetilde{Y}_1)_{12}(0) = 0$. The result then follows from Theorem 3.1 and Proposition 2.1. **QED**

From Theorem 3.2, we see that the asymptotic analyticity of an off-central path for SDLCP as a function of $\sqrt{\mu}$ depends only on the asymptotic analyticity of one of its off-diagonal entries. This is a rather surprising result. From [8], we know that not all off-central paths are analytic at the solution of SDLCP. The above theorem gives a criteria as to when an off-central path for general SDLCP is analytic at the solution.

To end this section, we remark that similar theorem as Theorem 3.2 can also be stated for HKM direction.

4   Boundedness of First Derivative of Off-Central Path.

In [8], the authors show through a simple example that most off-central paths for SDLCP, $(X(\mu), Y(\mu))$, have unbounded first derivatives as μ tends to zero. This suggests an undesirable consequence on the local convergence behavior of IPM, using the dual HKM direction, on SDLCP given the close relation between the boundedness of derivatives of off-central paths and the local behavior of interior point path-following algorithm when iterates are near the solution of SDLCP. It turns out that an off-central path for SDLCP, $(X(\mu), Y(\mu))$, does not behave too badly if we perform a slight transformation on the parameter μ. We show in this section that if we consider $(X_1(t), Y_1(t)) = (X(t^2), Y(t^2))$, where $t = \sqrt{\mu}$, then the first derivatives of $X_1(t)$ and $Y_1(t)$ are bounded as t approaches zero. Note that we consider only the case when SDLCP (1) has a unique solution. That is, we have an additional assumption on SDLCP (1):

Assumption 4.1 *SDLCP (1) has a unique solution (X^*, Y^*), which is strictly complementary.*

In this section, we assume, without loss of generality, that the SDLCP (1) that we consider has already undergone the various equivalent transformations that we made in Section 2. Hence, the unique solution (X^*, Y^*) can be written as $\left(\begin{pmatrix} X_{11}^* & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & Y_{22}^* \end{pmatrix} \right)$, where $X_{11}^*, Y_{22}^* \succ 0$.

By uniqueness of the solution to the given SDLCP, we have the following lemma:

Lemma 4.1 *If $(U_{11}, V_{22}) \in S^m \times S^{n-m}$ is such that* $\begin{pmatrix} (A_1)_{11} \bullet U_{11} + (B_1)_{22} \bullet V_{22} \\ \vdots \\ (A_{i_1})_{11} \bullet U_{11} + (B_{i_1})_{22} \bullet V_{22} \end{pmatrix} = q_1,$

then $U_{11} = X_{11}^*$ and $V_{22} = Y_{22}^*$.

Proof. Suppose $(U_{11}, V_{22}) \in S^m \times S^{n-m}$ is such that $\begin{pmatrix} (A_1)_{11} \bullet U_{11} + (B_1)_{22} \bullet V_{22} \\ \vdots \\ (A_{i_1})_{11} \bullet U_{11} + (B_{i_1})_{22} \bullet V_{22} \end{pmatrix} = q_1.$

Then

$$X = s \begin{pmatrix} X_{11}^* & 0 \\ 0 & 0 \end{pmatrix} + (1-s) \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = s \begin{pmatrix} 0 & 0 \\ 0 & Y_{22}^* \end{pmatrix} + (1-s) \begin{pmatrix} 0 & 0 \\ 0 & V_{22} \end{pmatrix}$$

satisfy $\mathcal{A}svec(X) + \mathcal{B}svec(Y) = q$ - the feasibility condition, and the complementarity slackness condition for SDLCP. Furthermore, they satisfy positive semidefiniteness for some $s \neq 1$. Hence, (X, Y) is a solution to SDLCP for some $s \neq 1$ and by Assumption 4.1, $U_{11} = X_{11}^*$ and $V_{22} = Y_{22}^*$. **QED**

The above lemma plays an important role in the proof of the boundedness of the first derivatives of $X_1(t)$ and $Y_1(t)$ for t close to zero.

We have in Section 2 an ODE system for $(X_1(t), Y_1(t))$ given by

$$\frac{1}{2} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ I & X_1 \otimes_s Y_1^{-1} \end{pmatrix} \begin{pmatrix} svec(X_1') \\ svec(Y_1') \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 0 \\ svec(X_1) \end{pmatrix}.$$

To analyze the behavior of X_1' and Y_1' as $t \to 0$, let us first invert the matrix on the left-hand side of (10) (or the above system).

The inverse of the matrix on the extreme left of (10) (or the above system) is given by

$$\begin{pmatrix} -(X_1 \otimes_s Y_1^{-1})\mathcal{G}_1^{-1} & I + (X_1 \otimes_s Y_1^{-1})\mathcal{G}_1^{-1}\mathcal{A} \\ \mathcal{G}_1^{-1} & -\mathcal{G}_1^{-1}\mathcal{A} \end{pmatrix},$$

where $\mathcal{G}_1 := \mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1})$.

Therefore, (10) can be written as

$$\begin{pmatrix} svec(X_1') \\ svec(Y_1') \end{pmatrix} = 2 \begin{pmatrix} svec(X_1) + (X_1 \otimes_s Y_1^{-1})\mathcal{G}_1^{-1}\mathcal{A}svec(X_1) \\ -\mathcal{G}_1^{-1}\mathcal{A}svec(X_1) \end{pmatrix}. \tag{33}$$

Let us now simplify the right-hand side of (33). We have

$$2(svec(X_1) + (X_1 \otimes_s Y_1^{-1})\mathcal{G}_1^{-1}\mathcal{A}svec(X_1))$$
$$= 2svec(X_1) + 2(\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}\mathcal{A}svec(X_1)$$
$$= svec(X_1) + (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} - (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}\mathcal{B}svec(Y_1) +$$
$$\quad svec(X_1) + (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}\mathcal{A}svec(X_1)$$
$$= svec(X_1) + (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} -$$
$$\quad (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}(\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})svec(X_1) + svec(X_1)$$
$$= svec(X_1) + (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix},$$

where the second equality follows from the feasibility condition $(\mathcal{A}svec(X_1) + \mathcal{B}svec(Y_1) = (q_1^T, 0, 0)^T)$ of SDLCP.

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Similarly, we have

$$-2\mathcal{G}_1^{-1}\mathcal{A}svec(X_1) = svec(Y_1) - (\mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1}))^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, (33) becomes

$$\begin{pmatrix} svec(X_1') \\ svec(Y_1') \end{pmatrix} = \frac{1}{t} \begin{pmatrix} (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} + svec(X_1) \\ \\ -(\mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1}))^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} + svec(Y_1) \end{pmatrix}. \tag{34}$$

Observe that since $(\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}$ and $(\mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1}))^{-1}$ in (34) are undefined as t approaches zero, the behavior of $X_1'(t), Y_1'(t)$ as t approaches zero cannot be analyzed using (34). We have the following proposition to remedy this:

Proposition 4.1

$$(\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) (\tilde{X}_1 \otimes_s \tilde{Y}_1^{-1})\tilde{\mathcal{G}}_1^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$(\mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1}))^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} = \left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) \tilde{\mathcal{G}}_1^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix},$$

where $\tilde{\mathcal{G}}_1 := \mathcal{B}(t) - \mathcal{A}(t)(\tilde{X}_1 \otimes_s \tilde{Y}_1^{-1})$.

Proof. We have, by Lemma 2.6 and (14),

$$\left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) (\tilde{X}_1 \otimes_s \tilde{Y}_1^{-1})\mathcal{G}_1^{-1} = (\mathcal{B}(X_1 \otimes_s Y_1^{-1})^{-1} - \mathcal{A})^{-1}\text{diag}(I, tI, t^2 I). \tag{35}$$

Similarly,

$$\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) \mathcal{G}_1^{-1} = (\mathcal{B} - \mathcal{A}(X_1 \otimes_s Y_1^{-1}))^{-1}\text{diag}(I, tI, t^2 I). \tag{36}$$

The results then follow from (35) and (36) above. **QED**

Therefore, using Proposition 4.1, (34) becomes

$$\begin{pmatrix} svec(X_1') \\ svec(Y_1') \end{pmatrix} = \frac{1}{t} \begin{pmatrix} \left(\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \otimes_s \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \right) (\tilde{X}_1 \otimes_s \tilde{Y}_1^{-1})\tilde{\mathcal{G}}_1^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} + svec(X_1) \\ \\ -\left(\begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \otimes_s \begin{pmatrix} tI & 0 \\ 0 & I \end{pmatrix} \right) \tilde{\mathcal{G}}_1^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix} + svec(Y_1) \end{pmatrix}, \tag{37}$$

where $\widetilde{\mathcal{G}}_1 := \mathcal{B}(t) - \mathcal{A}(t)(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})$.

Note that it is advantageous to use (37) to analyze the behavior of X_1' and Y_1' near t equals to zero since $\widetilde{\mathcal{G}}_1$ is invertible for all $t \geq 0$ and $\widetilde{X}_1, \widetilde{Y}_1$ positive definite, by Proposition 2.4. Hence the vector on the right-hand side of (37) is defined in the limit as t tends to zero for $(X_1(t), Y_1(t))$ of an off-central path.

We are now ready to state and prove the main theorem in this section:

Theorem 4.1 *Under Assumptions 2.1 and 4.1, given an off-central path for SDLCP (1), $(X(\mu), Y(\mu))$. Let $X_1(t) = X(t^2)$ and $Y_1(t) = Y(t^2)$. We have $X_1'(t), Y_1'(t)$ are bounded near $t = 0$.*

Proof. We observe from (37) and using Lemma 2.2 that besides the upper left block of $X_1'(t)$, namely, $(X_1')_{11}(t)$ and the lower right block of $Y_1'(t)$, namely $(Y_1')_{22}(t)$, the rest of $X_1'(t)$ and $Y_1'(t)$ are bounded, near t equals to zero. Hence, to show the boundedness of $X_1'(t)$ and $Y_1'(t)$ near t equals to zero, we need only show that $(X_1')_{11}(t)$ and $(Y_1')_{22}(t)$ are bounded near t equals to zero.

Now we know by Remark 2.2 that $(\mathcal{A}(0)\ \mathcal{B}(0))$ has full row rank. Therefore, there exists a $\tilde{n} \times \tilde{n}$ submatrix of $(\mathcal{A}(0)\ \mathcal{B}(0))$ which is nonsingular. By the Implicit Function Theorem, given any $U_0, V_0 \in S^n$ such that $\mathcal{A}(0)svec(U_0) + \mathcal{B}(0)svec(V_0) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$, there exists a bounded open neighborhood $\mathcal{U}$ of $(0, U_0^s, V_0^s)$, where U_0^s, V_0^s consist of those entries in $svec(U_0)$, $svec(V_0)$ respectively, not corresponding to the columns of the nonsingular submatrix of $(\mathcal{A}(0)\ \mathcal{B}(0))$, and analytic functions U and V defined in $\mathcal{U}$ such that for every $(t, U^s, V^s) \in \mathcal{U}$, $U = U(t, U^s, V^s)$ and $V = V(t, U^s, V^s)$ satisfy $\mathcal{A}(t)svec(U) + \mathcal{B}(t)svec(V) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$.

Since U, V are analytic functions of (t, U^s, V^s), they can, in particular, be written as

$$U(t, U^s, V^s) = U(0, U^s, V^s) + tU_1(t, U^s, V^s), \quad V(t, U^s, V^s) = V(0, U^s, V^s) + tV_1(t, U^s, V^s), (38)$$

where U_1 and V_1 are bounded in $\mathcal{U}$.

Now, $U(0, U^s, V^s)$ and $V(0, U^s, V^s)$ satisfy $\mathcal{A}(0)svec(U) + \mathcal{B}(0)svec(V) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$. Hence, by the uniqueness property Lemma 4.1, we have $U_{11}(0, U^s, V^s) = X_{11}^* = $ constant and $V_{22}(0, U^s, V^s) = Y_{22}^* = $ constant. In particular, $U_{11}(t, U^s, V^s) = X_{11}^* + t(U_1)_{11}(t, U^s, V^s)$ and $V_{22}(t, U^s, V^s) = Y_{22}^* + t(V_1)_{22}(t, U^s, V^s)$ for all $(t, U^s, V^s) \in \mathcal{U}$.

Using the above arguments, let us now consider $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$, where $X_1(t), \widetilde{X}_1(t)$ and $Y_1(t), \widetilde{Y}_1(t)$ are related by (8) and (9) respectively.

Suppose $(\widetilde{X}_1^*, \widetilde{Y}_1^*)$, $\widetilde{X}_1^*, \widetilde{Y}_1^* \succ 0$, is an accumulation point of $(\widetilde{X}_1(t), \widetilde{Y}_1(t))$ as t approaches zero. Let $\widetilde{X}_1(t_k) \to \widetilde{X}_1^*$ and $\widetilde{Y}_1(t_k) \to \widetilde{Y}_1^*$.

Since $(\widetilde{X}_1^*, \widetilde{Y}_1^*)$ satisfies $\mathcal{A}(0)svec(U) + \mathcal{B}(0)svec(V) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$, therefore, there exists a bounded open neighborhood $\mathcal{U}$ of $(0, (\widetilde{X}_1^*)^s, (\widetilde{Y}_1^*)^s)$ such that we can write $U(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = $

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$U(0, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) + t_k U_1(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k))$ and $V(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = V(0, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) + t_k V_1(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k))$ for k large, by (38).

Now $(\mathcal{A}(t) \quad \mathcal{B}(t))$, for t sufficiently small, has a nonsingular submatrix, whose columns occupy the same positions in $(\mathcal{A}(t) \quad \mathcal{B}(t))$ as that of the nonsingular submatrix in $(\mathcal{A}(0) \quad \mathcal{B}(0))$.

Therefore, with $\mathcal{A}(t_k) svec(\widetilde{X}_1(t_k)) + \mathcal{B}(t_k) svec(\widetilde{Y}_1(t_k)) = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$, by uniqueness, we have

$U(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = \widetilde{X}_1(t_k)$, $V(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = \widetilde{Y}_1(t_k)$ for k large.

Hence, we have

$$(\widetilde{X}_1)_{11}(t_k) = U_{11}(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = X_{11}^* + t_k(U_1)_{11}(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k))$$

and

$$(\widetilde{Y}_1)_{22}(t_k) = V_{22}(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k)) = Y_{22}^* + t_k(V_1)_{22}(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k))$$

for k large.

Let us now consider $g(t, \widetilde{X}_1, \widetilde{Y}_1) := (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})\widetilde{\mathcal{G}}_1^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$ in (37).

Observe that g is analytic for all $t \geq 0$ and $\widetilde{X}_1, \widetilde{Y}_1$ positive definite. Therefore, we can write $g(t, \widetilde{X}_1, \widetilde{Y}_1) = g(0, \widetilde{X}_1, \widetilde{Y}_1) + t g_1(t, \widetilde{X}_1, \widetilde{Y}_1)$ for t close to zero and $\widetilde{X}_1, \widetilde{Y}_1 \succ 0$, where g_1 is bounded on bounded sets.

Now, let $g(0, \widetilde{X}_1, \widetilde{Y}_1)$ be equal to $svec \begin{pmatrix} W_{11}(\widetilde{X}_1, \widetilde{Y}_1) & W_{12}(\widetilde{X}_1, \widetilde{Y}_1) \\ W_{12}^T(\widetilde{X}_1, \widetilde{Y}_1) & W_{22}(\widetilde{X}_1, \widetilde{Y}_1) \end{pmatrix}$.

In other words,

$$svec \begin{pmatrix} W_{11}(\widetilde{X}_1, \widetilde{Y}_1) & W_{12}(\widetilde{X}_1, \widetilde{Y}_1) \\ W_{12}^T(\widetilde{X}_1, \widetilde{Y}_1) & W_{22}(\widetilde{X}_1, \widetilde{Y}_1) \end{pmatrix} = (\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})(\mathcal{B}(0) - \mathcal{A}(0)(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1}))^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= (\mathcal{B}(0)(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})^{-1} - \mathcal{A}(0))^{-1} \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$-\mathcal{A}(0) svec \begin{pmatrix} W_{11}(\widetilde{X}_1, \widetilde{Y}_1) & W_{12}(\widetilde{X}_1, \widetilde{Y}_1) \\ W_{12}^T(\widetilde{X}_1, \widetilde{Y}_1) & W_{22}(\widetilde{X}_1, \widetilde{Y}_1) \end{pmatrix} +$$

$$\mathcal{B}(0)(\widetilde{X}_1 \otimes_s \widetilde{Y}_1^{-1})^{-1} svec \begin{pmatrix} W_{11}(\widetilde{X}_1, \widetilde{Y}_1) & W_{12}(\widetilde{X}_1, \widetilde{Y}_1) \\ W_{12}^T(\widetilde{X}_1, \widetilde{Y}_1) & W_{22}(\widetilde{X}_1, \widetilde{Y}_1) \end{pmatrix} = \begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, by the uniqueness property Lemma 4.1 again, we have $W_{11}(\widetilde{X}_1, \widetilde{Y}_1) = -X_{11}^*$.

Hence $(smat(g(t_k, \widetilde{X}_1(t_k), \widetilde{Y}_1(t_k))))_{11} = -X_{11}^* + t_k(smat(g_1(t_k, \widetilde{X}_1(t_k), \widetilde{Y}_1(t_k))))_{11}$ for k large.

We therefore have, from (37) and $(X_1)_{11}(t) = (\widetilde{X}_1)_{11}(t)$, that

$$(X_1')_{11}(t_k) = \frac{1}{t_k}((smat(g(t_k, \widetilde{X}_1(t_k), \widetilde{Y}_1(t_k))))_{11} + (X_1)_{11}(t_k))$$

$$= (smat(g_1(t_k, \widetilde{X}_1(t_k), \widetilde{Y}_1(t_k))))_{11} + (U_1)_{11}(t_k, \widetilde{X}_1^s(t_k), \widetilde{Y}_1^s(t_k))$$

is bounded for k large.

Similarly, we can show that the same boundedness property holds for $(Y_1')_{22}(t_k)$ for large k.

Therefore, by a contradiction argument, we show that $(X_1')_{11}(t)$, $(Y_1')_{22}(t)$ must be bounded for all t close to zero, and we are done. **QED**

5   Conclusion and Future Directions.

In this paper we study the asymptotic behavior of off-central path for SDLCP, using the dual HKM direction. A purpose of this paper is to provide a framework upon which the asymptotic behavior of off-central path for SDLCP, using the dual HKM direction, can be analyzed, using (29) and (37), which can reveal more about the properties of off-central path than (10) near $t = 0$. From a practical point of view, we are left with the following open questions:

1. Given a problem in a specific class of SDLCP, how to determine if its paths are all analytic, all non-analytic, or a mixture?

2. If a problem has both analytic and non-analytic paths, what are conditions to distinguish them?

We do not attempt to answer these questions directly in this paper. In Section 3, we give a necessary and sufficient condition for when an off-central path is analytic as a function of $\sqrt{\mu}$ at the solution of SDLCP. This condition is closely related to the analysis of asymptotic analytic behavior of paths for the example in [8]. In [8], we obtain an algebraic condition for asymptotic analyticity of paths for the example consider there. Here, we are unable to obtain a similar algebraic condition and further analysis needs to be done in future to obtain a more practical necessary and sufficient condition for asymptotic analyticity. The asymptotic analyticity of off-central paths as a function of μ will also be investigated as future work. In Section 4, we show that the off-central path for SDLCP, when viewed as a function of $t = \sqrt{\mu}$, has bounded first derivative as t approaches zero. Here, we assume that SDLCP has a unique solution which is strictly complementary. Whether the same result holds without uniqueness assumption is still an open question. In [8], it indicates, through an example, that the usual interior point path-following algorithm, based on paths as a function of μ (where μ represents the duality gap between the primal and dual variables), may not converge fast to the solution of the SDLCP in general, since the first derivatives of the paths for the example are unbounded as μ tends to zero. The result in this section suggests that it may be worthwhile to investigate and design interior point path-following algorithm, using underlying paths as a function of $\sqrt{\mu}$, instead of μ, whose iterates possibly converge rapidly to the unique solution of SDLCP. Similar study on such new interior point path-following algorithm has been done for LCP in [12, 13, 21], where a parametrization different from the usual one is used for the underlying paths, as in this paper.

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