Theory of Algebraic Invariants of Vector Spaces of Killing Tensors: Methods for Computing the Fundamental Invariants

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We review computational techniques that can be used in the determination of fundamental invariants of vector spaces of Killing tensors under the action of the isometry group. In particular, for the first time in this study, we employ for these purposes the method of moving frames introduced by Fels and Olver. Illustrative examples are provided.

1 Introduction

Conceived as an extension of classical invariant theory of homogeneous polynomials [1, 2], the theory of algebraic invariants of Killing tensors has been introduced recently [3–10] as the study of invariant properties of vector spaces of Killing tensors under the action of the isometry group. According to the dictates of the mathematical structure of the classical theory, the theory of algebraic invariants of Killing tensors shares many of its essential features. Thus, the pivotal problem in both theories is the determination of the set of fundamental invariants for a given vector space under the action of a group. Furthermore, in both theories, a solution to the problem is a precursor to solving the intimately related problems of equivalence and finding canonical forms (see [2, 7] for more details).

As is well known, the geometrical properties of Killing two-tensors play a vital role in the Hamilton–Jacobi theory of orthogonal separation of variables [11–15] as well as in the field theory (see, for example [16, 17]). In this spirit, the fundamental invariants of vector spaces of Killing tensors have been effectively employed in classification problems arising in the Hamilton–Jacobi theory [3, 4, 6–10]. It has also been shown that the fundamental invariants of Killing tensors of higher valences can be applied to other classification problems of Hamiltonian mechanics [5].

The main aim of this article is to discuss computational techniques that can be employed within the framework of the theory to determine the fundamental invariants of vector spaces of Killing tensors defined on pseudo-Riemannian spaces of constant curvature.

2 A glimpse into the theory of algebraic invariants of vector spaces of Killing tensors

Assuming only a minimal familiarity with the classical invariant theory (see [2] for an introduction), this section describes how the basic concepts of the classical theory of algebraic invariants of homogeneous polynomials can be naturally adapted to the study of vector spaces of Killing tensors under the action of the isometry group $I(M)$.

Let $(M, \boldsymbol{g})$ be a pseudo-Riemannian manifold of constant curvature. We begin by defining the concept of a Killing tensor (vector).

Definition 1. A *Killing tensor* $\boldsymbol{K}$ *of valence* p *defined in* $(M, \boldsymbol{g})$ is a symmetric $(p, 0)$ tensor satisfying the Killing tensor equation

$$[\boldsymbol{K}, \boldsymbol{g}] = 0, \tag{1}$$

where $[\ ,\ ]$ denotes the Schouten bracket. When $p = 1$, $\boldsymbol{K}$ is said to be a *Killing vector (infinitesimal isometry)* and the equation (1) reads $\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g} = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

In the case when $(M, \boldsymbol{g})$ is a pseudo-Riemannian manifold of constant curvature, the dimension d of $\mathcal{K}^p(M)$ is determined by the *Delong–Takeuchi–Thompson (DTT) formula* $d = \dim \mathcal{K}^p(M) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$, $p \geq 1$, where $\mathcal{K}^p(M)$ denotes the vector space of Killing tensor of valence p defined on $(M, \boldsymbol{g})$. In this study we treat Killing tensors as elements of their respective vector spaces. This approach differs from the more conventional view of Killing tensors defined on spaces of constant curvature as symmetrized sums of Killing vectors. Accordingly, taking into account the DTT formula, a Killing tensor of valence p is an algebraic object determined by its d parameters, where d is the dimension of the vector space $\mathcal{K}^p(M)$.

We have exploited this idea by making a natural link between the study of Killing tensors and the classical invariant theory. Indeed, whenever the action of a group is defined properly on a vector space, one can pose the question of determining the induced action of the group in the space of the parameters of the vector space under consideration and then proceed to find the functions of the parameters that remain unchanged under the action of the group (i.e. the group invariants). Thus, for a fixed $p \geq 1$, consider the corresponding vector space $\mathcal{K}^p(M)$ on $(M, \boldsymbol{g})$. It is spanned by d arbitrary parameters $a_1, \ldots, a_d$, where d is determined by the DTT formula. Alternatively, this fact can be established by solving the corresponding Killing tensor equation (1) with respect to a fixed coordinate system. Then, the parameters $a_1, \ldots, a_d$ will appear as constants of integration in the general form of the Killing tensor in $\mathcal{K}^p(M)$. The action of the isometry group $I(M)$ on $\mathcal{K}^p(M)$ induces transformation laws for the d parameters $a_1, \ldots, a_d$ of the following form:

$$\begin{aligned}
&\tilde{a}_1 = \tilde{a}_1(a_1, \ldots, a_d, g_1, \ldots, g_r),\\
&\tilde{a}_2 = \tilde{a}_2(a_1, \ldots, a_d, g_1, \ldots, g_r),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\tilde{a}_d = \tilde{a}_d(a_1, \ldots, a_d, g_1, \ldots, g_r),
\end{aligned} \tag{2}$$

where d is given by the DTT formula, $g_1, \ldots, g_r$ are local coordinates on $I(M)$ that parametrize the group and $r = \dim I(M) = \frac{1}{2}n(n+1)$. Let $\Sigma \simeq \mathbb{R}^d$ be the space of parameters $a_1, \ldots, a_d$. By analogy with the classical theory of invariants, we formulate the following problem.

Problem 1. *Let* $\Sigma \simeq \mathbb{R}^d$ *be the space determined by the* d *parameters* $a_1, \ldots, a_d$ *that define the vector space* $\mathcal{K}^p(M)$ *in* $(M, \boldsymbol{g})$ *for some fixed* $p \geq 1$. *In the space of functions in* Σ, *describe the subspace of all functions that remain fixed under the induced action of the corresponding isometry group* $I(M)$.

The functions having this property are called *$I(M)$-invariants of the vector space $\mathcal{K}^p(M)$.* More specifically, they must satisfy the condition $F(\tilde{a}_1,\ldots,\tilde{a}_d) = F(a_1,\ldots,a_d)$ under the transformation laws (2) induced by the action of the isometry group $I(M)$. Solving Problem 1 then reduces to finding a set of *fundamental invariants* with the property that any other invariant is a (analytic) function of the fundamental invariants. The Fundamental Theorem on invariants of a regular Lie group action [2] determines the number of fundamental invariants required to define the whole of the space of $I(M)$-invariants:

Theorem 1. *Let G be a Lie group acting regularly on an m-dimensional manifold X with s-dimensional orbits. Then, in a neighbourhood N of each point $x_0 \in M$, there exist $m-s$ functionally independent G-invariants $\Delta_1,\ldots,\Delta_{m-s}$. Any other G-invariant $\mathcal{I}$ defined near x_0 can be locally uniquely expressed as an analytic function of the fundamental invariants through $\mathcal{I} = F(\Delta_1,\ldots,\Delta_{m-s})$.*

In all of the examples studied to this point, the isometry group acts regularly on the corresponding subsets $\Sigma\backslash\{\mathbf{0}\}$, where Σ is the space of the parameters of the vector space of Killing tensors under consideration, with the origin $a_1 = a_2 = \cdots = a_d = 0$, being one of the orbits.

3 Computation of the fundamental $I(M)$-invariants

In this section we briefly describe the methods of computing fundamental invariants of vector spaces of Killing tensors defined in pseudo-Riemannian spaces of constant curvature under the action of the isometry group.

3.1 Method of moving frames

The method of moving frames introduced recently by Fels and Olver [18, 19] is a very useful direct method for computing fundamental invariants. In this paper we use it for the first time within the framework of the theory of algebraic invariants of Killing tensors. As the following example shows, at least in some instances, it can be much more effective than the method of infinitesimal generators (see below) which has been used in the cases already studied. For example, consider the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ of non-trivial Killing tensors of valence two defined in the Euclidean plane $\mathbb{R}^2$ under the action of $I(\mathbb{R}^2)$. Here "non-trivial" means that none of the elements of $\mathcal{NK}^2(\mathbb{R}^2)$ is a multiple of the corresponding metric tensor $\boldsymbol{g}$: $\mathcal{NK}^2(\mathbb{R}^2) = \{\boldsymbol{K} \in \mathcal{K}(\mathbb{R}^2) | \boldsymbol{K} \neq \ell\boldsymbol{g}, \ell \in \mathbb{R}\}$. This is a five-dimensional vector space. Without loss of generality we can assume any element of $\mathcal{NK}^2(\mathbb{R}^2)$ has the following general form:

$$\begin{bmatrix} a_1 + 2a_3y + a_5y^2 & a_2 - a_3x - a_4y - a_5xy \\ a_2 - a_3x - a_4y - a_5xy & 2a_4x + a_5x^2 \end{bmatrix}, \tag{3}$$

where (x,y) is some fixed system of Cartesian coordinates in $\mathbb{R}^2$ and a_i, $i = 1,\ldots,5$ are arbitrary parameters that determine the dimension of the vector space $\mathcal{NK}^2(\mathbb{R}^2)$. In view of the standard parametrization of the isometry group $I(\mathbb{R}^2)$, we have the transformation of the coordinates (x,y)

$$\tilde{\boldsymbol{x}} = R_\theta\boldsymbol{x} + \boldsymbol{a}, \tag{4}$$

where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2)$ and $\boldsymbol{a} = (a,b) \in \mathbb{R}^2$. The tensor transformation laws induce the corresponding transformation laws (2) on the parameters appearing in (3):

$$\begin{aligned} \tilde{a}_1 = {} & a_1(\cos^2\theta - \sin^2\theta) - 4a_2\sin\theta\cos\theta + 2(aa_4 - ba_3)\cos\theta \\ & - 2(ba_4 + aa_3)\sin\theta + a_5(b^2 - a^2), \end{aligned} \tag{5}$$

$$\tilde{a}_2 = a_1 \sin\theta\cos\theta + a_2(\cos^2\theta - \sin^2\theta) + (aa_3 + ba_4)\cos\theta + (aa_4 - ba_3)\sin\theta - a_5 ab, \tag{6}$$
$$\tilde{a}_3 = a_3\cos\theta + a_4\sin\theta - a_5 b, \tag{7}$$
$$\tilde{a}_4 = a_4\cos\theta - a_3\sin\theta - a_5 a, \tag{8}$$
$$\tilde{a}_5 = a_5. \tag{9}$$

We immediately observe that $\Delta_1 = a_5$ is a fundamental $I(\mathbb{R}^2)$-invariant. According to Theorem 1, we must have in total 5 (the dimension of the vector space) $-$ 3 (the dimension of the orbits) $=$ 2 fundamental invariants. (The fact that the orbits of the isometry group are 3-dimensional on the space $\Sigma\backslash\{\mathbf{0}\}$ was established earlier [7, 10].) To find the second fundamental invariant we employ the method of moving frames (see [2] for more details). Consider the following coordinate cross-section K in the space $\Sigma \simeq \mathbb{R}^5$ defined by the parameters a_i, $i = 1, \dots, 5$, $K = \{a_2 = a_3 = a_4 = 0\}$. Next, we find the solutions to the *normalization equations* $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$, to determine the *moving frame map*: $\gamma : \Sigma \to I(\mathbb{R}^2)$. Solving (7) and (8) for a and b respectively, we get

$$a = \frac{a_4\cos\theta - a_3\sin\theta}{a_5}, \qquad b = \frac{a_3\cos\theta + a_4\sin\theta}{a_5}. \tag{10}$$

Next, substituting the formulas (10) into (6) we obtain

$$\tan 2\theta = -\frac{2(a_5 a_2 + a_3 a_4)}{a_1 a_5 - a_3^2 + a_4^2}. \tag{11}$$

Therefore we conclude that the equations (10) and (11) define in this case the moving frame map $\gamma : (a_1, \dots, a_5) \to (a, b, \theta)$. Finally, substituting (10) and (11) into the equation (5) and using elementary trigonometric formulas, we obtain the second fundamental invariant

$$\Delta_2 = \frac{\sqrt{(a_5 a_1 - a_3^2 + a_4^2)^2 + 4(a_5 a_2 + a_3 a_4)^2}}{a_5}. \tag{12}$$

Since $\Delta_1 = a_5$ is also a fundamental invariant, we can conclude that the pair (Δ_1, Δ_2) constitutes a complete system of fundamental invariants in this case. We note that these results are consistent with the results obtained in [10] by the method of infinitesimal generators. Hence, by making use of Theorem 1 and employing the method of moving frames, we have solved Problem 1 for the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ and hence proven the following theorem:

Theorem 2. *Any algebraic $I(\mathbb{R}^2)$-invariant $\mathcal{I}$ of the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ can locally be uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \Delta_2)$ of the fundamental $I(\mathbb{R}^2)$-invariants* (9) *and* (12).

We note that in this case the method of moving frames appears to be more effective in comparison with the method of infinitesimal generators used previously to solve this problem. More specifically, the fundamental invariants have been found by a *direct algebraic* method without having to solve a system of PDEs.

Recall that non-trivial Killing tensors define orthogonal coordinate webs in $\mathbb{R}^2$. From this perspective, the invariant $k^2 = \Delta_2/\Delta_1$ is the square of the (half) distance between the foci in the case when the corresponding Killing tensor gives rise to the elliptic-hyperbolic coordinate web (see [7, 10]) for more details).

3.2 Method of infinitesimal generators

Let $\mathcal{K}^p(M)$ be the vector space of Killing tensors of valence $p \geq 1$ defined on a pseudo-Riemannian manifold M of constant curvature. Consider the action of the isometry group $I(M)$

on $\mathcal{K}^p(M)$. The method of infinitesimal generators for finding fundamental invariants rests on the following idea: One finds the infinitesimal generators of the Lie algebra $i(M)$ of $I(M)$ in the space Σ determined by the parameters $a_1, \ldots, a_d$ of the vector space $\mathcal{K}^p(M)$, $d = \dim \mathcal{K}^p(M)$. Since invariants are annihilated by the generators (Killing vectors) of the Lie algebra of the isometry group, determining the fundamental invariants reduces to solving the corresponding system of PDEs in the space $\Sigma \simeq \mathbb{R}^d$. To find the generators in Σ we employ the following technique. Consider $\mathrm{Diff}\,\Sigma$, the space of all diffeomorphisms of Σ. It yields $\mathrm{Diff}\,\mathcal{K}^p(M)$, the space of Killing tensors of valence p whose parameters are determined by all possible diffeomorphisms of Σ. Thus, an element $\tilde{\boldsymbol{K}} \in \mathrm{Diff}\,\mathcal{K}^p(M)$ is a Killing tensor of valence p, whose d parameters $\tilde{a}_1, \ldots, \tilde{a}_d$ are defined by a diffeomorphism of the coordinates $a_1, \ldots, a_d$ of Σ: $\tilde{a}_i = \tilde{a}_i(a_1, \ldots, a_d)$, $i = 1, \ldots, d$. Define now the following map $\pi : \mathrm{Diff}\,\mathcal{K}^p(M) \to \mathcal{X}(\Sigma)$, where $\mathcal{X}(\Sigma)$ is the space of all vector fields on Σ, as follows:

$$\pi(\tilde{\boldsymbol{K}}) = \sum_{i=1}^{d} \tilde{a}_i(a_1, \ldots, a_d) \frac{\partial}{\partial a_i}, \qquad i = 1, \ldots, d, \tag{13}$$

where $\tilde{\boldsymbol{K}} \in \mathrm{Diff}\,\mathcal{K}^p(M)$, $a_1, \ldots, a_d$ are the coordinates of the space Σ (see Section 2 for more details) and $\tilde{a}_i$, $i = 1, \ldots, d$ determine $\tilde{\boldsymbol{K}}$. Now let $\boldsymbol{K} \in \mathcal{K}^p(M)$ be the general Killing tensor determined by $a_1, \ldots, a_d$ of Σ (in essence, $\boldsymbol{K}$ is the solution of the corresponding Killing tensor equation (1)) and $\boldsymbol{X}_i$, $i = 1, \ldots, r$, $r = \frac{1}{2}n(n+1)$ be the generators of $i(M)$. Define the following vector fields in $\mathcal{X}(\Sigma)$:

$$\boldsymbol{V}_i = \pi(\mathcal{L}_{\boldsymbol{X}_i}\boldsymbol{K}), \qquad i = 1, \ldots, r, \tag{14}$$

where $\mathcal{L}$ denotes the Lie derivative operator. Note that the Killing tensors $\boldsymbol{K}_{\boldsymbol{X}_i} = \mathcal{L}_{\boldsymbol{X}_i}\boldsymbol{K} \in \mathrm{Diff}\,\mathcal{K}^p(M)$, $i = 1, \ldots, r$ and so the vector fields $\boldsymbol{V}_i \in \mathcal{X}(\Sigma)$, $i = 1, \ldots, r$ are well-defined. The vector fields $\boldsymbol{V}_i$ represent the generators $\boldsymbol{X}_i$, $i = 1, \ldots, r$ in $\mathcal{X}(\Sigma)$ in the sense that they satisfy the same commutator relationships. This fact can be established on a case by case basis. It demonstrates that the Lie algebras generated by the vector fields $\boldsymbol{X}_i$ and $\boldsymbol{V}_i$, $i = 1, \ldots, r$ are in fact isomorphic and the vector fields $\boldsymbol{V}_i$, $i = 1, \ldots, r$ can be used in determining the fundamental invariants. Indeed, it follows from this fact that $I(M)$-invariants are annihilated by the generators of the corresponding Lie algebra. The following example will illustrate the procedure. Consider the six-dimensional vector space $\mathcal{K}^1(\mathbb{R}^3)$ which consists of all Killing vectors in Euclidean space $\mathbb{R}^3$. The most general element of $\mathcal{K}^1(\mathbb{R}^3)$ in terms of Cartesian coordinates is given by

$$\boldsymbol{K} = (a_1 + a_5 z - a_6 y)\frac{\partial}{\partial x} + (a_2 + a_6 x - a_4 z)\frac{\partial}{\partial y} + (a_3 + a_4 y - a_5 x)\frac{\partial}{\partial z}. \tag{15}$$

The generators of the Lie algebra $\mathcal{K}^1(\mathbb{R}^3)$ in terms of Cartesian coordinates are given by

$$\begin{aligned} &\boldsymbol{X}_1 = \frac{\partial}{\partial x}, \qquad \boldsymbol{X}_2 = \frac{\partial}{\partial y}, \qquad \boldsymbol{X}_3 = \frac{\partial}{\partial z} \qquad \text{(translations)}, \\ &\boldsymbol{R}_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad \boldsymbol{R}_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad \boldsymbol{R}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad \text{(rotations)}. \end{aligned} \tag{16}$$

They satisfy the following commutator relations.

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = 0, \qquad [\boldsymbol{X}_i, \boldsymbol{R}_j] = -\epsilon_{ijk}\boldsymbol{X}_k, \qquad [\boldsymbol{R}_i, \boldsymbol{R}_j] = -\epsilon_{ijk}\boldsymbol{R}_k, \tag{17}$$

where ϵ_{ijk} is the permutation tensor and $i, j, k = 1, 2, 3$. Applying the technique described above (see (14)) we find the corresponding vector fields in $\mathcal{X}(\Sigma)$, where $\Sigma \simeq \mathbb{R}^6$ is determined by the parameters $a_1, \dots, a_6$ of (15)

$$\begin{aligned}
\boldsymbol{U}_1 &= a_6\frac{\partial}{\partial a_2} - a_5\frac{\partial}{\partial a_3},\\
\boldsymbol{U}_2 &= -a_6\frac{\partial}{\partial a_1} + a_4\frac{\partial}{\partial a_3},\\
\boldsymbol{U}_3 &= a_5\frac{\partial}{\partial a_1} - a_4\frac{\partial}{\partial a_2},\\
\boldsymbol{V}_1 &= a_3\frac{\partial}{\partial a_2} - a_2\frac{\partial}{\partial a_3} + a_6\frac{\partial}{\partial a_5} - a_5\frac{\partial}{\partial a_6},\\
\boldsymbol{V}_2 &= -a_3\frac{\partial}{\partial a_1} + a_1\frac{\partial}{\partial a_3} - a_6\frac{\partial}{\partial a_4} + a_4\frac{\partial}{\partial a_6},\\
\boldsymbol{V}_3 &= a_2\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial a_2} + a_5\frac{\partial}{\partial a_4} - a_4\frac{\partial}{\partial a_5}.
\end{aligned} \tag{18}$$

Note that the vector fields (18) satisfy the same commutator relations (17)

$$[\boldsymbol{U}_i, \boldsymbol{U}_j] = 0, \qquad [\boldsymbol{U}_i, \boldsymbol{V}_j] = -\epsilon_{ijk}\boldsymbol{U}_k, \qquad [\boldsymbol{V}_i, \boldsymbol{V}_j] = -\epsilon_{ijk}\boldsymbol{V}_k. \tag{19}$$

Therefore, in view of (17) and (19), we have established the following result:

Lemma 1. *The Lie algebras* $i(\mathbb{R}^3) \simeq \mathcal{K}^1(\mathbb{R}^3)$ *and* $i_\Sigma(\mathbb{R}^3)$ *are isomorphic, where the Lie algebra* $i_\Sigma(\mathbb{R}^3)$ *is spanned by* (18)*.*

Next, we observe that the coefficient matrix of the generators (18)

$$\begin{bmatrix}
0 & a_6 & -a_5 & 0 & 0 & 0\\
-a_6 & 0 & a_4 & 0 & 0 & 0\\
a_5 & -a_4 & 0 & 0 & 0 & 0\\
0 & a_3 & -a_2 & 0 & a_6 & -a_5\\
-a_3 & 0 & a_1 & -a_6 & 0 & a_4\\
a_2 & -a_1 & 0 & a_5 & -a_4 & 0
\end{bmatrix} \tag{20}$$

has rank four almost everywhere. Therefore we conclude that the isometry group has 4-dimensional orbits in the space $\Sigma\backslash\{\mathbf{0}\}$, and so, in view of Theorem 1, we have 6 (the dimension of the group) $-$ 4 (the dimension of the orbits) $=$ 2 fundamental invariants. Indeed, taking into account the result of Lemma 1 and employing the *method of undetermined coefficients* (see Section 3.3) we solve the system of linear PDEs generated by (18)

$$\boldsymbol{V}_i(F) = 0, \qquad \boldsymbol{U}_i(F) = 0, \qquad i = 1, 2, 3 \tag{21}$$

and arrive at the following result:

Theorem 3. *Any algebraic* $I(\mathbb{R}^3)$*-invariant* $\mathcal{I}$ *of the vector space* $\mathcal{K}^1(\mathbb{R}^3)$ *can locally be uniquely expressed as an analytic function* $\mathcal{I} = F(\Delta_1, \Delta_2)$*, where*

$$\Delta_1 = a_4^2 + a_5^2 + a_6^2, \qquad \Delta_2 = a_1a_4 + a_2a_5 + a_3a_6 \tag{22}$$

are the corresponding fundamental $I(\mathbb{R}^3)$*-invariants.*

As expected, we have obtained two fundamental invariants; they can be effectively employed in a classification of the elements of the vector space $\mathcal{K}^1(\mathbb{R}^3)$ [3].

3.3 Method of undetermined coefficients

One typically employs the method of characteristics when solving systems of linear PDEs, such as (21). Unfortunately, this procedure does not always yield a result, and, in particular, becomes intractable when applied to (21). However, an alternate solution method, namely the *method of undetermined coefficients* has helped to alleviate the difficulties. Thus, based on previous problems that we have solved so far, it has been conjectured that there always exists a set of fundamental invariants consisting of homogeneous polynomials in $a_1, \ldots, a_d$. We can take advantage of this by constructing a suitable trial function for the system of PDEs. This is the general idea of the method of undetermined coefficients.

The method works as follows. We construct a general homogeneous polynomial in $a_1, \ldots, a_d$ of some fixed degree with coefficients to be determined. Substituting this trial polynomial into the system of PDEs leads to a system of linear equations in the undetermined coefficients. Thus, any non-trivial solution of this linear system yields a solution of the PDE system, and hence an $I(M)$-invariant of $\mathcal{K}^p(M)$. This process can be repeated with higher degree polynomials until the required number of functionally independent invariants asserted by Theorem 1 is derived. In this way, a set of fundamental $I(M)$-invariants which are functionally independent is obtained.

It turns out that the resulting linear system is overdetermined. There are approximately $r = \frac{1}{2}n(n+1)$ times as many linear equations as there are unknowns. This is understandable since there are r infinitesimal generators in the system. We also emphasize that the linear system is sparse, in the sense that the corresponding coefficient matrix of the system consists primarily of zero entries. As a result, the linear system can easily be solved using a symbolic sparse linear solver.

We have successfully implemented this algorithm in Maple to solve the resulting system of linear equations. In particular, we have computed the complete set of fundamental invariants for a number of vector spaces of Killing tensors, notably for those where the method of characteristics has failed [3]. For the amusement of the reader, we remark that in the space $\mathcal{K}^2(\mathbb{R}^3)$, an arbitrary quintic polynomial in $a_1, \ldots, a_{20}$ must be constructed, which results in a linear system of approximately 250 000 equations in 50 000 unknowns with about 1 000 000 non-zero entries. The computation took approximately ninety hours to complete on a 300 MHz Sun Ultra-5. The exact details of this calculation are discussed in [3]. This paper also examines how a more restrictive trial function can be constructed which further enhances the efficiency of the algorithm.

4 Method of reducing invariants

In some instances, fundamental $I(N)$-invariants of a vector space $\mathcal{K}^p(N)$ can be obtained from fundamental $I(M)$-invariants of $\mathcal{K}^p(M)$, where $N \subset M$. For example, a complete set of fundamental $I(\mathbb{S}^2)$-invariants for $\mathcal{K}^2(\mathbb{S}^2)$ has been obtained [4] from the fundamental $I(\mathbb{R}^3)$-invariants of $\mathcal{K}^2(\mathbb{R}^3)$ derived in [3], based on the result of the Delong theorem [13] describing explicitly the elements of a general space $\mathcal{K}^2(\mathbb{R}^{n+1})$, $n \geq 2$ which are the elements of $\mathcal{K}^2(\mathbb{S}^n)$.

5 Conclusion

We have demonstrated how the method of moving frames [18, 19] can be used to solve the problem of finding the complete sets of fundamental algebraic invariants of vector spaces of Killing tensors under the action of the isometry group. In the study of Killing tensors defined on low-dimension pseudo-Riemannian spaces of constant curvature the method appears to be more effective than the method of infinitesimal generators. However, in finding fundamental invariants of vector spaces of Killing tensors of higher valences defined on spaces of higher dimensions the above mentioned methods prove to be less effective. This prompts one to use the method of undetermined coefficients combined with a computer algebra system such as Maple to solve the problem of finding complete sets of fundamental $I(M)$-invariants for the vector spaces $\mathcal{K}^p(M)$.

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