J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain. © 1973

LETTER TO THE EDITOR

Critical exponents of the modified F model

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Received 27 February 1973

Abstract. Critical exponents of an eight-vertex model, the modified F model, are obtained. The results $\beta = \frac{1}{8}$, $\gamma = \gamma' = \frac{7}{4}$, $\eta = \frac{1}{4}$, $\nu = \nu' = 1$ together with the established indices $\alpha = \alpha' = 0$ obey the scaling relations.

It has been argued (Kadanoff and Wegner 1971) that the existence of continuously variable critical exponents α and α' (Baxter 1971, 1972) in the eight-vertex model (Fan and Wu 1970) is not inconsistent with the scaling hypothesis. To see whether the scaling relations are indeed valid in the eight-vertex model, it is necessary to know exponents in addition to the ones that have been computed. Unfortunately, other indices of the eight-vertex model are not known to this date (see Note Added in Proof).

We consider here a special case of the eight-vertex model and report the result on its exponents and the verification of the scaling relations. This is the modified F model introduced by one of us (Wu 1969). In zero external electric field this model is equivalent to a nearest-neighbour Ising model and hence has the exponents $\alpha = \alpha' = 0$. To obtain other exponents, it is necessary to include a nonzero staggered quadrupole electric field to remove the degeneracy of the two ordered antiferroelectric ground states. Thus, following the notation of Wu (1969), we take

$$e_1 = e_2 = \epsilon_1 > 0, \qquad e_3 = e_4 = \epsilon_2 > 0$$
$$e_5 = -e_6 = \pm s, \qquad e_7 = e_8 = \epsilon_1 + \epsilon_2 \tag{1}$$

where the two signs of e_5 and e_6 refer to vertices belonging to the two sublattices A and B respectively. The spontaneous staggered polarization and the zero-field polarizability are given by

$$P_0 = -\left(\frac{\partial f}{\partial s}\right)_{s=0+}, \qquad \chi = -\left(\frac{\partial^2 f}{\partial s^2}\right)_{s=0+} \tag{2}$$

where f is the free energy per vertex. The exponents β, γ and γ' are then defined as usual by the critical behaviours of P_0 and χ near the transition temperature T_c.

Our first observation is that the eight-vertex model defined by (1) is equivalent to an

§ On leave of absence from the University of Groningen, The Netherlands. Supported by The Netherlands Organisation for Pure Scientific Research (ZWO).
|| Supported by the Swiss National Foundation for Scientific Research.
¶ Supported in part by the National Science Foundation.

Ising model on a square lattice with first-neighbour interactions $\frac{1}{2}s$ and second-neighbour interactions $\frac{1}{2}\epsilon_1$ and $\frac{1}{2}\epsilon_2$ respectively along the two diagonal directions. This equivalence is most conveniently seen by applying a sequence of two transformations to the model. Firstly, in the modified F model we reverse the arrows along every other zigzag path in the northeast-southwest direction: see, for example, figure 2 of Fan and Wu (1970). The model is then transformed into a *ferroelectric* model (that is, vertices (1) and (2) favoured) with horizontal and vertical electric fields both equal to $\frac{1}{2}s$. The stated result now follows from the established equivalence of the latter model to an Ising model with two- and four-spin interactions (Wu 1971). The four-spin interactions vanish in this case because of the choice of the energy parameters in (1). We pause to remark here that, as a consequence of the first transformation, the *antiferroelectric* eight-vertex model in a staggered field is completely equivalent to a *ferroelectric* eight-vertex model in a direct field at 45° direction in the first quadrant. Consequently, the respective critical indices of both models are the same.

To compute the spontaneous staggered polarization P_0 we use standard convexity argument to derive the inequality

$$P_+(0+) \geqslant P_0 \geqslant P_+(0) \tag{3}$$

where $P_+(s)$ is the staggered polarization computed at the '+' boundary condition that all spins on the boundary are fixed at $\sigma = +1$. Using the Griffiths inequality (Griffiths 1966) and some recent results on the Ising model by Martin-Löf (1972) and Benettin *et al* (1973), it can be rigourously established that

$$P_+(0+) = P_+(0) \equiv \lim_{N\to\infty} (2N)^{-1} \sum_{(\mathrm{NN})} \langle\sigma^A\sigma^B\rangle_+ = m^2 \tag{4}$$

where N is the total number of vertices, $\langle\sigma^A\sigma^B\rangle_+$ denotes the thermal average of the nearest-neighbour (NN) spin correlation under the boundary condition $\sigma = +1$, and m is the spontaneous magnetization per vertex of the Ising model. It follows then that

$$P_0 = m^2. \tag{5}$$

Since it is known that $m \sim (T_c - T)^{1/8}$ at T_c-, we find $P_0 \sim (T_c - T)^{1/4}$ and hence $\beta = \frac{1}{4}$. This result is to be compared with the $(T_c - T)^{1/2}$ behaviour of the correlation between two vertical arrows (Barouch 1971).

In order to compute the zero-field polarizability we use the fluctuation relation

$$\chi = \tfrac{1}{4} \sum_{r,\delta,\delta'} [\langle\sigma_0{}^A\sigma_\delta{}^B\sigma_r{}^A\sigma_{r+\delta'}{}^B\rangle - \langle\sigma_0{}^A\sigma_\delta{}^B\rangle\langle\sigma_r{}^A\sigma_{r+\delta'}{}^B\rangle]_{s=0+} \tag{6}$$

where $\langle\ \rangle$ denotes the thermal averages in the thermodynamic limit; σ and σ' are respectively the nearest-neighbour vectors of the vertices at the origin and at r. The four-spin correlation function decouples into the product $\langle\sigma^A\sigma^A\rangle\langle\sigma^B\sigma^B\rangle$ in the $s = 0$ limit, while the $\langle\sigma^A\sigma^B\rangle$ correlation is evaluated by the use of the '+' boundary condition as above. Using the result (Kadanoff 1966a,b) that for large r

$$\langle\sigma_0{}^A\sigma_r{}^A\rangle - \langle\sigma_0{}^A\rangle_+\langle\sigma_r{}^A\rangle_+ \sim (a/r)^{1/4} D^\pm(\kappa r), \qquad T\to T_c\pm \tag{7}$$

where κ^{-1} is the correlation length of the nearest-neighbour Ising model which diverges as $|T - T_c|^{-1}$, we obtain the singular behaviour of χ by replacing the sum Σ_r by an integral. The result leads to

$$\chi \sim |T - T_c|^{-3/2}, \qquad T\to T_c\pm \tag{8}$$

which gives $\gamma = \gamma' = \frac{3}{2}$.

If we identify the quantity inside the square bracket of (6) as the vertex correlation function which, by (7), decays as $D^+/r^{1/2}$ at T_c+, we then obtain the critical index $\eta = \frac{1}{2}$. Furthermore, all the two-point vertex–vertex correlation functions can be expressed in terms of the Ising spin correlations. The critical index of the vertex–vertex correlation length is therefore the same as that of the Ising model, $\nu = \nu' = 1$. In this connection our result agrees, as expected, with that of Johnson *et al* (1972), who computed ν for the arrow–arrow correlation in the general eight-vertex model.

It is readily verified that our results on the exponents obey the scaling relations. To complete the list of critical indices we include the values of further exponents obtained by assuming scaling: $\sigma = 7$, $\Delta = \Delta' = \frac{7}{4}$.

We would like to thank Dr N H Kuiper for the hospitality extended to us at the Institut des Hautes Etudes Scientifiques where this work was performed.

Note Added in Proof. After the submission of this paper for publication, we were informed by R J Baxter of an earlier discussion on the validity of the scaling relations in the eight-vertex model. In a forthcoming publication, Baxter (1973) has obtained the critical index μ of the interfacial tension for the general eight-vertex model and found the result in agreement with the scaling predictions.

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