The Coupler Surface of the RSRS Mechanism

Nicolas Rojas[1]
Department of Mechanical Engineering and Materials Science,
Yale University,
9 Hillhouse Avenue,
New Haven, CT 06511
e-mail: nicolas.rojas@yale.edu

Aaron M. Dollar
Department of Mechanical Engineering and Materials Science,
Yale University,
15 Prospect Street,
New Haven, CT 06511
e-mail: aaron.dollar@yale.edu

Two degree-of-freedom (2-DOF) closed spatial linkages can be useful in the design of robotic devices for spatial rigid-body guidance or manipulation. One of the simplest linkages of this type, without any passive DOF on its links, is the revolute-spherical-revolute-spherical (RSRS) four-bar spatial linkage. Although the RSRS topology has been used in some robotics applications, the kinematics study of this basic linkage has unexpectedly received little attention in the literature over the years. Counteracting this historical tendency, this work presents the derivation of the general implicit equation of the surface generated by a point on the coupler link of the general RSRS spatial mechanism. Since the derived surface equation expresses the Cartesian coordinates of the coupler point as a function only of known geometric parameters of the linkage, the equation can be useful, for instance, in the process of synthesizing new devices. The steps for generating the coupler surface, which is computed from a distance-based parametrization of the mechanism and is algebraic of order twelve, are detailed and a web link where the interested reader can download the full equation for further study is provided. It is also shown how the celebrated sextic curve of the planar four-bar linkage is obtained from this RSRS dodecic.
[DOI: 10.1115/1.4030776]

1 Introduction

When the end links of two revolute-spherical (RS) serial chains are rigidly connected, an RSSR four-bar spatial linkage is obtained. This closed chain mechanism has mobility two and, regardless of the considered kinematic inversion, one of these DOF corresponds to the passive rotation of the link connecting the two spherical pairs about the axis defined by their centers of rotation. The kinematic analysis and synthesis of this spatial linkage has been object of extensive study over the years. The literature on the topic is indeed vast, but a good introduction to relevant results and its applications can be found in Refs. [1–4].

The linkage resulting from the topological permutation of one of the RS couples in the RSSR mechanism corresponds to a four-bar spatial linkage of mobility two without any passive DOF on its links: the spatial RSRS linkage, where all kinematic inversions are equivalent (Fig. 1). Surprisingly, this 2-DOF four-bar spatial linkage has received little attention in the literature except for cursory descriptions [5,6, p. 180]. In terms of applications, the mechanism has been employed in the design of force-reflecting devices [7,8], and closed spatial linkages of mobility two can generally be used for the design of high performance pick-and-place robots [9] or for the development of milling tool manipulators [10]. The scarce prior work related to this mechanism has not formally described its coupler surface.

This work aims to direct the attention of the mechanism community to the interesting, and basic, RSRS spatial linkage by presenting the derivation of the general implicit equation of the surface generated by a point on its coupler link. This equation expresses the Cartesian coordinates of the coupler point as a function only of known geometric parameters of the mechanism. The coupler surface of a RSRS mechanism determines its full work region, that is, its reachable workspace. In other words, at any spatial point that belongs to the coupler surface of a RSRS mechanism, the linkage can be assembled. Thus, for instance, this surface equation can be useful in the process of synthesizing new devices based on the RSRS topology for any application that implies spatial rigid-body guidance or manipulation.

Parametric equations of coupler curves/surfaces of planar and spatial linkages are in general simpler to derive than implicit equations—think, for example, in the coupler curve equation of the planar four-bar linkage in parametric form. The deduction of implicit expressions for the motion of coupler links is indeed far more complicated. These kinds of equations are important in theoretical kinematics of mechanisms for several reasons, for instance, implicit equations allow the determination of singular points, e.g., ordinary double points—in the workspace of the mechanism and other algebraic, geometric, and kinematic properties of it, e.g., circularity [4,11]. Moreover, in implicit expressions, the point membership determination is trivial; then it can be straightforwardly detected if a point belongs to the workspace of the linkage. This is a relevant question for any designer, and specialized software can use the property to guide the design not only of the mechanism itself but also of the surrounding environment to avoid potential collisions. Other aspects that show the relevance of implicit coupler equations include the synthesis of mechanisms for known workspaces [12], the development of alternative approaches for kinematic synthesis [13], and their use in the validation of numerical approaches for the computation of the workspace of linkages and robotic systems, e.g., Refs. [14,15].

The RSRS coupler surface, which is computed from a distance-based parametrization of the mechanism, is algebraic of order twelve, i.e., a dodecic surface. The highest-order term of the equation is $(1/16)s_{1,2}^4 A_{4,5,6}^4 (x^2+z^2)^2 (x^2+y^2+z^2)^4$, where $s_{1,2}$ and $A_{4,5,6}$ correspond to known squared distance and area values of the spatial linkage. The details for generating such surface are provided, as well as a web link where the interested reader can download the full equation for further study. The presented result is additionally validated by showing how the celebrated sextic curve of the planar four-bar linkage is obtained from the derived RSRS dodecic. Both results as well as the distance-based technique introduced for the computation of coupler equations of spatial linkages constitute original contributions to theoretical kinematics of mechanisms.

2 The RSRS Coupler Surface

Using a distance-based parametrization, a link connecting two skew revolute axes can be modeled by taking two points on each of these axes and connecting them all with edges to form a tetrahedron. Similarly, a link connecting a revolute axis and a ball joint can be modeled by taking two points on the axis and the center of rotation of the spherical pair and connecting them all with edges to form a triangle. Then, a general RSRS mechanism can be modeled as the bar-and-joint framework involving seven vertices and 13 edges shown in Fig. 1, where P_7 corresponds to a point on the coupler link.

Given the point sequence P_i, P_j, P_k, P_l, P_m in some Euclidean space, the Cayley–Menger determinant of the involved points is defined as

[1]Corresponding author.
Manuscript received February 16, 2015; final manuscript received May 27, 2015; published online August 18, 2015. Assoc. Editor: Raffaele Di Gregorio.

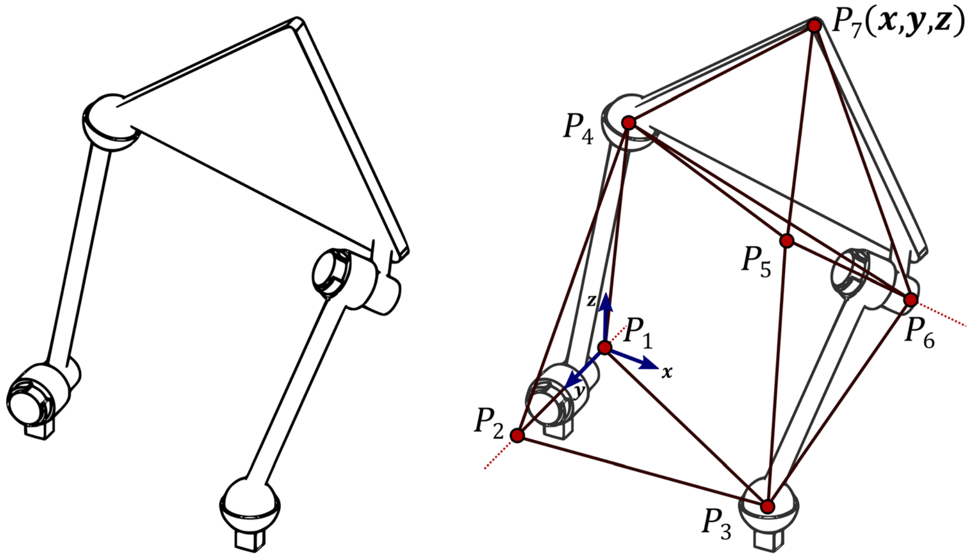

Fig. 1 General RSRS spatial mechanism and its associated notation

$$D(i,j,k,l,m) = -\frac{1}{16}\left|\Delta_{i,j,k,l,m}\right| = -\frac{1}{16}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & S_{i,j} & S_{i,k} & S_{i,l} & S_{i,m} \\ 1 & S_{i,j} & 0 & S_{j,k} & S_{j,l} & S_{j,m} \\ 1 & S_{i,k} & S_{j,k} & 0 & S_{k,l} & S_{k,m} \\ 1 & S_{i,l} & S_{j,l} & S_{k,l} & 0 & S_{l,m} \\ 1 & S_{i,m} & S_{j,m} & S_{k,m} & S_{l,m} & 0 \end{vmatrix} \quad (1)$$

where $s_{i,j} = d_{i,j}^2 = \|\mathbf{p}_j - \mathbf{p}_i\|^2$ is the squared distance between P_i and P_j, with $\mathbf{p}_i$ being the position vector of point P_i in the global reference frame. Equation (1) is a quadratic polynomial in $s_{l,m}$ that via Laplace expansion can be expressed as

$$D(i,j,k,l,m) = -\frac{1}{16}\left(a_{i,j,k,l,m}s_{l,m}^2 + b_{i,j,k,l,m}s_{l,m} + c_{i,j,k,l,m}\right) \quad (2)$$

with the coefficients $a_{i,j,k,l,m}$, $b_{i,j,k,l,m}$, and $c_{i,j,k,l,m}$ as given in Appendix A. For the general point sequence $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$, the Cayley–Menger determinant gives $(n-1)!^2$ times the squared hypervolume of the simplex spanned by the points in $\mathbb{E}^{n-1}$ [16]. Hence, $D(i,j,k,l,m) = 0$ in $\mathbb{E}^3$.

According to the notation of Fig. 1, let $f_1 = D(1,2,7,3,4)$ and $f_2 = D(5,6,7,3,4)$. Then,

$$f_1 = a_{1,2,7,3,4}s_{3,4}^2 + b_{1,2,7,3,4}s_{3,4} + c_{1,2,7,3,4} = 0 \text{ and} \quad (3)$$

$$f_2 = a_{5,6,7,3,4}s_{3,4}^2 + b_{5,6,7,3,4}s_{3,4} + c_{5,6,7,3,4} = 0 \quad (4)$$

since points $P_1, \ldots, P_7$ are all embedded in $\mathbb{E}^3$. Now, by eliminating $s_{3,4}$ in the system formed by Eqs. (3) and (4), we get

$$a_1^2c_2^2 - 2a_1a_2c_1c_2 - a_1b_1b_2c_2 + a_1b_2^2c_1 + a_2^2c_1^2 + a_2b_1^2c_2 - a_2b_1b_2c_1 = 0 \quad (5)$$

where $a_1 = a_{1,2,7,3,4}$, $b_1 = b_{1,2,7,3,4}$, $c_1 = c_{1,2,7,3,4}$, $a_2 = a_{5,6,7,3,4}$, $b_2 = b_{5,6,7,3,4}$, and $c_2 = c_{5,6,7,3,4}$. Equation (5) is a closure polynomial of the RSRS mechanism in terms of the squared lengths of the edges of its associated framework and the unknown squared distances: $s_{1,7}$, $s_{2,7}$, and $s_{3,7}$. This equation is solely satisfied in the points in $\mathbb{E}^3$ where the spatial four-bar linkage can be assembled with the corresponding bar dimensions and orientation of the revolute pairs.

Finally, to derive the locus of point P_7 with coordinates $\mathbf{p}_7 = (x,y,z)$, the location of the ground link has to be defined. Then, we can assume that P_1 equals the origin of the global reference frame with P_2 located in the positive side of the $y-$ axis and that P_3 is located in the positive quadrant of the $xy-$ plane. Thus, we have $\mathbf{p}_1 = (0,0,0)$, $\mathbf{p}_2 = \left(0, d_{1,2}, 0\right)$, and

$$\mathbf{p}_3 = \left(\frac{2A_{1,2,3}}{d_{1,2}}, \frac{s_{1,2} - s_{2,3} + s_{1,3}}{2d_{1,2}}, 0\right)$$

where $A_{1,2,3} = \frac{1}{4}\sqrt{-s_{2,3}^2 + 2s_{1,2}s_{2,3} + 2s_{1,3}s_{2,3} - s_{1,3}^2 + 2s_{1,3}s_{1,2} - s_{1,2}^2}$ is the area of the triangle defined by points P_1, P_2, and P_3. Therefore,

$$\begin{aligned} s_{1,7} &= x^2 + y^2 + z^2 \\ s_{2,7} &= x^2 + y^2 + z^2 - 2d_{1,2}y + s_{1,2}, \text{ and} \\ s_{3,7} &= x^2 + y^2 + z^2 - \frac{4A_{1,2,3}}{d_{1,2}}x - \frac{s_{1,2} - s_{2,3} + s_{1,3}}{d_{1,2}}y + s_{1,3} \end{aligned} \quad (6)$$

Substituting Eq. (6) into Eq. (5), fully expanding the result and rearranging terms, we get

$$\begin{aligned} \Gamma(x,y,z) \stackrel{\text{def}}{=} &\frac{1}{16}s_{1,2}^4A_{4,5,6}^4\left(x^2+z^2\right)^2\left(x^2+y^2+z^2\right)^4 \\ &+ \frac{1}{32}d_{1,2}^7A_{4,5,6}^2p_3\left(x^2+z^2\right)\left(x^2+y^2+z^2\right)^3 \\ &+ \frac{1}{2^{13}}s_{1,2}^3p_6\left(x^2+y^2+z^2\right)^2 + \frac{1}{2^{12}}d_{1,2}^5p_7\left(x^2+y^2+z^2\right) \\ &+ \frac{1}{2^{16}}s_{1,2}^2q_8 + \frac{1}{2^{14}}d_{1,2}^3q_7 + \frac{1}{2^{15}}s_{1,2}q_6 + \frac{1}{2^{14}}d_{1,2}q_5 \\ &+ q_4 + \frac{1}{2^{14}}d_{1,2}\left(s_{1,4} - s_{4,7}\right)q_3 + \frac{1}{2^{15}}s_{1,2}\left(s_{1,4} - s_{4,7}\right)^2q_2 \\ &+ \frac{1}{2^6}d_{1,2}^3A_{1,2,3}^2A_{5,6,7}^2q_1 + \frac{1}{2^8}s_{1,2}^2A_{5,6,7}^4\left(s_{1,4} - s_{4,7}\right)^4q_0 = 0 \end{aligned} \quad (7)$$

where p_i and q_i are homogeneous polynomials of the form $\sum_{a+b+c=i}\alpha_{a,b,c}x^ay^bz^c$ with at most $\binom{i+2}{i}$ monomials and $\alpha_{a,b,c}$ a polynomial in $s_{1,2} = d_{1,2}^2$, $s_{1,3}$, $s_{1,4}$, $s_{2,3}$, $s_{2,4}$, $s_{3,5}$, $s_{3,6}$,

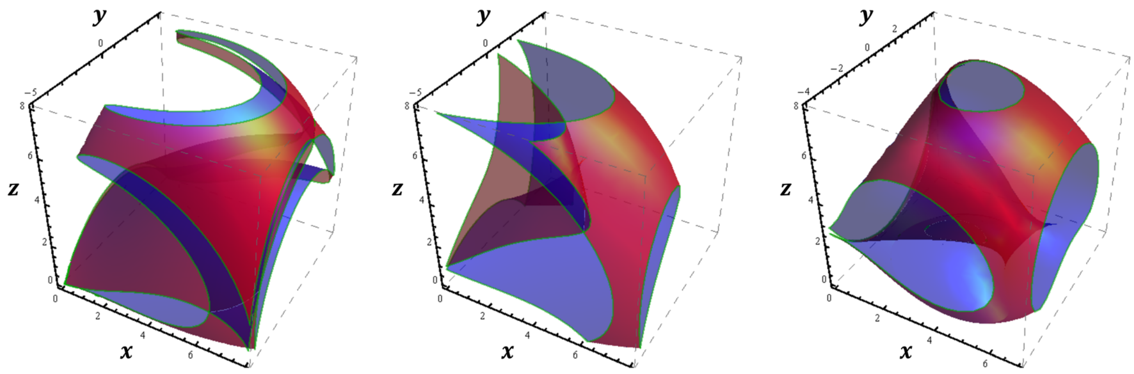

Fig. 2 Examples of the RSRS dodecic: (a) $s_{1,2}=4, s_{1,3}=65, s_{1,4}=17, s_{2,3}=65, s_{2,4}=21, s_{3,5}=30, s_{3,6}=19, s_{4,5}=66, s_{4,6}=41, s_{4,7}=42, s_{5,6}=9, s_{5,7}=36, s_{6,7}=49$, **(b)** $s_{1,2}=4, s_{1,3}=73, s_{1,4}=21, s_{2,3}=65, s_{2,4}=33, s_{3,5}=53, s_{3,6}=35, s_{4,5}=69, s_{4,6}=37, s_{4,7}=33, s_{5,6}=14, s_{5,7}=62, s_{6,7}=66$, **and (c)** $s_{1,2}=4, s_{1,3}=50, s_{1,4}=25, s_{2,3}=50, s_{2,4}=29, s_{3,5}=21, s_{3,6}=34, s_{4,5}=36, s_{4,6}=21, s_{4,7}=22, s_{5,6}=9, s_{5,7}=22, s_{6,7}=21$

$s_{4,5}, s_{4,6}, s_{4,7}, s_{5,6}, s_{5,7}$, and $s_{6,7}$ that can be zero. Additionally, $A_{4,5,6}=(1/4)\sqrt{-s_{5,6}^2+2s_{5,6}s_{4,6}-s_{4,6}^2+2s_{4,5}s_{4,6}+2s_{4,5}s_{5,6}-s_{4,5}^2}$ and $A_{5,6,7}=(1/4)\sqrt{-s_{5,6}^2+2s_{5,7}s_{5,6}-s_{5,7}^2+2s_{5,6}s_{6,7}+2s_{6,7}s_{5,7}-s_{6,7}^2}$. $\Gamma(x,y,z)$ is an algebraic surface of degree 12 (a dodecic) that corresponds to the coupler surface, traced by point P_7, of the general RSRS spatial mechanism. The expressions of the polynomials p_i and q_i cannot be included here due to space limitations. However, they can be easily reproduced using a computer algebra system following the steps given above and the interested reader can download the full expression of $\Gamma(x,y,z)$ from the website.[2] Examples of the RSRS dodecic are depicted in Fig. 2.

3 The Planar Four-Bar Sextic

According to the notation of Fig. 1, if

(i) the axes of the revolute pairs, defined by points P_1 and P_2, and P_5 and P_6, respectively, are parallel and
(ii) the points P_1, P_3, P_4, P_6, and P_7 lie on the $xz-$ plane,

the RSRS spatial mechanism reduces to a planar four-bar linkage, namely, the kinematic chain constituted by the revolute centers P_1, P_3, P_4, and P_6 with the triangle formed by P_4, P_6, and P_7 defining the coupler link. It is well known that the coupler equation of a planar four-bar linkage, i.e., the curve traced by point P_7—is a curve of order six [1]. Next we show how such sextic can be obtained from $\Gamma(x,y,z)$, the RSRS dodecic presented in Eq. (7).

Using projective geometry arguments, condition (i) implies that the axes of the revolute pairs meet at a point at infinity, say P^∞. Hence, $P_2=P_5=P^\infty$, that is,

$$d_{1,2}=d_{2,3}=d_{2,4}=d_{3,5}=d_{4,5}=d_{5,6}=d_{5,7}=\delta \tag{8}$$

with $\delta>0, \delta\to\infty$. Moreover, condition (ii) implies that the motion of the mechanism is restricted to the surface

$$y=0 \tag{9}$$

After substituting Eqs. (8) and (9) into Eq. (7), the RSRS coupler surface reduces to a curve that can be rewritten as a radical equation in δ. Explicitly,

$$\Gamma(x,y,z)\left[\begin{array}{l} y=0, s_{1,2}=\delta^2 \\ s_{2,3}=\delta^2, s_{2,4}=\delta^2 \\ s_{3,5}=\delta^2, s_{4,5}=\delta^2 \\ s_{5,6}=\delta^2, s_{5,7}=\delta^2 \end{array}\right. = \begin{array}{c} \gamma_{12}(x,z)\delta^{12}+\sqrt{s_{1,3}(4\delta^2-s_{1,3})}\gamma_{11}(x,z)\delta^{11} \\ +\gamma_{10}(x,z)\delta^{10}+\cdots+\sqrt{s_{1,3}(4\delta^2-s_{1,3})}\gamma_1(x,z)\delta \\ +\gamma_0(x,z)=0 \end{array} \tag{10}$$

Now, factoring out δ^{12} in the above expression, we get

$$\delta^{12}\left(\gamma_{12}(x,z)+\frac{\sqrt{s_{1,3}(4\delta^2-s_{1,3})}\gamma_{11}(x,z)}{\delta}+\frac{\gamma_{10}(x,z)}{\delta^2}+\cdots+\frac{\sqrt{s_{1,3}(4\delta^2-s_{1,3})}\gamma_1(x,z)}{\delta^{11}}+\frac{\gamma_0(x,z)}{\delta^{12}}\right)=0 \tag{11}$$

Then, the curve equation can be expressed as

$$\gamma_{12}(x,z)+\frac{\sqrt{s_{1,3}(4\delta^2-s_{1,3})}}{\delta}\gamma_{11}(x,z)+\frac{1}{\delta^2}\gamma_{10}(x,z)+\ldots+\frac{\sqrt{s_{1,3}(4\delta^2-s_{1,3})}}{\delta^{11}}\gamma_1(x,z)+\frac{1}{\delta^{12}}\gamma_0(x,z)=0 \tag{12}$$

[2] http://www.robot-mechanics.com/rsrsmechanism.html

Since $\delta \to \infty$ and

$$\lim_{\delta\to\infty} \frac{\sqrt{s_{1,3}\left(4\delta^2 - s_{1,3}\right)}}{\delta^n} = \frac{2\sqrt{s_{1,3}}}{\lim_{\delta\to\infty} \delta^{n-1}} = \begin{cases} 2d_{1,3}, & \text{if } n=1 \\ 0, & \text{if } n>1 \end{cases} \tag{13}$$

it is concluded that the curve equation traced by point P_7 is

$$\gamma_{12}(x,z) + 2d_{1,3}\gamma_{11}(x,z) = 0 \tag{14}$$

This curve is algebraic of order twelve and factorizes as

$$\begin{aligned}\Omega(x,z) &\stackrel{\text{def}}{=} \gamma_{12}(x,z) + 2d_{1,3}\gamma_{11}(x,z) \\ &= \frac{1}{256}\left(\psi(x,z) - \phi(x,z)A_{4,6,7}\right)\left(\psi(x,z) + \phi(x,z)A_{4,6,7}\right) = 0\end{aligned} \tag{15}$$

where $A_{4,6,7}=(1/4)\sqrt{-s_{6,7}^2+2s_{4,6}s_{6,7}+2s_{6,7}s_{4,7}-s_{4,7}^2+2s_{4,7}s_{4,6}-s_{4,6}^2}$ with $\psi(x,z)$ and $\phi(x,z)$ as given in Appendix B.

It can be straightforwardly verified that the expression deduced by S. Roberts for the coupler curve of the planar four-bar linkage [17] (a succinct reproduction of this deduction can be found in Ref. [1]) is equivalent to $\psi(x,z) - \phi(x,z)A_{4,6,7} = 0$. This equation corresponds to the coupler curve of the linkage in the case that, in the triangle associated to the coupler link, P_7 is located to the left of the vector defined by points P_4 and P_6. The equation $\psi(x,z) + \phi(x,z)A_{4,6,7} = 0$ corresponds to the coupler curve in the case P_7 is located to the right of such vector.

From the topology of the RSRS mechanism, according to the notation of Fig. 1, it can seem that a valid configuration of the linkage depends on the orientation of the tetrahedrons $P_1P_2P_3P_4$ and $P_4P_5P_6P_7$ thus making the condition of Eq. (5) necessary but not sufficient. Nevertheless, this is not the case. Note that the tetrahedron $P_1P_2P_3P_4$ is not a geometric parameter of the RSRS linkage. Then, an RSRS mechanism may have valid configurations with positive and negative orientations of such tetrahedron. In a planar four-bar linkage, this is equivalent to the case of the triangle formed by the crank and the ground link in a given configuration.

The case of the tetrahedron $P_4P_5P_6P_7$ is different because it defines the coupler link of the mechanism, being evidently a geometric parameter of the linkage. In this circumstance, observe that at any given configuration of an RSRS mechanism with, for instance, a positive-oriented coupler link tetrahedron, a second valid configuration with opposite orientation can always be obtained since the points P_5 and P_6 can be mirrored with respect to the plane defined by the points P_3, P_4, and P_7. In other words, a valid configuration of an RSRS mechanism does not depend on the orientation of the coupler link. However, note that this is not the case in a planar four-bar linkage; since, in general, the coupler link cannot be reflected with respect to a fixed reference at a particular configuration. This is indeed the reason why Eq. (15) factorizes in two factors of degree six.

4 Conclusion

The derivation of the general implicit equation of the coupler surface of the RSRS spatial mechanism has been presented and its full expression has been made publicly available for those interested in its further study. Despite that this closed chain linkage belongs to the family of four-bar spatial mechanisms, whose members are the simplest movable closed chains and have been the object of extensive analyses over the years, the kinematics study of the RSRS topology has certainly received little attention. Additionally, it has been also shown how the celebrated sextic curve of the planar four-bar linkage can be obtained from the derived RSRS coupler expression. To the authors' knowledge, all these results are original contributions to theoretical kinematics of mechanisms; they are obtained thanks to a novel distance-based technique introduced herein to compute implicit coupler equations of spatial linkages. Future work on the topic could focus on determining algebraic, geometric, and kinematic properties of the RSRS mechanism from its coupler equation as well as on exploring the application of the presented technique to the unification of the computation of coupler surfaces of 2-DOF four-bar spatial mechanisms.

Acknowledgment

This work was supported in part by the National Science Foundation Grant No. IIS-1317976.

Appendix A

This appendix presents the expressions of the coefficients $a_{i,j,k,l,m}$, $b_{i,j,k,l,m}$, and $c_{i,j,k,l,m}$ of Eq. (2). They can be straightforwardly obtained from iteratively developing $D(i,j,k,l,m) = -(1/16)\left|\Delta_{i,j,k,l,m}\right|$ according to Laplace along the last row. Thus,

$$a_{i,j,k,l,m} = -M_{65,55}, \tag{A1}$$

$$\begin{aligned}b_{i,j,k,l,m} = {}& -M_{61,55} + s_{i,m}M_{62,55} - s_{j,m}M_{63,55} + s_{k,m}M_{64,55} \\ & - M_{65,51} + s_{i,l}M_{65,52} - s_{j,l}M_{65,53} + s_{k,l}M_{65,54}\end{aligned} \tag{A2}$$

$$\begin{aligned}c_{i,j,k,l,m} = {}& -s_{i,l}M_{61,51} + s_{j,l}M_{61,52} - s_{k,l}M_{61,53} \\ & + s_{i,m}\left(M_{62,51} - s_{j,l}M_{62,52} + s_{k,l}M_{62,53}\right) \\ & - s_{j,m}\left(M_{63,51} - s_{i,l}M_{63,52} + s_{k,l}M_{63,53}\right) \\ & + s_{k,m}\left(M_{64,51} - s_{i,l}M_{64,52} + s_{j,l}M_{64,53}\right)\end{aligned} \tag{A3}$$

where $M_{op,qr}$ is the (q,r) minor of the submatrix constructed by removing row o and column p from $\Delta_{i,j,k,l,m}$. For instance,

$$M_{65,55} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & s_{i,j} & s_{i,k} \\ 1 & s_{i,j} & 0 & s_{j,k} \\ 1 & s_{i,k} & s_{j,k} & 0 \end{vmatrix}$$

Appendix B

This appendix presents the expressions of the polynomials $\psi(x,z)$ and $\phi(x,z)$ of Eq. (15)

$$\begin{aligned}\psi(x,z) = {}& s_{1,3}\left(x^2+z^2\right)^3 - d_{1,3}\left(s_{4,7} + 3s_{4,6} - s_{6,7}\right)x\left(x^2+z^2\right)^2 \\ & + r_2\left(x^2+z^2\right) - d_{1,3}\big(-3s_{4,7}s_{3,6} - s_{1,4}s_{6,7} + s_{6,7}s_{3,6} \\ & - s_{3,6}s_{4,6} - s_{4,7}s_{4,6} - 3s_{4,6}s_{1,4} + 3s_{1,4}s_{4,7} + s_{4,6}s_{1,3} \\ & - 3s_{4,6}s_{6,7} - s_{6,7}s_{1,3} + 2s_{4,6}^2 + 3s_{4,7}s_{1,3} \\ & + s_{6,7}^2 - s_{4,7}^2\big)x\left(x^2+z^2\right) + c_1x^2 + c_2z^2 \\ & + d_{1,3}\left(s_{1,4} - s_{4,7}\right)c_3x + s_{1,3}s_{6,7}\left(s_{1,4} - s_{4,7}\right)^2\end{aligned} \tag{B1}$$

$$\begin{aligned}\phi(x,z) = {}& -4d_{1,3}z\left(x^2+z^2\right)^2 + 8s_{1,3}xz\left(x^2+z^2\right) \\ & - 4d_{1,3}\left(2s_{4,6} - s_{6,7} + s_{1,3} - s_{3,6} - s_{4,7} - s_{1,4}\right)z\left(x^2+z^2\right) \\ & - 8s_{1,3}\left(s_{6,7} + s_{1,4} - s_{4,6}\right)xz \\ & + 4d_{1,3}\left(s_{1,3} + s_{6,7} - s_{3,6}\right)\left(s_{1,4} - s_{4,7}\right)z\end{aligned} \tag{B2}$$

with

$$\begin{aligned}r_2 = {}& \big(-s_{4,7}s_{3,6} - s_{4,7}s_{4,6} - s_{3,6}s_{4,6} + 3s_{4,6}s_{1,3} - 2s_{6,7}s_{1,3} \\ & - s_{1,4}s_{6,7} + s_{1,4}s_{4,7} - s_{4,6}s_{6,7} + s_{4,6}^2 - s_{4,6}s_{1,4} \\ & + 3s_{4,7}s_{1,3} + s_{6,7}s_{3,6}\big)x^2 + \big(-s_{4,7}s_{3,6} - s_{4,6}s_{6,7} - s_{1,4}s_{6,7} \\ & + s_{4,7}s_{1,3} - s_{4,7}s_{4,6} - s_{3,6}s_{4,6} + s_{4,6}s_{1,3} + s_{1,4}s_{4,7} \\ & + s_{4,6}^2 - s_{4,6}s_{1,4} + s_{6,7}s_{3,6}\big)z^2\end{aligned}$$

$$c_1 = s_{1,4}^2 s_{6,7} + s_{6,7}^2 s_{1,4} + s_{1,3} s_{6,7}^2 + s_{4,7} s_{4,6} s_{1,3} - 2 s_{4,7}^2 s_{1,3} - 3 s_{4,6} s_{1,4} s_{1,3} - s_{1,4} s_{4,7} s_{6,7} + s_{4,6} s_{6,7} s_{4,7} + s_{1,4} s_{1,3} s_{6,7} + s_{4,7} s_{1,3}^2 + s_{4,7} s_{3,6}^2 - 2 s_{4,6} s_{6,7} s_{1,3} - s_{4,7} s_{1,3} s_{6,7} - s_{4,7} s_{3,6} s_{4,6} - 2 s_{4,7} s_{1,3} s_{3,6} - s_{1,4} s_{6,7} s_{3,6} - s_{1,4} s_{4,7} s_{3,6} - s_{6,7} s_{4,7} s_{3,6} + s_{4,7}^2 s_{3,6} + 3 s_{1,4} s_{1,3} s_{4,7} - s_{1,4} s_{6,7} s_{4,6} + s_{4,6}^2 s_{1,3} + s_{4,6} s_{1,4} s_{3,6}$$

$$c_2 = -2 s_{4,7} s_{1,3} s_{3,6} - s_{1,4} s_{6,7} s_{4,6} + s_{4,7} s_{1,3}^2 - s_{1,3} s_{6,7}^2 + s_{4,7} s_{1,3} s_{6,7} - s_{1,4} s_{4,7} s_{6,7} + s_{6,7}^2 s_{1,4} + s_{4,7} s_{3,6}^2 + s_{4,6} s_{6,7} s_{4,7} - s_{1,4} s_{6,7} s_{3,6} + s_{4,7}^2 s_{3,6} - s_{1,4} s_{4,7} s_{3,6} - 2 s_{4,7}^2 s_{1,3} - s_{6,7} s_{4,7} s_{3,6} - s_{4,7} s_{3,6} s_{4,6} - s_{1,4} s_{1,3} s_{6,7} + 3 s_{4,7} s_{4,6} s_{1,3} + s_{1,4} s_{1,3} s_{4,7} + s_{4,6} s_{1,4} s_{3,6} + 2 s_{4,6} s_{6,7} s_{1,3} + s_{1,4}^2 s_{6,7} - s_{4,6}^2 s_{1,3} - s_{4,6} s_{1,4} s_{1,3}$$

$$c_3 = -2 s_{1,4} s_{6,7} + s_{4,6} s_{6,7} + s_{6,7} s_{3,6} - s_{3,6} s_{4,6} - s_{4,7} s_{1,3} + s_{6,7} s_{4,7} - s_{6,7} s_{1,3} + s_{4,6} s_{1,3} + s_{4,7} s_{3,6} - s_{6,7}^2$$

References

[1] Hunt, K. H., 1978, *Kinematic Geometry of Mechanisms*, Oxford University Press, New York.

[2] Merlet, J. P., 2006, *Parallel Robots*, Vol. 128, Springer, Dordrecht, The Netherlands.

[3] McCarthy, J. M., and Soh, G. S., 2010, *Geometric Design of Linkages*, Vol. 11, Springer, New York.

[4] Nawratil, G., 2010, "On the Spin Surface of RSSR Mechanisms With Parallel Rotary Axes," ASME J. Mech. Rob., **2**(1), p. 011009.

[5] Bagci, C., 1971, "Degrees of Freedom of Motion in Mechanisms," ASME J. Manuf. Sci. Eng., **93**(1), pp. 140–148.

[6] Dukkipati, R. V., 2001, *Spatial Mechanisms: Analysis and Synthesis*, CRC, Narosa Publishing House, New Delhi, India.

[7] Hayward, V., 1995, "Toward a Seven Axis Haptic Device," IEEE/RSJ International Conference on Intelligent Robots and Systems, Vol. 3, pp. 133–139.

[8] Zanganeh, K. E., and Angeles, J., 1996, "Displacement Analysis of a Six-Degree-of-Freedom Hybrid Hand Controller," IEEE International Conference on Robotics and Automation (ICRA), Minneapolis, MN, Apr. 22–28, Vol. 2, pp. 1281–1286.

[9] Pierrot, F., Krut, S., Baradat, C., and Nabat, V., 2011, "Par2: A Spatial Mechanism for Fast Planar Two-Degree-of-Freedom Pick-and-Place Applications," Meccanica., **46**(1), pp. 239–248.

[10] Shen, Q., Russell, K., and Sodhi, R. S., 2012, "Analytical Displacement and Velocity Modeling of the RSSR-SS Linkage," *Advances in Mechanisms Design*, Springer, Dordrecht, pp. 151–156.

[11] Pennock, G. R., and Hasan, A., 2002, "A Polynomial Equation for a Coupler Curve of the Double Butterfly Linkage," ASME J. Mech. Des. **124**(1), pp. 39–46.

[12] Bai, S., 2015, "Determination of Linkage Parameters From Coupler Curve Equations," *Mechanisms, Transmissions and Applications*, Springer International, Cham, Switzerland, pp. 49–57.

[13] Blechschmidt, J. L., and Uicker, J. J., 1986, "Linkage Synthesis Using Algebraic Curves," ASME J. Mech. Des., **108**(4), pp. 543–548.

[14] Rojas, N., and Thomas, F., 2013, "Application of Distance Geometry to Tracing Coupler Curves of Pin-Jointed Linkages," ASME J. Mech. Rob., **5**(2), p. 021001.

[15] Porta, J. M., Ros, L., Bohigas, O., Manubens, M., Rosales, C., and Jaillet, L., 2014, "The CUIK Suite: Motion Analysis of Closed-Chain Multibody Systems," IEEE Rob. Autom. Mag., **21**(3), pp. 105–114.

[16] Menger, K., 1931, "New Foundation of Euclidean Geometry," Am. J. Math., **53**(4), pp. 721–745.

[17] Roberts, S., 1875, "On Three-Bar Motion in Plane Space," Proc. London Math. Soc., **s1–7**(1), pp. 14–23.