Mumtaz K. Kassir

Professor,
Department of Civil Engineering,
The City College of the City University
of New York,
New York, N. Y. Mem. ASME

Stress-Intensity Factors for an Insulated Half-Plane Crack

This paper is concerned with determining the stress-intensity factors due to disturbance of a uniform flow of heat by an insulated half-plane crack in an elastic solid. The spatial thermoelastic problem is formulated in terms of Papkovich-Neuber displacement potentials and is solved by the application of Kontorovich-Lebedev integral transform and certain singular solutions of Laplace equation in three dimensions. The analysis reveals that four distinct displacement potentials are needed to satisfy the finite displacement and inverse square root stress-singularity at the edge of the crack. Closed-form expressions are obtained for the stress-intensity factors (k_2 and k_3) and their variations along the crack border are shown in curves.

Introduction

When an undisturbed heat flow with constant thermal gradient is diverted round a sharp edge of an insulated crack or flaw, there is local intensification of the temperature gradient accompanied by singular thermal stress which may cause crack propagation resulting in serious damage to structural members. Since the critical value of the intensity of the local stress field can be associated with the fracture toughness of the material, it follows that by knowing the stress-intensity factors as functions of the temperature gradient, material properties, and flaw size, it is possible to predict a critical temperature gradient which will not result in failure of the material. Confining attention to three-dimensional cracks or planes of discontinuities, the local intensity of the thermal stress has been determined for the so-called "penny-shaped" crack [1][1] and the more general elliptical crack [2]. In case of the elliptical geometry, both factors k_2 and k_3 (associated with the edge sliding and tearing modes of fracture [3]), are found to be operating, while for the axially symmetric crack, the local stress field is controlled by k_2 only. A summary of the solutions to these and other closely related crack problems is given in [4].

The objective of this investigation is to provide stress-intensity factor solutions to the thermoelastic problem of a uniform flow of heat disturbed by an insulated half-plane crack embedded in an elastic solid. The solid is assumed to be isotropic and homogeneous and the effects of both inertia and coupling between temperature and strain fields are neglected. Fig. 1 shows a sketch of the crack. In terms of the coordinates indicated, the crack occupies the region $x \leq 0, |z| < \infty$ of the midplane $y = 0$. The general thermal stress problem requires first finding the temperature, $T(x, y, z)$, at every point of the solid from Laplace equation in three dimensions

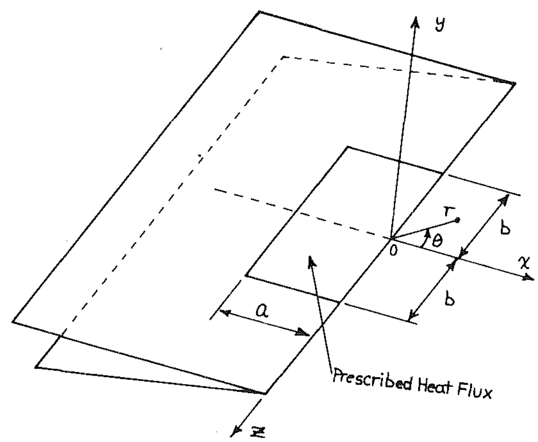

Fig. 1 Semi-infinite plane crack

$$\nabla^2 T(x, y, z) = 0 \tag{1}$$

and consequently the induced displacements from the Navier equations of static equilibrium

$$\nabla \cdot \nabla \mathbf{u} + (1 - 2\nu)\nabla^2 \mathbf{u} = 2(1 + \nu)\alpha \nabla T \tag{2}$$

In equations (1) and (2), $\mathbf{u}$ designates the displacement vector, ν and α are, respectively, the Poisson's ratio and the coefficient of thermal expansion of the material of the solid, and ∇, ∇^2 are the

[1] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until June 1, 1976. Readers who need more time to prepare a Discussion should request an extension of the deadline from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, March, 1975. Paper No. 76-APM-D.

usual del and Laplacian operators. Once the displacements are known, the stresses can be computed in straightforward manner from the Duhamel-Neumann stress-displacement relations.

The analysis shows that four distinct harmonic functions are required to satisfy the finite displacement and inverse square root singularity conditions at the crack edge ($r \to 0$). The state of the disturbed temperature field and the associated stresses throughout the solid is determined mainly in closed form. These are obtained by employing the techniques of Fourier and Kontorovich-Lebedev integral transforms and certain singular solutions of Laplace equation [5]. Explicit expressions for the stress-intensity factors are found and their variations along the crack border are displayed graphically. It should also be mentioned that the Green's functions due to concentrated normal and shear forces applied to the surfaces of the half-plane crack are treated in the works of Uflyand [5] and Kassir and Sih [6].

Temperature Field

The disturbed temperature field at every point of the solid is obtained from a solution of equation (1) in the region $y \geqslant 0$ subject to the conditions

$$\frac{\partial T}{\partial y} = -Q(x, z), \; y = 0, \theta = \pi \tag{3a}$$

$$T = 0, y = 0, \theta = 0 \tag{3b}$$

in which $Q(x, z)$ is the specified temperature gradient. The desired expression for the temperature is conveniently constructed by employing a Fourier cosine transform in the variable z and a Kontorovich-Lebedev transform in the variable r. Toward this end, let

$$T = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \int_0^\infty A(s, t) \frac{\sinh(\theta t)}{t \cosh(\pi t)} K_{it}(sr) \cos(sz) ds dt \tag{4}$$

where $K_{it}(sr)$ is the modified Bessel function of the second kind of imaginary order [7] and $A(s, t)$ is an arbitrary function such that equation (3b) is satisfied. Inserting (4) into (3a) and carrying out the appropriate inversions, $A(s, t)$ is determined as

$$A(s, t) = \frac{2^{3/2}}{\pi^{5/2}} t \sinh(\pi t) \int_0^\infty Q^*(r, s) K_{it}(sr) dr \tag{5a}$$

where

$$Q^*(r, s) = \int_0^\infty Q(r, z) \cos(sz) dz \tag{5b}$$

In order to establish the Green's function for the temperature field, suppose that there is a temperature gradient of magnitude $Q_0 \delta(x + a)\delta(z)$ at the point $r = a$, $\theta = \pi$, $z = 0$ of the crack surface, then equations (5) yield

$$A(s, t) = \frac{\sqrt{2}}{\pi^{5/2}} Q_0 t \sinh(\pi t) K_{it}(as) \tag{6}$$

Inserting equation (6) into (4) and performing the resulting integrals [4], the temperature field is obtained

$$T(r, \theta, z) = \frac{Q_0}{\pi^2 \rho} \tan^{-1}\left(\frac{\sqrt{2a(r - x)}}{\rho}\right) \tag{7}$$

where ρ represents the distance of any point in $y \geqslant 0$ to the point $r = a$, $\theta = \pi$, $z = 0$, i.e.,

$$\rho = [(x + a)^2 + y^2 + z^2]^{1/2} \tag{8}$$

Having obtained the temperature field, the induced displacements and stresses in the solid can be found. This will be done in the next section.

Thermal Stresses

The thermal displacements and stresses are governed by equations (2) where the temperature is known from equations (4) and (5). Denoting the projections of the displacement vector in the directions of the cylindrical coordinates by (u_r, u_θ, u_z), the Papkovich-Neuber potential representation of equations (2) gives the displacement field

$$u_r(r, \theta, z) = 4(1 - \nu)(f_1 \cos\theta + f_2 \sin\theta) - \frac{\partial F}{\partial r} + \sin\theta \frac{\partial}{\partial r}(r\Omega) \tag{9a}$$

$$u_\theta(r, \theta, z) = 4(1 - \nu)(f_2 \cos\theta - f_1 \sin\theta) - \frac{1}{r}\frac{\partial F}{\partial \theta} + \frac{\partial}{\partial \theta}(\sin\theta\,\Omega) \tag{9b}$$

$$u_z(r, \theta, z) = 4(1 - \nu) f_3 - \frac{\partial F}{\partial z} + r \sin\theta \frac{\partial \Omega}{\partial z} \tag{9c}$$

in which the following abbreviation has been introduced:

$$F = f_0 + (r \cos\theta) f_1 + (r \sin\theta) f_2 + z f_3 \tag{10}$$

and $f_n(r, \theta, z)$, $n = 0, 1, 2, 3$, are space harmonic functions. The potential $\Omega(r, \theta, z)$ can be determined from a knowledge of the temperature through the relation

$$\frac{\partial \Omega}{\partial y} = \frac{(1 + \nu)\alpha}{2(1 - \nu)} T(r, \theta, z) \tag{11}$$

The corresponding stress field is readily obtained from equations (9) and (10) and the usual Duhamel-Neumann stress-displacement relations in linear thermoelasticity [8]. In particular, the stresses associated with the θ-plane are found as

$$\begin{aligned}\frac{\sigma_\theta}{2\mu} = &-(1 - 2\nu)\left(\cos\theta \frac{\partial f_1}{\partial r} + \sin\theta \frac{\partial f_2}{\partial r}\right) \\ &+ 2(1 - \nu)\frac{1}{r}\left(\cos\theta \frac{\partial f_2}{\partial \theta} - \sin\theta \frac{\partial f_1}{\partial \theta}\right) + 2\nu \frac{\partial f_3}{\partial z} + \frac{\partial^2 f_0}{\partial r^2} + \frac{\partial^2 f_0}{\partial z^2} \\ &- \frac{1}{r}\left(\cos\theta \frac{\partial^2 f_1}{\partial \theta^2} + \sin\theta \frac{\partial^2 f_2}{\partial \theta^2}\right) + z\left(\frac{\partial^2 f_3}{\partial r^2} + \frac{\partial^2 f_3}{\partial z^2}\right) \\ &- \sin\theta\left(\frac{\partial \Omega}{\partial r} - \frac{1}{r}\frac{\partial^2 \Omega}{\partial \theta^2}\right)\end{aligned} \tag{12a}$$

$$\frac{\tau_{\theta r}}{2\mu} = \frac{\partial G}{\partial r} + 2(1 - \nu)\frac{1}{r}\left(\cos\theta \frac{\partial f_1}{\partial \theta} + \sin\theta \frac{\partial f_2}{\partial \theta}\right) + \frac{\partial^2}{\partial r \partial \theta}(\Omega \sin\theta) \tag{12b}$$

$$\frac{\tau_{\theta z}}{2\mu} = \frac{\partial G}{\partial z} + 2(1 - \nu)\frac{1}{r}\frac{\partial f_3}{\partial \theta} + \frac{\partial^2}{\partial z \partial \theta}(\Omega \sin\theta) \tag{12c}$$

In equations (12), μ designates the shear modulus of the solid and the function $G(r, \theta, z)$ is defined by means of the relation

$$G = (1 - 2\nu)(f_2 \cos\theta - f_1 \sin\theta) - \frac{1}{r}\frac{\partial f_0}{\partial r} - \cos\theta \frac{\partial f_1}{\partial \theta} - \sin\theta \frac{\partial f_2}{\partial \theta} - \frac{z}{r}\frac{\partial f_3}{\partial \theta} \tag{13}$$

By virtue of the fact that the induced deformation is skew-symmetric with respect to the variable θ, the quantities u_r, u_z, σ_θ are odd in θ while u_θ, $\tau_{\theta r}$, and $\tau_{\theta z}$ are even in the same variable. This circumstance suggests that the problem can be formulated for the upper half space $y \geqslant 0$ with appropriate boundary conditions prescribed in the regions $\theta = 0$ and $\theta = \pi$. In view of these observations, the continuity of the solid outside the crack region implies

$$u_r(r, 0, z) = 0 \tag{14a}$$

$$u_z(r, 0, z) = 0 \tag{14b}$$

$$\sigma_\theta(r, 0, z) = 0 \tag{14c}$$

On the other hand, as the crack surface is assumed to be free from mechanical loading, the following conditions must be satisfied in the region $\theta = \pi$

$$\sigma_\theta(r, \pi, z) = 0 \tag{15a}$$

$$\tau_{\theta r}(r, \pi, z) = 0 \tag{15b}$$

$$\tau_{\theta z}(r, \pi, z) = 0 \tag{15c}$$

These conditions need to be accompanied by the regularity requirements at infinity, namely, the vanishing of displacements and stresses in that vicinity. Furthermore, near the crack border ($r \to 0$) the displacements must be finite and the stresses are expected to have the usual square root singularity $r^{-1/2}$.

Making use of equations (9*a*), (9*c*), and (12*a*) in conjunction with (10), it is readily confirmed that the boundary conditions given in equations (14) are satisfied by

$$f_0(r, 0, z) = 0 \tag{16a}$$

$$f_1(r, 0, z) = 0 \tag{16b}$$

$$f_3(r, 0, z) = 0 \tag{16c}$$

$$\frac{\partial f_2}{\partial \theta}(r, 0, z) = 0 \tag{16d}$$

With reference to the region $\theta = \pi$, equations (15) when inserted in the appropriate stress expressions yield the following relations:

$$(1-2\nu)\frac{\partial f_1}{\partial r} - \frac{2(1-\nu)}{r}\frac{\partial f_2}{\partial \theta} + 2\nu\frac{\partial f_3}{\partial z} + \frac{\partial^2 f_0}{\partial r^2} + \frac{\partial^2 f_0}{\partial z^2} + \frac{1}{r}\frac{\partial^2 f_1}{\partial \theta^2} + z\left(\frac{\partial^2 f_3}{\partial r^2} + \frac{\partial^2 f_3}{\partial z^2}\right) = 0 \tag{17a}$$

$$\frac{\partial G}{\partial r} - \frac{2(1-\nu)}{r}\frac{\partial f_1}{\partial \theta} = \frac{\partial \Omega}{\partial r} \tag{17b}$$

$$\frac{\partial G}{\partial z} + \frac{2(1-\nu)}{r}\frac{\partial f_3}{\partial \theta} = \frac{\partial \Omega}{\partial z} \tag{17c}$$

in which G can be found from

$$G = -(1-2\nu)f_2 - \frac{1}{r}\frac{\partial f_0}{\partial r} + \frac{\partial f_1}{\partial \theta} - \frac{z}{r}\frac{\partial f_3}{\partial \theta}, \quad \theta = \pi \tag{18}$$

Further simplification may be achieved by setting

$$G = 0, \quad \theta = \pi \tag{19}$$

and as a consequence of (19) it follows that

$$\frac{\partial G}{\partial r} = \frac{\partial G}{\partial z} = 0, \quad \theta = \pi \tag{20}$$

By invoking the identities

$$\left.\begin{aligned} &\frac{\partial}{\partial x} = -\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} = -\frac{1}{r}\frac{\partial}{\partial \theta} \\ &\frac{\partial^2}{\partial y^2} = -\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) \end{aligned}\right\} \quad \theta = \pi \tag{21}$$

in connection with equations (2), the relations (17) are transformed into

$$2(1-\nu)\frac{\partial f_2}{\partial y} - (1-2\nu)\frac{\partial f_1}{\partial x} + 2\nu\frac{\partial f_3}{\partial z} - \frac{\partial^2 f_0}{\partial y^2} - \frac{\partial^2 f_1}{\partial \theta \partial y} - z\frac{\partial^2 f_3}{\partial y^2} = 0, \quad \theta = \pi \tag{22a}$$

$$\frac{\partial f_1}{\partial y} = -\frac{1}{2(1-\nu)}\frac{\partial \Omega}{\partial x}, \quad \theta = \pi \tag{22b}$$

$$\frac{\partial f_3}{\partial y} = -\frac{1}{2(1-\nu)}\frac{\partial \Omega}{\partial z}, \quad \theta = \pi \tag{22c}$$

In a similar manner, equations (18) and (19) render

$$(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} - \frac{\partial f_1}{\partial \theta} - z\frac{\partial f_3}{\partial y} = 0, \quad \theta = \pi \tag{23}$$

The mixed relations (16*b*), (16*c*), (22*b*), and (22*c*) provide the necessary information for the evaluation of f_1 and f_3. In these relations the derivatives of the thermoelastic potential are determined from a knowledge of the temperature in the solid already treated in the previous section. In particular, when a constant heat flux is present at the point $(a, \pi, 0)$ equations (7) and (11) yield

$$\frac{\partial \Omega}{\partial x} + i\frac{\partial \Omega}{\partial z} = \frac{(1+\nu)\alpha Q_0}{2(1-\nu)\pi^2(x+a-iz)}\left[\left(\frac{2a}{x-a-iz}\right)^{1/2} \times \ln\left(\frac{\sqrt{r+x}+\sqrt{x-a-iz}}{\sqrt{r+a+iz}}\right) - \frac{y}{\rho}\tan^{-1}\left(\frac{\sqrt{2a(r-x)}}{\rho}\right)\right] \tag{24}$$

The character of equations (16*b*), (16*c*), (22*b*), (22*c*), and (24) suggests that the functions f_1 and f_3 can be expressed by[2]

$$f_1(r, \theta, z) = -\frac{1}{2(1-\nu)}\left[y\frac{\partial \Omega}{\partial x} - (x+a)\frac{\partial \Omega}{\partial y}\right] + C_1\frac{\sin(\theta/2)}{\sqrt{r}}\,\mathrm{Re}\,[g(\zeta)] \tag{25}$$

$$f_3(r, \theta, z) = -\frac{1}{2(1-\nu)}\left[y\frac{\partial \Omega}{\partial z} - z\frac{\partial \Omega}{\partial y}\right] \tag{26}$$

where C_1 is a constant introduced for convenience and Re designates the real part of an analytic function of the variable ζ defined by

$$\zeta = r + a + iz, \quad i = \sqrt{-1} \tag{27}$$

In equation (25), the term associated with C_1 is a singular solution of Laplace equation which satisfies conditions (16*b*) and (22*b*) and conform to the usual regularity requirement at infinity. It has been introduced to insure finiteness of displacements at the crack border ($r \to 0$).

The next step in the analysis is to derive another set of relations which govern the remaining potentials f_0 and f_2. In order to achieve this, it is expedient to add the term $y(\partial f_3/\partial z)$ (which vanishes in the region $\theta = \pi$) to both sides of equations (23), i.e.,

$$(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} - \frac{\partial f_1}{\partial \theta} + y\frac{\partial f_3}{\partial z} - z\frac{\partial f_3}{\partial y} = 0, \quad \theta = \pi \tag{28}$$

Moreover, utilizing equations (16) and the identities

$$\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \frac{\partial^2}{\partial y^2} = -\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right), \quad \theta = 0 \tag{29}$$

it is not difficult to verify that

$$\frac{\partial}{\partial y}\left[(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} - \frac{\partial f_1}{\partial \theta} + y\frac{\partial f_3}{\partial z} - z\frac{\partial f_3}{\partial y}\right] = 0, \quad \theta = 0 \tag{30}$$

A glance at the quantity inside the bracket in equation (30) reveals that it is harmonic. It follows that equations (28) and (30) suggest the following relation involving the singular solution introduced in (25)

$$(1-2\nu)f_2 - \frac{\partial f_0}{\partial y} - \frac{\partial f_1}{\partial \theta} + y\frac{\partial f_3}{\partial z} - z\frac{\partial f_3}{\partial y} = C_2\frac{\cos(\theta/2)}{\sqrt{r}}\,\mathrm{Re}\,[g(\zeta)] \tag{31}$$

A second relation between f_0 and f_2 may be derived from equations (16), (22*a*), and (29). The mathematical manipulation necessary for obtaining the second relation is identical to that used by Kassir and Sih [6] in analyzing the problem of the considered crack under concentrated shear load parallel to the crack edge. Without going into the details, it can be shown that

$$2(1-\nu)f_2 - \frac{\partial f_0}{\partial y} = \frac{\partial f_1}{\partial \theta} + z\frac{\partial f_3}{\partial y} - y\frac{\partial f_3}{\partial z} + (1-2\nu)\int_\infty^y\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_3}{\partial z}\right)dy, \quad y \geq 0 \tag{32}$$

exists throughout the entire region $y \geq 0$ (see equation (22) of [6]). Solving equations (31) and (32) simultaneously and utilizing equations (25) and (26) result in

$$f_2 = [(1-2\nu)C_1 - C_2]\frac{\cos(\theta/2)}{\sqrt{r}}\,\mathrm{Re}\,[g(\zeta)]$$

[2] The results in equations (25) and (26) can also be reached by applications of Fourier and Kontorovich-Lebedev integral transforms to equations (16*b*), (16*c*), (22*b*), and (22*c*).

$$- (1 - 2\nu)C_1 \int_\infty^y \frac{\sin(\theta/2)}{\sqrt{r}} \mathrm{Re}\,[g'(\zeta)]dy + \frac{(1-2\nu)}{2(1-\nu)}\left[\Omega + (x+a)\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} + z\frac{\partial \Omega}{\partial z}\right] \quad (33a)\ (Cont.)$$

and

$$\frac{\partial f_0}{\partial y} = \left[\left(\frac{1}{2} - 4\nu + 4\nu^2\right)C_1 - 2(1-\nu)C_2\right]\frac{\cos(\theta/2)}{\sqrt{r}}\mathrm{Re}\,[g(\zeta)]$$

$$-(1-2\nu)^2 C_1 \int_\infty^y \frac{\sin(\theta/2)}{\sqrt{r}}\mathrm{Re}\,[g'(\zeta)]dy + \frac{1}{2(1-\nu)}$$

$$\times \left\{\left[(1-2\nu)^2 + y\frac{\partial}{\partial y}\right]\left[\Omega + (x+a)\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} + z\frac{\partial \Omega}{\partial z}\right] + x\left[\frac{\partial \Omega}{\partial x} + y\frac{\partial^2 \Omega}{\partial x \partial y} - (x+a)\frac{\partial^2 \Omega}{\partial y^2}\right] + z\left[\frac{\partial \Omega}{\partial z} + y\frac{\partial^2 \Omega}{\partial y \partial z} - z\frac{\partial^2 \Omega}{\partial y^2}\right]\right\}, \quad (33b)$$

where $g'(\zeta) = (d/d\zeta)g(\zeta)$ and Ω is given in (24).

The remaining conditions to be satisfied are equations (16*a*) and the requirement that the displacements are finite at $r = 0$. These conditions are utilized to determine the function $g(\zeta)$ as well as the constants C_1 and C_2 in equations (25) and (33). However, instead of applying the condition in equation (16*a*), it is easier to subject (33*b*) to the equivalent condition

$$\frac{\partial f_0}{\partial z} = 0, \theta = 0 \quad (34)$$

Differentiating (33*b*) with respect to z and then integrating in the variable y between the limits y and ∞, making use of equation (24) and the results

$$\int_\infty^y \frac{\cos(\theta/2)}{\sqrt{r}} g'(\zeta)dy = \frac{1}{\sqrt{2}}\int_\infty^\zeta \frac{g'(t)dt}{\sqrt{\zeta - \zeta_0}} \quad (35a)$$

$$\int_\infty^y dy \int_\infty^y \frac{\sin(\theta/2)}{\sqrt{r}} g''(\zeta)dy = \frac{1}{2\sqrt{2}}\left\{\int_\infty^\zeta \frac{g'(t)dt}{\sqrt{t - \zeta_0}} - 2y\int_\infty^\zeta g'(t)\frac{d}{dt}(t - \zeta_0 + 2x)^{-1/2}dt\right\} \quad (35b)$$

$$\Omega + (x+a)\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} + z\frac{\partial \Omega}{\partial z} = -\frac{(1+\nu)\alpha Q_0\sqrt{a}}{2\sqrt{2}\,(1-\nu)\pi^2} \times \mathrm{Re}\left\{\frac{1}{\sqrt{2x - \zeta_0}}\ln\left|\frac{\sqrt{r+x} - \sqrt{2x - \zeta_0}}{\sqrt{r+x} + \sqrt{2x-\zeta_0}}\right|\right\} \quad (35c)$$

it is found that condition (34) results in

$$(\nu C_1 + C_2)\int_{\zeta_0}^\infty \frac{g'(t)dt}{\sqrt{t - \zeta_0}} = -\frac{\nu(1+\nu)\alpha Q_0\sqrt{a}}{4(1-\nu)\pi\sqrt{\zeta_0}} \quad (36)$$

where the variable ζ_0 is defined as

$$\zeta_0 = [\zeta]_{\theta=0} = x + a + iz \quad (37)$$

Equation (36) is a standard integral equation of Abel type whose solution is [9]

$$g(\zeta) = \ln \zeta \quad (38)$$

provided that

$$\nu c_1 + C_2 = -\frac{\nu(1+\nu)\alpha Q_0\sqrt{a}}{4(1-\nu)^2\pi^2} \quad (39)$$

The final step in the analysis is to relate the constants C_1 and C_2. This may be done by imposing the regularity condition on the displacements at the crack edge. Inserting the appropriate values of the potentials in equations (9), making use of (24) and (38), expanding asymtotically for small r and retaining the lowest order terms, it is found that the displacements near the crack edge assume the form

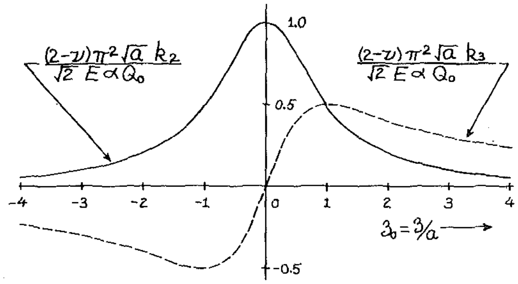

Fig. 2 Variations of k_2 and k_3 along crack border

$$u_r = \frac{\sin\frac{\theta}{2}}{4\sqrt{r}}\ln(a^2 + z^2)[2(1-\nu)C_1 - C_2][3 - 4\nu + (7 - 8\nu)\cos\theta] + 0(r^0) \quad (40a)$$

$$u_\theta = -\frac{\cos\frac{\theta}{2}}{4\sqrt{r}}\ln(a^2+z^2)[2(1-\nu)C_1 - C_2] \times [3 - 4\nu - (5 - 8\nu)\cos\theta] + 0(r^0) \quad (40b)$$

$$u_z = 0(r^0) \quad (40c)$$

Now, the finite displacements requirement at the crack edge gives

$$2(1-\nu)C_1 - C_2 = 0 \quad (41)$$

and it follows from equation (39) that

$$C_1 = -\frac{\nu(1+\nu)\alpha Q_0\sqrt{a}}{4(2-\nu)(1-\nu)^2\pi^2} \quad (42a)$$

$$C_2 = -\frac{\nu(1+\nu)\alpha Q_0\sqrt{a}}{2(2-\nu)(1-\nu)\pi^2} \quad (42b)$$

This, basically, completes the solution of the problem. The induced displacements and stresses in the solid may be readily computed from equations (9)–(12) when use is made of equations (24)–(26), (33), (38), and (42). These quantities serve as the Green's functions for any distribution of heat flux applied to arbitrary regions of the crack surface.

Stress-Intensity Factors

The shear stresses across the surface $\theta = 0$ are computed from equations (12*b*) and (12*c*) as

$$\tau_{\theta r}(r, 0, z) = \frac{E\alpha Q_0\sqrt{a}\,(x+a)}{(2-\nu)\pi^2\sqrt{x}\,[(x+a)^2 + z^2]} + \ldots\ldots \quad (43a)$$

$$\tau_{\theta z}(r, 0, z) = \frac{E\alpha Q_0\sqrt{a}z}{(2-\nu)\pi^2\sqrt{x}\,[(x+a)^2 + z^2]} + \ldots\ldots \quad (43b)$$

where E is Young's modulus of the material, $E = 2\mu(1+\nu)$, and the nonsingular terms have been neglected. Equations (43) may be expressed in the standard form

$$\tau_{\theta r}(r, 0, z) = \frac{k_2}{\sqrt{2x}} + 0(x^0) \quad (44a)$$

$$\tau_{\theta z}(r, 0, z) = \frac{k_3}{\sqrt{2x}} + 0(x^0) \quad (44b)$$

in which the stress-intensity factors, k_2 and k_3, associated with the edge-sliding and tearing modes of crack extension, respectively, are given in terms of the nondimensional parameter $z_0 = z/a$ by way of the relations

$$k_2 = \frac{\sqrt{2}\,E\alpha Q_0}{(2-\nu)\pi^2\sqrt{a}}\frac{1}{1 + {z_0}^2} \quad (45a)$$

$$k_3 = \frac{\sqrt{2}\,E\alpha Q_0}{(2-\nu)\pi^2\sqrt{a}}\frac{z_0}{1 + {z_0}^2} \quad (45b)$$

Fig. 2 shows the variation of equations (45) with z_0.

The formulas in equations (45) can be used to generate results for any distribution of heat flux applied to arbitrary regions of the crack surface. As an example, suppose that the heat flux is applied to a rectangular region, $-a \leq x \leq 0$, $y = 0$, $|z| \leq b$, of the crack surface, Fig. 1, then double integrations performed on equations (45) yield the stress-intensity factors

$$k_2 = \frac{2\sqrt{2}\,E\alpha Q_0}{3(2-\nu)\pi^2}\left[(|z|+b)^{3/2}H_1\left(\frac{a}{|z|+b}\right) - \operatorname{sgn}(|z|-b)||z|-b|^{3/2}H_1\left(\frac{a}{||z|-b|}\right)\right] \quad (46a)$$

$$k_3 = \frac{\sqrt{2}\,E\alpha Q_0}{3(2-\nu)\pi^2}\left[a^{3/2}\ln\langle a^2+(|z|+b)^2\rangle - (|z|+b)^{3/2}H_2\left(\frac{a}{|z|+b}\right) - \operatorname{sgn}(|z|-b)\left\langle a^{3/2}\ln[a^2+||z|-b|^2] - ||z|-b|^{3/2}H_2\left(\frac{a}{||z|-b|}\right)\right\rangle\right] \quad (46b)$$

in which the functions H_1 and H_2 are given by

$$H_1(s) = s^{3/2}\tan^{-1}\frac{1}{s} + 2s^{1/2} - \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{\sqrt{2s}}{1-s}\right) + \ln\left(\frac{1+\sqrt{2s}+s}{\sqrt{1+s^2}}\right)\right] \quad (47a)$$

$$H_2(s) = 2s^{3/2} - \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2s}}{1-s}\right) + \frac{3}{\sqrt{2}}\ln\left(\frac{1+\sqrt{2s}+s}{\sqrt{1+s^2}}\right) \quad (47b)$$

and the signum function, sgn (z), is $+1$, 0, or -1 depending on whether the argument, z, is positive, zero, or negative, respectively. Other solutions can be generated in a similar manner.

Conclusion

The linear thermoelastic problem of a uniform heat flow disturbed by an insulated semi-infinite plane crack embedded in a three-dimensional elastic solid has been formulated and solved. The method of analysis involves the application of the technique of Fourier-Kontorovich-Lebedev Integral transforms and certain singular solutions of Laplace equation. The Green's functions for the distribution of the temperature field as well as the induced displacements and stresses in the solid are derived mainly in closed form. These results are useful in examining theories of brittle fracture for crack propagation caused by passage of steady-state heat in solids.

References

1 Florence, A. L., and Goodier, J. N., "The Linear Thermoelastic Problem of Uniform Heat Flow Disturbed by a Penny-Shaped Insulated Crack," *International Journal of Engineering Science*, Vol. 1, 1963, pp. 533–540.

2 Kassir, M. K., and Sih, G. C., "Three-Dimensional Thermoelastic Problems of Planes of Discontinuities or Cracks in Solids," *Developments in Theoretical and Applied Mechanics*, ed., Shaw, W. A., Vol. 3, 1967, Pergamon Press, New York, p. 117.

3 Irwin, G. R., "Fracture Mechanics," *Structural Mechanics*, ed., Goodier, J. N., and Hoff, N. J., 1960, Pergamon Press, New York, p. 557.

4 Kassir, M. K., and Sih, G. C., *Three-Dimensional Crack Problems*, Noordhoff International Publishing, Leydon, The Netherlands, 1975.

5 Uflyand, Ya. S., *Survey of Articles on the Applications of Integral Transforms in the Theory of Elasticity*, North Carolina State University, 1965.

6 Kassir, M. K., and Sih, G. C., "Application of Papkovich-Neuber Potentials to a Crack Problem," *International Journal of Solids and Structures*, Vol. 9, 1973, pp. 643–654.

7 Lebedev, N. N., *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

8 Nowacki, W., *Thermoelasticity*, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962.

9 Whittaker, E. I., and Watson, G. N., *A Course of Modern Analysis*, 4th ed., Cambridge University Press, 1962.