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Transient Dynamic Analysis of Rotors Using the Combined Methodologies of Finite Elements and Transfer Matrix

A new approach is proposed to predict the dynamic behavior of rotor-bearing systems in time domain using the combined methodologies of finite elements and transfer matrices. This approach makes use of the finite element method to model symmetric shafts and then transforms the system properties to transfer matrix mode. The formulation provides flexibility to include both linear and nonlinear system models, often encountered in rotor dynamic applications. Few example rotor cases had been studied and the results were compared with those obtained using finite element method. This establishes that considerable savings in computational effort can be achieved without losing any accuracy.

Introduction

Transfer matrix approach had been adopted by many researchers in the past (Lund, 1974; Rao, 1983) to solve rotor-dynamic problems in the frequency domain. While this approach is adequate to study the steady state behavior of the rotor, sometimes, it is important to obtain the information concerning the instantaneous behavior of the subsystems, especially for those regions close to the rotor instability condition. Under this situation, a time domain approach will be most appropriate to obtain the system informations precisely. Towards this, a combined time domain finite element transfer matrix method, namely Transient Property Transfer Approach (TPTA), is introduced here.

At the current state of rotor dynamic technology, the finite element method (FEM) is the only validated tool available for nonlinear time domain analysis (Ruhl et al., 1972; Nelson et al., 1976). For large rotor systems, however, the use of finite element method leads to prohibitively higher computation time and costs. In order to minimize this, a continuing effort is made by various researchers to limit the dynamic degrees of freedom without reducing the accuracy of the results. Of these, the combined finite element transfer matrix technique introduced by Dokainish (1972) is of importance since it results in an exact condensation of matrix size without any loss of accuracy. In recent years, other researchers (Chiatti et al., 1979; Mucino et al., 1981; Ohga et al., 1983; Degan et al., 1985) have improved and generalized this methodology for other applications. Inspite of this, because of the nature of transfer matrix relations used in the formulation, the use of this combined methodology is still limited to linear frequency domain analysis only. Moreover, a rotor system supported on hydrodynamic bearings is nonconservative, nonsymmetric, and possibly nonlinear due to the asymmetric cross coupled stiffness and damping properties of the oil film bearings. Thus, the Dokainish's combined finite element transfer matrix methodology, as it is, cannot be directly used for rotor-dynamic applications. Under these circumstances, it will be more appropriate to develop a combined finite element transfer matrix method.

In this respect the Discrete Time Transfer Matrix Method (DT-TMM), introduced recently by Kumar et al. (1986) for general structural dynamics response calculations, is of importance. They showed that with the use of appropriate time marching numerical integration algorithms, the application of transfer matrix methods can be extended to time domain and nonlinear analysis of dynamic systems. Following this, Subbiah et al. (1987) applied this method for transient nonlinear rotor dynamic applications and found that even for a simple rotor system the response results obtained using finite element method and discrete time transfer matrix method showed little discrepancy. The reason for such discrepancy can be attributed to the fact that the transfer matrix model for a shaft section is formulated using cantilever beam theory whereas the finite element model is developed with consistent property of the system (Archer, 1963). This observation led to the development of the present work wherein, following the strategy used by Dokainish, the finite element methodology and discrete time transfer matrix methodology are combined. This formulation provides flexibility to include both linear and nonlinear system models, often encountered in rotor-dynamic applications.

In the following sections the formulation of the proposed

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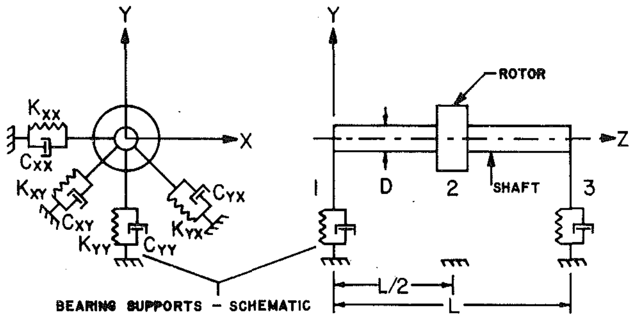

Fig. 1 Model rotor system

combined methodology is first presented. A comparison of the response results obtained using the proposed method with those obtained using finite element method is also carried out to establish the validity of the method.

Theory

Finite Element Formulation. Consider the three station rotor model which is discretized into shaft, disk and bearing elements as shown in Fig. 1. Based on this finite element model the matrix equations of motion for the element 1 with nodes 1 and 2 can be written as (Nelson et al., 1976):

$$[M^e]\{\ddot{q}(t)\}+[C^e]\{\dot{q}(t)\}+[K^e]\{q(t)\}=\{F(t)\} \quad (1)$$

where M^e, C^e and K^e are the element mass, damping and stiffness matrices (8×8), respectively, and are defined as,

$$M^e=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad C^e=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad K^e=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

and the generlized force and displacement vectors (8×1) are given by

$$\{F(t)\}=\{f_1|f_2\}^T,$$

and

$$\{q(t)\}=\{x_1,\theta_1,y_1,\phi_1|x_2,\theta_2,y_2,\phi_2\}^T.$$

Similarly, the matrix equations of motion for all the elements in the model, including the point elements at unbalance locations, bearing locations, etc., can be derived. In the conventional finite element method these component element equations of motions are assembled into a large global matrix equation of motion which is then solved using any suitable time marching integration scheme.

Discrete Time Transfer Matrix Method Formulation. The DT-TMM (Kumar et al., 1986) is based on the assumption that at any given time instant, say, t_i, the acceleration and velocity in any given degree of freedom, say, q_n of subsystem n, can be expressed as a linear function of the displacement q_n with reasonable accuracy. That is,

$$\ddot{q}_n(t_i)=A_n(t_i)q_n(t_i)+B_n(t_i) \quad (2)$$

$$\dot{q}_n(t_i)=D_n(t_i)q_n(t_i)+E_n(t_i) \quad (3)$$

Derivation of these types of relationships can generally be carried out based on truncated Taylor series as given below.

The starting point for most of the numerical integration schemes used in structural response analysis is the truncated Taylor series of order 3. That is,

$$q_n(t_i)=q_n(t_{i-1})+\Delta T\dot{q}_n(t_{i-1}) + \frac{\Delta T^2}{2}\ddot{q}_n(t_{i-1})+\frac{\Delta T^3}{6}\dddot{q}_n(t_{i-1}) \quad (4)$$

where interval $\Delta T = (t_i - t_{i-1})$. Different integration schemes with varying sophistication and accuracy are then derived by replacing the derivatives in equation (4) by finite differences. Here, one of the simplest finite difference schemes available is chosen to explain the methodology. In addition, it is assumed that the acceleration is constant during the time interval $(t_i - t_{i-1})$ and is equal to the average of the acceleration values at t_i and t_{i-1}. Thus,

$$\ddot{q}_n=a=\frac{\ddot{q}_n(t_i)+\ddot{q}_n(t_{i-1})}{2} \quad (5)$$

$$\dddot{q}_n=0 \quad (6)$$

$$\dot{q}_n=a\Delta T=\frac{\ddot{q}_n(t_i)+\ddot{q}_n(t_{i-1})}{2}\Delta T \quad (7)$$

Substitution of these relationships into equation (4) results in,

$$q_n(t_i)=q_n(t_{i-1})+\Delta T\dot{q}_n(t_{i-1})+\Delta T^2\frac{\ddot{q}_n(t_i)+\ddot{q}_n(t_{i-1})}{4} \quad (8)$$

which can be rewritten in the form equation (2), as,

$$\ddot{q}_n(t_i)=A_n(t_i)q_n(t_i)+B_n(t_i) \quad (9)$$

where

$$A_n(t_i)=\frac{4}{\Delta T^2} \quad (10)$$

and

$$B_n(t_i)=-A_n(t_i)\left[q_n(t_{i-1})+\Delta T\dot{q}_n(t_{i-1})+\frac{\Delta T^2}{4}\ddot{q}_n(t_{i-1})\right] \quad (11)$$

Similarly, by substituting equation (9) into equation (7), it can be rewritten in the form of equation (3), as,

$$\dot{q}_n(t_i)=D_n(t_i)q_n(t_i)+E_n(t_i) \quad (12)$$

where

$$D_n(t_i)=\frac{2}{\Delta T} \quad (13)$$

and

$$E_n(t_i)=-D_n(t_i)\left[q_n(t_{i-1})+\frac{\Delta T}{2}\dot{q}_n(t_{i-1})\right] \quad (14)$$

From equations (10), (11), (13), and (14), it can be seen that the coefficients $A_n(t_i)$, $B_n(t_i)$, $D_n(t_i)$, and $E_n(t_i)$ are all functions of the system properties at time t_i, and the response quantities $q_n(t_{i-1})$, $\dot{q}_n(t_{i-1})$, and $\ddot{q}_n(t_{i-1})$ at the previous time instant which are all known at time instant t_i. Thus, the coefficients $A_n(t_i)$, $B_n(t_i)$, $D_n(t_i)$, and $E_n(t_i)$ are all definable for any subsystem n for the time interval $(t_i - t_{i-1})$. It should be noted that the simple finite difference scheme and the constant average acceleration assumption used here in the formulation are only for explanatory purposes. Instead, any of the many more accurate and commonly available numerical integration procedures can be used.

The coefficients A_n, B_n, D_n, and E_n for various commonly used integrating procedures are tabulated by Kumar et al., (1986). They developed and employed this technique to replace the time derivative quantities in the subsystem equations of motion. The resulting relationships were then rewritten as transfer relations leading to discrete time transfer matrix of each subsytem. Then, following the conventional transfer matrix solution procedure, they computed the overall transfer matrix and solved for the unknown state vector quantities by applying the proper boundary conditions. For details refer to Kumar et al., (1986). This novel approach led to the development of a useful transient analysis tool for large dynamic systems. However, subsequent application of this technique to rotor dynamic analysis (Subbiah et al., 1987) showed that the lumped parameter characteristics and the massless cantilever beam theory used in this formulation led to some errors in response calculations. In order to eliminate this source of error the following methodology is proposed.

Transient Property Transfer Approach (TPTA) Formulation. The formulation of TPTA, as given below, combines the finite element formulation and the discrete time transfer matrix formulation. Consider the element matrix equation of motion given by the equation (1) at time $t = t_i$.

$$[M^e]\{\ddot{q}(t_i)\} + [C^e]\{\dot{q}(t_i)\} + [K^e]\{q(t_i)\} = \{F(t_i)\} \quad (15)$$

Substitution of the equations (2) and (3) into equation (15) will result in,

$$[M^e]\{A_n(t_i)q_n(t_i) + B_n(t_i)\} + [C^e]\{D_n(t_i)q_n(t_i) + E_n(t_i)\} + [K^e]\{q(t_i)\} = \{F(t_i)\} \quad (16)$$

or

$$[[M^e]A_n(t_i) + [C^e]D_n(t_i) + [K^e]]\{q_n(t_i)\} + [M^e]\{B_n(t_i)\} + [C^e]\{E_n(t_i)\} = \{F(t_i)\} \quad (17)$$

Expansion of this equation in terms of submatrices gives,

$$\begin{bmatrix} K_{11} & K_{12} & v_1 \\ K_{21} & K_{22} & v_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ 1 \end{Bmatrix} \quad (18)$$

where,

$$K_{11} = m_{11}A_n + c_{11}D_n + k_{11}$$
$$K_{12} = m_{12}A_n + c_{12}D_n + k_{12}$$
$$K_{21} = m_{21}A_n + c_{21}D_n + k_{21}$$
$$K_{22} = m_{22}A_n + c_{22}D_n + k_{22}$$
$$v_1 = m_{11}B_1 + m_{12}B_2 + c_{11}E_1 + c_{12}E_2$$
$$v_2 = m_{21}B_1 + m_{22}B_2 + c_{21}E_1 + c_{22}E_2.$$

Equation (18) can then be rewritten in terms of left and right nodal displacements and forces such that,

$$\begin{Bmatrix} q_2 \\ f_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -K_{12}^{-1}K_{11} & K_{12}^{-1} & s_1 \\ K_{21} - K_{22}K_{12}^{-1}K_{11} & K_{22}K_{12}^{-1} & s_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ f_1 \\ 1 \end{Bmatrix} \quad (19)$$

where

$$s_1 = -K_{12}^{-1}(m_{11}B_1 + m_{12}B_2 + c_{11}E_1 + c_{12}E_2)$$
$$s_2 = -K_{22}K_{12}^{-1}(m_{11}B_1 + m_{12}B_2 + c_{11}E_1 + c_{12}E_2) + m_{21}B_1 + m_{22}B_2 + c_{21}E_1 + c_{22}E_2$$

or

$$\{u\}_2^L = [S]\{u\}_1^R \quad (20)$$

Thus, for an uniform shaft element with nodes 1 and 2, the matrix $[S]$ in equation (20) gives the transfer matrix and vector $\{u\}$ gives the state vector. Similarly, starting from point element equations of motion, the following point element transfer matrix relation can be derived, such that,

$$\{u\}_1^R = [P]\{u\}_1^L. \quad (21)$$

The transfer matrix $[T]$ for a general rotor-bearing finite element can then be formulated by combining equations (20) and (21). That is,

$$\{u\}_2^L = [T]\{u\}_1^L \quad (22)$$

where

$$[T] = [S][P].$$

Repetitive application of the transfer matrix relation (22) results in the following transfer relation for the entire rotor-bearing system:

$$\{u\}_n^L = [T]_n[T]_{n-1} \ldots\ldots [T]_2[T]_1\{u\}_1^L \quad (23)$$

With the application of proper boundary conditions and the initial conditions corresponding to time $t = t_{i-1}$, the unknown generalized displacements and forces at the left end of the rotor can be solved for. Knowing this, at time $t = t_i$, the displacements at all the subsequent nodes can be obtained with the use of the equation (23). The nodal displacements at subsequent time instants (i.e., $t = t_{i+1}$ onwards) can be solved for by repetitively following the same steps.

In the foregoing formulation, the model has been obtained by the transfer system properties and hence this approach is named as transient property transfer approach (TPTA). Equations (18) and (19) correspond to Dokainish's (1972) frequency domain finite element transfer matrix formulation.

It should be noted that , as with the discrete time transfer matrix formulation, the TPTA can be used with any suitable time marching integration scheme which can be rewritten in the form of equations (2) and (3). In this study, the well known Houbolt's algorithm is used. This selection is based on the detailed sensitivity tests carried out by Subbiah et al., (1987). They tested different time marching algorithms (Bathe et al., 1976; Kumar et al., 1986) with different rotor configurations and concluded that for rotor dynamic applications the Houbolt's algorithm provides required high stability and rapid convergence even with a time step of ΔT = period/30.

Table 1 Details of the rotor-bearing system shown in Fig. 1

Type of bearings	Plain cylindrical
Bearing diameter (m)	0.0254
Bearing L/D ratio	1.0
Viscosity of oil at 25.5 degree C (N.s/m^2)	0.024
Disk mass (kg)	11.82
Disk diameter (m)	0.2032
Disk eccentricity (m)	0.001
Shaft diameter (m)	0.022
Total length of rotor (m)	0.5105
Modulus of elasticity for the shaft material (N/m^2)	2.145×10^{11}

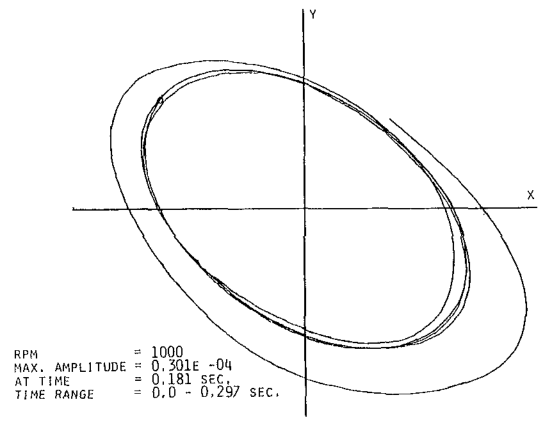

Fig. 2 Orbital response of rotor at bearing location (TPTA)

Results and Discussion

The TPTA has been used to study the dynamic response of a single rotor system supported on fluid film bearing which is shown in Fig. 1. The details of the rotor are given in Table 1. The transient orbital response obtained by the present investigation has been compared with those obtained using finite element method. In all the test cases considered, the number of elements, the type of elements, and the time step used are all kept the same for both the TPTA and the finite element approach. And as mentioned earlier, Houbolt's method is employed here as the integrating procedure. Coincidentally, the finite element code used, namely ANSYS, also employs Houbolt's method for integration. The initial conditions required to start the analysis are obtained using $A_n = -\omega^2$, and $B_n = D_n = E_n = 0$. This helps to reach steady state condi-

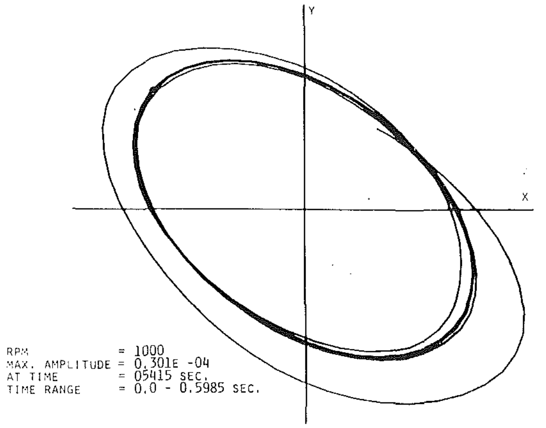

Fig. 3 Orbital response of rotor at bearing location (FEM)

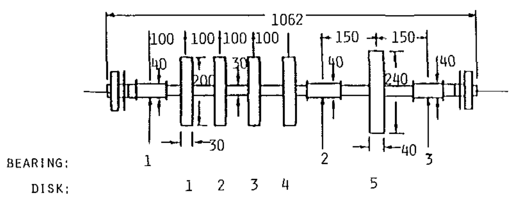

Fig. 4 Five-disk and three-bearing rotor (Kikuchi, 1970)

tions fast, resulting in considerable savings in computation time.

The orbital plots obtained for the rotor, using the present method and the finite element method, are shown in Figs. 2 and 3 for a rotor speed of 1000 rpm. The steady state of the rotor has been obtained within 5 cycles in both the cases. The discrepancy between these two results is almost negligible. In order to demonstrate the applicability of this approach to multi-span rotor-bearing system analysis, the multi rotor model of Kikuchi (1970) has been studied using the present approach. This rotor system consists of five disks and three bearings as shown in Fig. 4. The results obtained for this rotor using TPTA and finite element approach shows negligible discrepancy. Hence, only one such orbital plot obtained by TPTA is shown in Fig. 5. Finally, the nonlinear bearing model developed by Hashish et al. (1982a, 1982b), has been adapted to study dynamic response of the single rotor system shown in Fig. 1 and the nonlinear orbital response is shown in Fig. 6. For small amplitude motions, the total orbital amplitude in Fig. 6 closely corresponds to the linear result as shown already in Fig. 2.

The computational efficiency of TPTA is not appreciable in comparison with finite element approach when smaller rotor systems were studied. However, in the case of larger systems, the TPTA will lead to smaller computational effort. For example, in the case of Kikuchi's rotor, the computational time on a PRIME 950 comptuer was worked out to be 5.37 CPU minutes for TPTA and 7.35 CPU minutes for FEM.

Conclusions

1 A transient property transfer approach has been developed to study the linear and nonlinear dynamic behavior of complex rotor-bearing systems in both space and time using the combined methodologies of FEM, transfer matrix method and time marching numerical integration techniques.

2 The method presented in this investigation is capable of accommodating other subsystems such as pedestals, foundations, and mechanical couplings, etc.

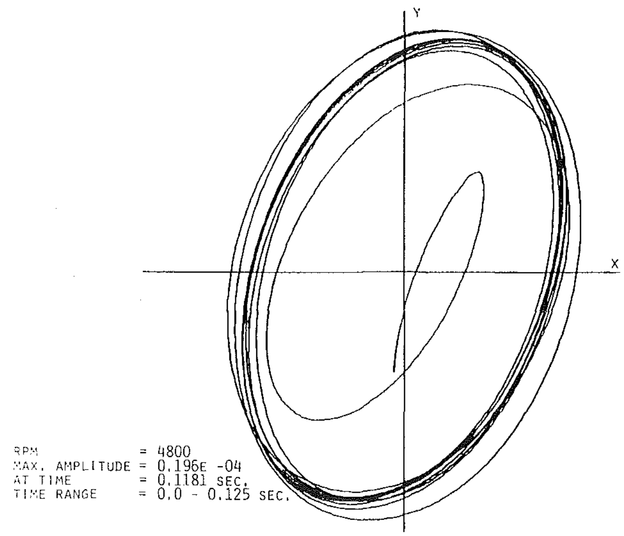

Fig. 5 Orbital response of Kikuchi rotor at disk location #4 (TPTA)

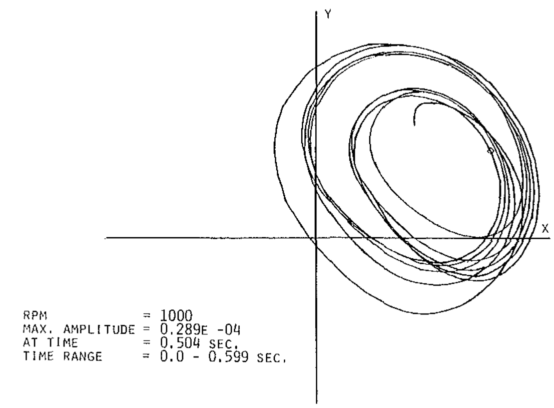

Fig. 6 Nonlinear orbital response of rotor at bearing location (TPTA)

3 The gyroscopic effects can be easily included in the model. System nonlinearities due to asymmetric shafts, nonlinear bearings and random pedestal motions can be studied by including their effects appropriately in the system model.

4 The method not only provides an excellent correlation with the finite element method, but also handles small and manageable forms of matrices and, hence, the computer effort is reduced to the largest extent possible.

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