An Exact Three-Dimensional Solution for Simply Supported Rectangular Piezoelectric Plates

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An exact three-dimensional solution for the problem of a simply supported rectangular homogeneous piezoelectric plate is obtained, in the framework of the linear theory of piezoelectricity. The plate is made of a transversely isotropic material, is earthed on the lateral boundary, and is subjected to prescribed surface charge and tractions on the end faces. The limit of this solution as the plate thickness aspect ratio approaches zero is explicitly carried out. The analytical results obtained may constitute a reference case when developing or applying two-dimensional plate theories for the analysis of more complex piezoelectric problems. A numerical investigation in the case of a square uniformly loaded plate is also performed, in order to evaluate the influence of the thickness-to-side ratio on the three-dimensional solution of the plate problem.

1 Introduction

At the beginning of the present century, piezoelectricity was applied to sonars, to frequency control, to acoustic interferometry (Mason, 1981). Presently, applications of piezoelectric materials are mainly oriented towards detecting deformations and motions of structures, as well as in active structural control. As a matter of fact, by bonding or embedding a piezoelectric element in a structure, it is possible both to measure strains and displacements, and to give localized strains through which the deformation of the structure can be controlled. In this way a so-called "smart" structure is obtained. An effective sensing and control of a smart structure can be achieved only through a careful modelization of the coupled electroelastic behavior of the structure. The literature on this topic is wide: extensive references on structural piezoelectric sensing and control can be found in a book by Tzou (1993).

There is a special interest in the structural problem of piezoelectric plates, since it arises in modeling flat piezoelectric sensors and actuators. In the analysis of this problem approximate theories are often adopted: indeed, simplifying a priori hypotheses are assumed, concerning the direction of the electric field, and the representation form of the displacement field. As a consequence, the interest for exact solutions (i.e., solutions obtained from the full three-dimensional theory) arises, for the purpose of verifying the accuracy of the results provided by approximate theories or computations.

In this paper the framework of the linear theory of piezoelectricity is adopted, and an exact three-dimensional solution for the problem of a simply supported rectangular homogeneous piezoelectric plate is obtained. The plate is made of a transversely isotropic material, is earthed on the lateral boundary, and is subjected to prescribed surface charge and tractions on the end faces. The given loads are represented in double-series trigonometric form. The expressions of all the mechanical and electric involved functions are explicitly found.

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The solving technique adopted here is based on the separation of the independent variables. It was applied to study plates regarded as three-dimensional bodies by some researchers. In particular, Vlasov (1957) and Levinson (1984) studied the problem of isotropic homogeneous elastic plates; Pan (1991) considered the problem of transversely isotropic homogeneous elastic plates; Lee (1967), Srinivas and Rao (1970), Pagano (1970), and Noor and Scott Burton (1990) studied the problem of orthotropic or anisotropic composite elastic plates. An application of the same technique to the problem of piezoelectric plates is due to Ray et al. (1992), which considered only the simplest case of cylindrical bending. In this paper the more general case of bidirectional bending is studied.

The paper is organized as follows. Section 2 is devoted to notation. The linear piezoelectric constitutive equations and the basic equations of the linear theory of piezoelectricty are recalled in Sections 3 and 4, respectively. The plate problem is introduced in Section 5 and a separate-variables three-dimensional solution is obtained in Section 6. This solution is then expanded into a power series of the thickness of the plate, and the lowest-order term of the expansion is explicitly carried out (Section 7). As a consequence, the limit of the three-dimensional solution of the plate problem as the plate thickness aspect ratio approaches zero is obtained, in a closed form. This analytical result is intended to serve as a reference case when developing or applying approximate theories for the analysis of more complex piezoelectric problems. Moreover, it may be useful in the development of a consistent theory of thin piezoelectric plates, since it supplies the orders of infinity of the mechanical and electric quantities as the plate thickness aspect ratio approaches zero. At authors' knowledge, an explicit calculation, for any load condition, of the limit of all the mechanical and electric quantities as the plate thickness aspect ratio approaches zero is carried out here for the first time. Levinson (1984) and Pan (1991) considered only the limit of the deflection of an elastic plate for one particular load condition.

Finally, a numerical investigation in the case of a square, uniformly loaded plate is performed in Section 8, in order to evaluate the influence of the thickness-to-side ratio on the three-dimensional solution of the plate problem.

2 Notation

Let $\mathcal{E}$ be the three-dimensional Euclidean space, and let $\mathcal{V}$ be the vector space associated to $\mathcal{E}$. Let $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be an orthonormal basis of $\mathcal{V}$ and let a_i ($i = 1 \dots 3$) denote the components of a vector $\mathbf{a} \in \mathcal{V}$ with respect to this basis. Let $O \in \mathcal{E}$ be a fixed origin and let (x_1, x_2, x_3) denote the coordinates of the typical point of $\mathcal{E}$ in the cartesian frame $(O, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

Let *Lin* be the vector space of the second-order tensors on $\mathcal{V}$, $Sym \subset Lin$ the subspace of the symmetric tensors, and $Orth \subset Lin$ the group of the isometries of $\mathcal{V}$. The identity of $\mathcal{V}$ is denoted by $\mathbf{I}$; the symmetric part of $\mathbf{A} \in Lin$ is denoted by sym $\mathbf{A}$. *Lin* is endowed with the usual inner product defined by the trace operator. The symbol "$\cdot$" denotes the scalar product both between vectors and between tensors. The symbol "$\otimes$" denotes the tensor product between vectors, between tensors, and between a vector and a tensor.

An orthogonal (not orthonormal) basis of *Sym* is $\{\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5, \mathbf{H}_6\}$, where

$$\mathbf{H}_1 := \mathbf{c}_1 \otimes \mathbf{c}_1, \quad \mathbf{H}_2 := \mathbf{c}_2 \otimes \mathbf{c}_2, \quad \mathbf{H}_3 := \mathbf{c}_3 \otimes \mathbf{c}_3,$$
$$\mathbf{H}_4 := \text{sym } \mathbf{c}_2 \otimes \mathbf{c}_3, \quad \mathbf{H}_5 := \text{sym } \mathbf{c}_3 \otimes \mathbf{c}_1,$$
$$\mathbf{H}_6 := \text{sym } \mathbf{c}_1 \otimes \mathbf{c}_2. \tag{1}$$

The components $\mathbf{A} \cdot \mathbf{H}_i$ of a tensor $\mathbf{A} \in Sym$ with respect to this basis are denoted by A_i $(i = 1 \ldots 6)$.

The symbols "∇" and "div" represent the gradient and the divergence operator, respectively.

3 Linear Piezoelectric Constitutive Equations

The framework of the infinitesimal strain theory is adopted. Let the strain tensor be $\mathbf{E} \in Sym$, the stress tensor be $\mathbf{T} \in Sym$, the electric field vector be $\mathbf{e} \in \mathcal{V}$ and the electric displacement vector be $\mathbf{d} \in \mathcal{V}$. The enthalpy $\mathcal{H}$ per unit volume of a piezoelectric material (Parton, 1984) can be expressed as a function of the strain $\mathbf{E}$ and of the electric field $\mathbf{e}$:

$$\mathcal{H} = \mathcal{H}(\mathbf{E}, \mathbf{e}). \tag{2}$$

The stress $\mathbf{T}$ and the electric displacement $\mathbf{d}$ are the dual quantities of $\mathbf{E}$ and $\mathbf{e}$, respectively:

$$\mathbf{T} = \frac{\partial \mathcal{H}}{\partial \mathbf{E}}, \quad \mathbf{d} = -\frac{\partial \mathcal{H}}{\partial \mathbf{e}}. \tag{3}$$

By definition, for a *linear* piezoelectric material the enthalpy $\mathcal{H}$ is a quadratic homogeneous function of $\mathbf{E}$ and $\mathbf{e}$: hence, it takes the form

$$\mathcal{H}(\mathbf{E}, \mathbf{e}) = \tfrac{1}{2}\mathcal{C}^{e}[\mathbf{E}] \cdot \mathbf{E} - L[\mathbf{E}] \cdot \mathbf{e} - \tfrac{1}{2}\epsilon^{\mathbf{E}}[\mathbf{e}] \cdot \mathbf{e}, \tag{4}$$

where $\mathcal{C}^{e}$ is a fourth-order symmetric tensor (elasticity tensor, evaluated at constant electric field), $\epsilon^{\mathbf{E}}$ is a second-order symmetric tensor (dielectric tensor, evaluated at constant strain) and L is a third-order tensor (piezoelectric tensor). As a consequence, the constitutive equations of a linear piezoelectric material can be formulated as

$$\begin{cases} \mathbf{T}(\mathbf{E}, \mathbf{e}) = \mathcal{C}^{e}[\mathbf{E}] - L^{t}[\mathbf{e}] \\ \mathbf{d}(\mathbf{E}, \mathbf{e}) = L[\mathbf{E}] + \epsilon^{\mathbf{E}}[\mathbf{e}], \end{cases} \tag{5}$$

where L^{t} denotes the transpose of L, defined by

$$\mathbf{A} \cdot L^{t}[\mathbf{b}] = L[\mathbf{A}] \cdot \mathbf{b}, \quad \mathbf{A} \in Lin, \quad \mathbf{b} \in \mathcal{V}. \tag{6}$$

The tensors $\mathcal{C}^{e}$, L, and $\epsilon^{\mathbf{E}}$ depend, respectively, upon 21, 18, and 6 material constants (which are called elastic, piezoelectric, and dielectric constants, respectively). The number of a priori different material constants is reduced by taking into account the symmetries of the material. The symmetry group of a piezoelectric material is defined as (Landau and Lifshitz, 1981):

$$\mathcal{G} := \{\mathbf{Q} \in Orth\colon \mathbf{Q}\mathbf{T}(\mathbf{E}, \mathbf{e})\mathbf{Q}^{t} = \mathbf{T}(\mathbf{Q}\mathbf{E}\mathbf{Q}^{t}, \mathbf{Q}\mathbf{e}); \quad \mathbf{Q}\mathbf{d}(\mathbf{E}, \mathbf{e}) = \mathbf{d}(\mathbf{Q}\mathbf{E}\mathbf{Q}^{t}, \mathbf{Q}\mathbf{e})\}. \tag{7}$$

In this paper, transversely isotropic piezoelectric materials (class ∞ mm), having the transverse isotropy axis parallel to $\mathbf{c}_3$, are considered: their symmetry group contains the rotations around $\mathbf{c}_3$ and the reflections with respect to planes parallel to $\mathbf{c}_3$ (Ikeda, 1990). The representation formulas of $\mathcal{C}^{e}$, L, and $\epsilon^{\mathbf{E}}$ for these materials are (Varadan et al., 1987)

$$\begin{cases} \mathcal{C}^{e} = C^{e}_{11}(\mathbf{H}_1 \otimes \mathbf{H}_1 + \mathbf{H}_2 \otimes \mathbf{H}_2) \\ \quad + C^{e}_{12}(\mathbf{H}_1 \otimes \mathbf{H}_2 + \mathbf{H}_2 \otimes \mathbf{H}_1) + C^{e}_{13}(\mathbf{H}_1 \otimes \mathbf{H}_3 \\ \quad + \mathbf{H}_3 \otimes \mathbf{H}_1 + \mathbf{H}_2 \otimes \mathbf{H}_3 + \mathbf{H}_3 \otimes \mathbf{H}_2) \\ \quad + C^{e}_{33}\mathbf{H}_3 \otimes \mathbf{H}_3 + 4C^{e}_{44}(\mathbf{H}_4 \otimes \mathbf{H}_4 + \mathbf{H}_5 \otimes \mathbf{H}_5) \\ \quad + 4C^{e}_{66}\mathbf{H}_6 \otimes \mathbf{H}_6 \\ L = L_{31}(\mathbf{c}_3 \otimes \mathbf{H}_1 + \mathbf{c}_3 \otimes \mathbf{H}_2) + L_{33}\mathbf{c}_3 \otimes \mathbf{H}_3 \\ \quad + 2L_{15}(\mathbf{c}_1 \otimes \mathbf{H}_5 + \mathbf{c}_2 \otimes \mathbf{H}_4) \\ \epsilon^{\mathbf{E}} = \epsilon^{\mathbf{E}}_{11}(\mathbf{H}_1 + \mathbf{H}_2) + \epsilon^{\mathbf{E}}_{33}\mathbf{H}_3, \end{cases} \tag{8}$$

where $C^{e}_{66} := (C^{e}_{11} - C^{e}_{12})/2$ and C^{e}_{11}, C^{e}_{12}, C^{e}_{13}, C^{e}_{33}, C^{e}_{44}, L_{31}, L_{33}, L_{15}, $\epsilon^{\mathbf{E}}_{11}$, $\epsilon^{\mathbf{E}}_{33}$ are ten independent material constants.

The representation formulas (8) hold also for hexagonal piezoelectric materials belonging to the class C_{6V} (or 6 mm) with the sixth-order symmetry axis parallel to $\mathbf{c}_3$ (Kiral and Eringen, 1990). As a consequence, the results obtained here hold also for the materials of this class.

In the technical literature one finds more frequently Eqs. (5) expressed in matrix notation. The components of $\mathbf{E}$, $\mathbf{T}$, $\mathbf{e}$, $\mathbf{d}$ are arranged in form of vectors, while the material constants are arranged in a square matrix. When the representation formulas (8) hold, it turns out

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} C^{e}_{11} & C^{e}_{12} & C^{e}_{13} & 0 & 0 & 0 & 0 & 0 & -L_{31} \\ C^{e}_{12} & C^{e}_{11} & C^{e}_{13} & 0 & 0 & 0 & 0 & 0 & -L_{31} \\ C^{e}_{13} & C^{e}_{13} & C^{e}_{33} & 0 & 0 & 0 & 0 & 0 & -L_{33} \\ 0 & 0 & 0 & C^{e}_{44} & 0 & 0 & 0 & -L_{15} & 0 \\ 0 & 0 & 0 & 0 & C^{e}_{44} & 0 & -L_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C^{e}_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{15} & 0 & \epsilon^{\mathbf{E}}_{11} & 0 & 0 \\ 0 & 0 & 0 & L_{15} & 0 & 0 & 0 & \epsilon^{\mathbf{E}}_{11} & 0 \\ L_{31} & L_{31} & L_{33} & 0 & 0 & 0 & 0 & 0 & \epsilon^{\mathbf{E}}_{33} \end{bmatrix} \cdot \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ 2E_4 \\ 2E_5 \\ 2E_6 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{9}$$

Table 1 Elastic, piezoelectric, and dielectric constants of some materials

	CdS	PZT-4	PZT-5H	PZT-8	C-24
C^e_{11} [GPa]	90.7	139.	126.	137.	150.
C^e_{33} [GPa]	93.8	115.	117.	124.	128.
C^e_{12} [GPa]	58.1	77.8	79.5	69.7	36.8
C^e_{13} [GPa]	50.9	74.3	84.1	71.6	30.9
C^e_{44} [GPa]	15.0	25.6	23.0	31.4	55.2
L_{31} [C/m^2]	−0.245	−5.2	−6.5	−4.0	1.51
L_{33} [C/m^2]	0.440	15.1	23.3	13.8	8.53
L_{15} [C/m^2]	−0.210	12.7	17.0	10.4	3.89
ϵ^E_{11} [nF/m]	79.8	6.46	15.0	7.95	1.82
ϵ^E_{33} [nF/m]	84.3	5.62	13.0	5.15	1.28

The material constants (at room temperature) of some piezoelectric materials are given in Table 1 (Eringen and Maugin, 1990; Smith, 1992; Toshiba, 1983; Vernitron, 1983): a 6 mm hexagonal crystal (Cadmium sulphure) and four transversely isotropic ceramics are considered. The units of the International System are used: gigaPascal for elastic constants, nanoFarad per meter for dielectric constants, and Coulomb per square meter for piezoelectric constants.

4 Basic Equations of the Linear Theory of Piezoelectricity

Let a body Ω be given, subjected to

(i) volume forces $\mathbf{b}$;
(ii) volume charge ρ;
(iii) surface forces $\mathbf{p}$ on a part $\partial_f\Omega$ of the boundary $\partial\Omega$;
(iv) surface charge ω on a part $\partial_\omega\Omega$ of $\partial\Omega$;
(v) prescribed displacement $\mathbf{s}_0$ on $\partial_s\Omega := \partial\Omega\backslash\partial_f\Omega$;
(vi) prescribed electric potential ϕ_0 on $\partial_\phi\Omega := \partial\Omega\backslash\partial_\omega\Omega$.

The strain field $\mathbf{E}$ corresponding to a displacement field $\mathbf{s}$ from the reference configuration of Ω to a deformed configuration is defined by the equation

$$\mathbf{E} = \text{sym}\,\nabla\mathbf{s}. \tag{10}$$

In this paper a time-independent (i.e., electrostatic) problem is considered: hence, the electric field can be derived from a potential, say ϕ:

$$\mathbf{e} = -\nabla\phi. \tag{11}$$

A displacement field $\mathbf{s}$ in Ω which equals $\mathbf{s}_0$ on $\partial_s\Omega$ and an electric potential field ϕ in Ω which equals ϕ_0 on $\partial_\phi\Omega$ are sought for, such that the stress field $\mathbf{T}$ and the electric displacement field $\mathbf{d}$, corresponding through Eqs. (5) to the strain field (10) and to the electric field (11), verify the balance equations (Maugin, 1988):

$$\begin{cases} \text{div}\,\mathbf{T} + \mathbf{b} = 0 & \text{in} \quad \Omega \\ \text{div}\,\mathbf{d} - \rho = 0 & \text{in} \quad \Omega \end{cases}$$

$$\text{and} \quad \begin{cases} \mathbf{Tn} = \mathbf{p} & \text{on} \quad \partial_f\Omega, \\ \{\mathbf{d}\}\cdot\mathbf{n} = \omega & \text{on} \quad \partial_\omega\Omega, \end{cases} \tag{12}$$

where $\mathbf{n}$ is the outward normal field to $\partial\Omega$ and $\{\mathbf{d}\}$ denotes the jump of $\mathbf{d}$ across $\partial\Omega$.

It is assumed that outside Ω the electric displacement field vanishes: e.g., this assumption holds true, if Ω is surrounded by conductors, and is usually acceptable, if Ω is surrounded by the vacuum (or air). Hence, the equation $\{\mathbf{d}\}\cdot\mathbf{n} = \omega$ becomes

$$\mathbf{d}\cdot\mathbf{n} = -\omega \quad \text{on} \quad \partial_\omega\Omega, \tag{13}$$

where $\mathbf{d}$ is the electric displacement field inside Ω.

As a consequence, the unknowns $\mathbf{s}$ and ϕ must satisfy the field equations

$$\begin{cases} \text{div}\,(\mathcal{C}^e[\text{sym}\,\nabla\mathbf{s}]) + \text{div}\,(L^t[\nabla\phi]) + \mathbf{b} = 0 \\ \text{div}\,(L[\text{sym}\,\nabla\mathbf{s}]) - \text{div}\,(\epsilon^E[\nabla\phi]) - \rho = 0 \end{cases} \text{in} \quad \Omega, \tag{14}$$

and the boundary conditions

$$\begin{cases} \mathcal{C}^e[\text{sym}\,\nabla\mathbf{s}]\mathbf{n} + L^t[\nabla\phi]\mathbf{n} = \mathbf{p} & \text{on} \quad \partial_f\Omega \\ L[\text{sym}\,\nabla\mathbf{s}]\cdot\mathbf{n} - \epsilon^E[\nabla\phi]\cdot\mathbf{n} = -\omega & \text{on} \quad \partial_\omega\Omega \\ \mathbf{s} = \mathbf{s}_0 & \text{on} \quad \partial_s\Omega \\ \phi = \phi_0 & \text{on} \quad \partial_\phi\Omega. \end{cases} \tag{15}$$

5 The Simply Supported Rectangular Plate Problem

A rectangular plate having in-plane dimensions L_1 and L_2 and thickness H is considered. Let the reference configuration of the plate be the region

$$\Omega = (-L_1/2, L_1/2) \times (-L_2/2, L_2/2) \times (-H/2, H/2) \subset \mathcal{E}. \tag{16}$$

The end faces of the plate, orthogonal to the x_3-axis at $x_3 = \pm H/2$, are denoted as upper or lower face, while the other faces constitute the lateral boundary.

The plate is assumed to be made of a transversely isotropic material, with the transverse-isotropy axis parallel to the x_3-axis.

The volume forces $\mathbf{b}$ and the volume charge ρ are assumed to vanish. The boundary conditions on $\partial\Omega$ are assigned in a slightly more general way than that specified in the previous section. Indeed, a so-called "general boundary value problem," or "mixed-mixed problem" (Gurtin, 1972) is considered here, in order to model a simply supported plate. In particular, on the lateral boundary of the plate, the tangential component $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{s}_0$ of the displacement and the normal component $(\mathbf{n} \otimes \mathbf{n})\mathbf{p}$ of the surface forces are assumed to vanish (see, e.g., Pagano, 1970; Srinivas and Rao, 1970; Levinson, 1984; Pan, 1991). In addition, the potential ϕ is assumed to be zero. On the upper and lower face the surface forces and the surface charge are given. In formulas, the boundary conditions are as follows:

(i) on the lateral faces orthogonal to the x_1 and x_2-axis, respectively:

$$\begin{cases} s_2(\pm L_1/2, x_2, x_3) = 0 \\ s_3(\pm L_1/2, x_2, x_3) = 0 \\ \phi(\pm L_1/2, x_2, x_3) = 0 \\ T_1(\pm L_1/2, x_2, x_3) = 0 \end{cases} \text{and} \quad \begin{cases} s_1(x_1, \pm L_2/2, x_3) = 0 \\ s_3(x_1, \pm L_2/2, x_3) = 0 \\ \phi(x_1, \pm L_2/2, x_3) = 0 \\ T_2(x_1, \pm L_2/2, x_3) = 0; \end{cases} \tag{17}$$

(ii) on the upper and lower face:

$$\begin{cases} T_5(x_1, x_2, \pm H/2) = \pm p_1^\pm(x_1, x_2) \\ T_4(x_1, x_2, \pm H/2) = \pm p_2^\pm(x_1, x_2) \\ T_3(x_1, x_2, \pm H/2) = \pm p_3^\pm(x_1, x_2) \\ d_3(x_1, x_2, \pm H/2) = \mp\omega^\pm(x_1, x_2), \end{cases} \tag{18}$$

where $p_1^\pm$, $p_2^\pm$, and $p_3^\pm$ are the components of the surface forces $\mathbf{p}^\pm$, $\omega^\pm$ is the surface charge, and upper (lower) sign applies to upper (lower) face. Equations (17) and (18) are intended to be read twice: one time taking all the upper signs, the other time taking all the lower signs.

The load acting on the plate is assumed to be symmetric with respect to the plane containing the x_1 and x_3-axes, and to the plane containing the x_2 and x_3-axes. In other words, the functions $p_3^{\pm}$ and $\omega^{\pm}$ are assumed to be even with respect to both x_1 and x_2; the functions $p_1^{\pm}$ are assumed to be odd with respect to x_1 and even with respect to x_2; the functions $p_2^{\pm}$ are assumed to be even with respect to x_1 and odd with respect to x_2.

With a view toward satisfying the boundary conditions (18), the functions $p_1^{\pm}$, $p_2^{\pm}$, $p_3^{\pm}$ and $\omega^{\pm}$ are expanded into a double Fourier series in the rectangle $(-L_1/2, L_1/2) \times (-L_2/2, L_2/2)$. As a consequence, it is sufficient to solve the problem of a plate subjected to surface forces and surface charge on the upper and lower face of the form

$$\begin{cases} p_1^{\pm}(x_1, x_2) = p_{1,n_1,n_2}^{\pm} \sin \dfrac{(2n_1+1)\pi x_1}{L_1} \cos \dfrac{(2n_2+1)\pi x_2}{L_2} \\ p_2^{\pm}(x_1, x_2) = p_{2,n_1,n_2}^{\pm} \cos \dfrac{(2n_1+1)\pi x_1}{L_1} \sin \dfrac{(2n_2+1)\pi x_2}{L_2} \\ p_3^{\pm}(x_1, x_2) = p_{3,n_1,n_2}^{\pm} \cos \dfrac{(2n_1+1)\pi x_1}{L_1} \cos \dfrac{(2n_2+1)\pi x_2}{L_2} \\ \omega^{\pm}(x_1, x_2) = \omega_{n_1,n_2}^{\pm} \cos \dfrac{(2n_1+1)\pi x_1}{L_1} \cos \dfrac{(2n_2+1)\pi x_2}{L_2}, \end{cases} \tag{19}$$

where n_1 and n_2 are fixed non-negative integers and $p_{1,n_1,n_2}^{\pm}$, $p_{2,n_1,n_2}^{\pm}$, $p_{3,n_1,n_2}^{\pm}$, $\omega_{n_1,n_2}^{\pm}$ are fixed scalars. Indeed, the case of a general load having the stipulated symmetry can be covered by superposition (e.g., Timoshenko and Woinowsky-Krieger, 1982). Hence, in what follows it is assumed that $p_1^{\pm}$, $p_2^{\pm}$, $p_3^{\pm}$, and $\omega^{\pm}$ have the form specified in Eqs. (19).

6 A Separate-Variables Three-Dimensional Solution

On account of the present rectangular geometry, it is natural to search for a three-dimensional solution of the plate problem such that each of the four unknown functions s_1, s_2, s_3 and ϕ is a product of functions, each depending on one variable only. Indeed, let the displacement and the potential field have the expressions

$$\begin{cases} s_1(x_1, x_2, x_3) = S_1(x_3) \sin q_1 x_1 \cos q_2 x_2 \\ s_2(x_1, x_2, x_3) = S_2(x_3) \cos q_1 x_1 \sin q_2 x_2 \\ s_3(x_1, x_2, x_3) = S_3(x_3) \cos q_1 x_1 \cos q_2 x_2 \\ \phi(x_1, x_2, x_3) = \Phi(x_3) \cos q_1 x_1 \cos q_2 x_2, \end{cases} \tag{20}$$

where

$$q_1 = (2n_1+1)\pi/L_1 \quad \text{and} \quad q_2 = (2n_2+1)\pi/L_2, \tag{21}$$

and S_1, S_2, S_3, Φ are functions to be determined, depending on x_3 only.

As a consequence of this choice, the boundary conditions (17) are trivially satisfied; the field Eqs. (14), using Eqs. (8), become (Tiersten, 1969)

$$\begin{cases} (-C_{11}^{e} q_1^2 - C_{66}^{e} q_2^2 + C_{44}^{e} d^2) S_1 - (C_{12}^{e} + C_{66}^{e}) q_1 q_2 S_2 \\ \qquad - (C_{13}^{e} + C_{44}^{e}) q_1 d S_3 - (L_{31} + L_{15}) q_1 d\Phi = 0 \\ -(C_{12}^{e} + C_{66}^{e}) q_1 q_2 S_1 + (-C_{11}^{e} q_2^2 - C_{66}^{e} q_1^2 + C_{44}^{e} d^2) S_2 \\ \qquad - (C_{13}^{e} + C_{44}^{e}) q_2 d S_3 - (L_{31} + L_{15}) q_2 d\Phi = 0 \\ (C_{13}^{e} + C_{44}^{e})(q_1 dS_1 + q_2 dS_2) + [-C_{44}^{e}(q_1^2 + q_2^2) + C_{33}^{e} d^2] S_3 \\ \qquad + [-L_{15}(q_1^2 + q_2^2) + L_{33} d^2]\Phi = 0 \\ (L_{31} + L_{15})(q_1 dS_1 + q_2 dS_2) + [-L_{15}(q_1^2 + q_2^2) + L_{33} d^2] S_3 \\ \qquad + [\epsilon_{11}^{E}(q_1^2 + q_2^2) - \epsilon_{33}^{E} d^2]\Phi = 0 \end{cases} \tag{22}$$

where d denotes the differential operator with respect to x_3; the boundary conditions (18), taking into account Eqs. (19), become

$$\begin{cases} C_{44}^{e}[dS_1(\pm H/2) - q_1 S_3(\pm H/2)] \\ \qquad - L_{15} q_1 \Phi(\pm H/2) = \pm p_{1,n_1,n_2}^{\pm} \\ C_{44}^{e}[dS_2(\pm H/2) - q_2 S_3(\pm H/2)] \\ \qquad - L_{15} q_2 \Phi(\pm H/2) = \pm p_{2,n_1,n_2}^{\pm} \\ C_{13}^{e}[q_1 S_1(\pm H/2) + q_2 S_2(\pm H/2)] + C_{33}^{e} dS_3(\pm H/2) \\ \qquad + L_{33} d\Phi(\pm H/2) = \pm p_{3,n_1,n_2}^{\pm} \\ L_{31}[q_1 S_1(\pm H/2) + q_2 S_2(\pm H/2)] + L_{33} dS_3(\pm H/2) \\ \qquad - \epsilon_{33}^{E} d\Phi(\pm H/2) = \mp \omega_{n_1,n_2}^{\pm}. \end{cases} \tag{23}$$

Hence, the plate problem is reduced to the boundary value problem (22)–(23), governed by an *ordinary* differential linear system in the unknown functions S_1, S_2, S_3, Φ.

For the sake of simplifying the problem, the following change of variables is made:

$$U := \nu_1 S_1 + \nu_2 S_2, \quad V := -\nu_2 S_1 + \nu_1 S_2, \quad X_3 := x_3/l, \tag{24}$$

where

$$l = 1/\sqrt{q_1^2 + q_2^2}, \quad \nu_1 = q_1 l \quad \text{and} \quad \nu_2 = q_2 l. \tag{25}$$

The quantity l, depending upon L_1, L_2, n_1, n_2 has the meaning of a characteristic in-plane dimension of the plate. It is a simple matter to verify that the system (22) is decoupled into the system

$$\begin{cases} (-C_{11}^{e} + C_{44}^{e} D^2) U - (C_{13}^{e} + C_{44}^{e}) D S_3 \\ \qquad - (L_{31} + L_{15}) D\Phi = 0 \\ (C_{13}^{e} + C_{44}^{e}) DU + (-C_{44}^{e} + C_{33}^{e} D^2) S_3 \\ \qquad + (-L_{15} + L_{33} D^2)\Phi = 0 \\ (L_{31} + L_{15}) DU + (-L_{15} + L_{33} D^2) S_3 \\ \qquad + (\epsilon_{11}^{E} - \epsilon_{33}^{E} D^2)\Phi = 0, \end{cases} \tag{26}$$

and the equation

$$(-C_{66}^{e} + C_{44}^{e} D^2) V = 0, \tag{27}$$

where $D = ld$ denotes the differential operator with respect to X_3.

It is worth noting that the coefficients of Eqs. (26) and (27) depend only upon the material constants and are independent of q_1 and q_2: hence, the general solution of Eqs. (26) and (27) can be found once for a given material, and then used for a rectangular plate made of that material, independently of the dimensions or the load.

The characteristic equation of the system (26), in the unknown λ, is

$$\det \begin{bmatrix} -C_{11}^{e} + C_{44}^{e}\lambda^2 & -(C_{13}^{e} + C_{44}^{e})\lambda & -(L_{31} + L_{15})\lambda \\ (C_{13}^{e} + C_{44}^{e})\lambda & -C_{44}^{e} + C_{33}^{e}\lambda^2 & -L_{15} + L_{33}\lambda^2 \\ (L_{31} + L_{15})\lambda & -L_{15} + L_{33}\lambda^2 & \epsilon_{11}^{E} - \epsilon_{33}^{E}\lambda^2 \end{bmatrix} = 0, \tag{28}$$

where "det" denotes the determinant. In a more explicit form, Eq. (28) can be written as

$$-a_6\lambda^6 + a_4\lambda^4 - a_2\lambda^2 + a_0 = 0, \tag{29}$$

where

$$\begin{cases} a_6 = C^{e}_{44}(C^{e}_{33}\epsilon^{E}_{33} + L^2_{33}), \\ a_4 = C^{e}_{44}C^{e}_{33}\epsilon^{E}_{11} + (C^{e}_{44})^2\epsilon^{E}_{33} + C^{e}_{11}C^{e}_{33}\epsilon^{E}_{33} + L^2_{33}C^{e}_{11} \\ \quad + 2L_{33}L_{15}C^{e}_{44} - 2(C^{e}_{13} + C^{e}_{44})(L_{31} + L_{15})L_{33} \\ \quad\quad + (L_{31} + L_{15})^2C^{e}_{33} - (C^{e}_{13} + C^{e}_{44})^2\epsilon^{E}_{33}, \\ a_2 = C^{e}_{33}C^{e}_{11}\epsilon^{E}_{11} + (C^{e}_{44})^2\epsilon^{E}_{11} + C^{e}_{11}C^{e}_{44}\epsilon^{E}_{33} + L^2_{15}C^{e}_{44} \\ \quad + 2L_{33}L_{15}C^{e}_{11} - 2(C^{e}_{13} + C^{e}_{44})(L_{31} + L_{15})L_{15} \\ \quad\quad + (L_{31} + L_{15})^2C^{e}_{44} - (C^{e}_{13} + C^{e}_{44})^2\epsilon^{E}_{11}, \\ a_0 = C^{e}_{11}(C^{e}_{44}\epsilon^{E}_{11} + L^2_{15}). \end{cases} \quad (30)$$

For the sake of simplicity, it is assumed that Eq. (29) has six distinct roots:

$$\lambda_1,\ -\lambda_1,\ \lambda_2,\ -\lambda_2,\ \lambda_3,\ -\lambda_3.$$

If $[U(X_3), S_3(X_3), \Phi(X_3)]$ is a solution of the system (26), then $[-U(-X_3), S_3(-X_3), \Phi(-X_3)]$ is a solution, too. Hence, the general solution of the system (26) can be given the form

$$\begin{cases} U(X_3) = U^e(X_3) + U^o(X_3) \\ S_3(X_3) = S^o_3(X_3) + S^e_3(X_3) \\ \Phi(X_3) = \Phi^o(X_3) + \Phi^e(X_3), \end{cases} \quad (31)$$

where

$$\begin{cases} U^e(X_3) = \sum_{i=1}^{3} A_i\alpha_i \cosh(\lambda_i X_3) \\ S^o_3(X_3) = \sum_{i=1}^{3} A_i\beta_i \sinh(\lambda_i X_3) \\ \Phi^o(X_3) = \sum_{i=1}^{3} A_i\gamma_i \sinh(\lambda_i X_3) \end{cases}$$

$$\text{and} \quad \begin{cases} U^o(X_3) = \sum_{i=1}^{3} B_i\alpha_i \sinh(\lambda_i X_3) \\ S^e_3(X_3) = \sum_{i=1}^{3} B_i\beta_i \cosh(\lambda_i X_3) \\ \Phi^e(X_3) = \sum_{i=1}^{3} B_i\gamma_i \cosh(\lambda_i X_3). \end{cases} \quad (32)$$

The superscripts "e" and "o" denote even and odd functions of X_3, respectively; A_i, B_i are arbitrary dimensionless (complex) constants. The vector $(\alpha_i, \beta_i, \gamma_i)$ spans the null space of the matrix at the left-hand side of Eq. (28) when λ is given the value λ_i, for $i = 1 \ldots 3$ (of course, this vector is defined up to a scalar multiple). In other words, the constants α_i, β_i, γ_i, satisfy the following equations, where $i = 1 \ldots 3$ is unsummed:

$$\begin{cases} (-C^{e}_{11} + C^{e}_{44}\lambda^2_i)\alpha_i - (C^{e}_{13} + C^{e}_{44})\lambda_i\beta_i \\ \quad - (L_{31} + L_{15})\lambda_i\gamma_i = 0 \\ (C^{e}_{13} + C^{e}_{44})\lambda_i\alpha_i + (-C^{e}_{44} + C^{e}_{33}\lambda^2_i)\beta_i \\ \quad + (-L_{15} + L_{33}\lambda^2_i)\gamma_i = 0 \\ (L_{31} + L_{15})\lambda_i\alpha_i + (-L_{15} + L_{33}\lambda^2_i)\beta_i + (\epsilon^{E}_{11} - \epsilon^{E}_{33}\lambda^2_i)\gamma_i = 0. \end{cases} \quad (33)$$

Equation (27) has the general solution

$$V(X_3) = V^e(X_3) + V^o(X_3), \quad (34)$$

where

$$V^e(X_3) = A_4 1 \cosh(\lambda_4 X_3) \quad \text{and}$$
$$V^o(X_3) = B_4 1 \sinh(\lambda_4 X_3), \quad (35)$$

$\lambda_4 = \sqrt{C^{e}_{66}/C^{e}_{44}}$, A_4, B_4 are arbitrary dimensionless constants, and 1 has the physical dimension of a length.

For the piezoelectric materials considered in Table 1, the values of the constants λ_i, α_i, β_i, and γ_i are given in Table 2, in the units of the International System (j denotes the imaginary unit).

The boundary conditions (23), according to the change of variables (24), are decoupled into

(i) a system of three equations involving U^e, S^o_3, Φ^o:

$$\begin{cases} C^{e}_{44}DU^e(h/2) - C^{e}_{44}S^o_3(h/2) - L_{15}\Phi^o(h/2) = lp^e \\ C^{e}_{13}U^e(h/2) + C^{e}_{33}DS^o_3(h/2) + L_{33}D\Phi^o(h/2) = lp^o_3 \\ L_{31}U^e(h/2) + L_{33}DS^o_3(h/2) - \epsilon^{E}_{33}D\Phi^o(h/2) = -l\omega^o; \end{cases} \quad (36)$$

(ii) a system of three equations involving U^o, S^e_3, Φ^e:

$$\begin{cases} C^{e}_{44}DU^o(h/2) - C^{e}_{44}S^e_3(h/2) - L_{15}\Phi^e(h/2) = lp^o \\ C^{e}_{13}U^o(h/2) + C^{e}_{33}DS^e_3(h/2) + L_{33}D\Phi^e(h/2) = lp^e_3 \\ L_{31}U^o(h/2) + L_{33}DS^e_3(h/2) - \epsilon^{E}_{33}D\Phi^e(h/2) = -l\omega^e; \end{cases} \quad (37)$$

(iii) one equation involving V^e:

$$C^{e}_{44}DV^e(h/2) = lq^e; \quad (38)$$

(iv) one equation involving V^o:

$$C^{e}_{44}DV^o(h/2) = lq^o, \quad (39)$$

where the plate thickness aspect ratio (i.e., the ratio between the thickness H and the characteristic in-plane dimension l of the plate) is introduced:

$$h := H/l \quad (40)$$

and the following positions are made:

$$\begin{aligned} p^e_1 &= \tfrac{1}{2}(p^+_{1,n_1,n_2} + p^-_{1,n_1,n_2}) \quad & p^o_1 &= \tfrac{1}{2}(p^+_{1,n_1,n_2} - p^-_{1,n_1,n_2}) \\ p^e_2 &= \tfrac{1}{2}(p^+_{2,n_1,n_2} + p^-_{2,n_1,n_2}) \quad & p^o_2 &= \tfrac{1}{2}(p^+_{2,n_1,n_2} - p^-_{2,n_1,n_2}) \\ p^e_3 &= \tfrac{1}{2}(p^+_{3,n_1,n_2} + p^-_{3,n_1,n_2}) \quad & p^o_3 &= \tfrac{1}{2}(p^+_{3,n_1,n_2} - p^-_{3,n_1,n_2}) \\ \omega^e &= \tfrac{1}{2}(\omega^+_{n_1,n_2} + \omega^-_{n_1,n_2}) \quad & \omega^o &= \tfrac{1}{2}(\omega^+_{n_1,n_2} - \omega^-_{n_1,n_2}) \\ p^e &= +\nu_1 p^e_1 + \nu_2 p^e_2 \quad & p^o &= +\nu_1 p^o_1 + \nu_2 p^o_2 \\ q^e &= -\nu_2 p^e_1 + \nu_1 p^e_2 \quad & q^o &= -\nu_2 p^o_1 + \nu_1 p^o_2. \end{aligned} \quad (41)$$

In a more explicit form, Eqs. (36) and (37) can be respectively rewritten as

$$\begin{cases} \sum_{i=1}^{3} A_i t_i \sinh(\lambda_i h/2) = lp^e \\ \sum_{i=1}^{3} A_i r_i \cosh(\lambda_i h/2) = lp^o_3 \\ \sum_{i=1}^{3} A_i z_i \cosh(\lambda_i h/2) = -l\omega^o \end{cases}$$

Table 2 The material constants λ_i, α_i, β_i, and γ_i

	CdS	PZT-4	PZT-5H	PZT-8	C-24
λ_1	$1.665+j0.$	$1.069+j0.200$	$1.029+j0.415$	$1.044+j0.242$	$1.032+j0.127$
α_1 [m]	$-0.913+j0.$	$0.254-j0.418$	$0.632-j0.248$	$0.594-j0.183$	$0.441-j0.516$
β_1 [m]	$0.409+j0.$	$-0.080+j0.459$	$-0.329+j0.464$	$-0.439+j0.350$	$-0.300+j0.630$
γ_1 [V]	$0.008+j0.$	$-0.659+j0.330$	$-0.327+j0.330$	$-0.407+j0.364$	$0.114+j0.199$
λ_2	$0.591+j0.$	$1.069-j0.200$	$1.029-j0.415$	$1.044-j0.242$	$1.032-j0.127$
α_2 [m]	$0.415+j0.$	$0.254+j0.418$	$0.632+j0.248$	$0.594+j0.183$	$0.441+j0.516$
β_2 [m]	$-0.910+j0.$	$-0.080-j0.459$	$-0.329-j0.464$	$-0.439-j0.350$	$-0.300-j0.630$
γ_2 [V]	$0.009+j0.$	$-0.659-j0.330$	$-0.327-j0.330$	$-0.407-j0.364$	$0.114-j0.199$
λ_3	$0.973+j0.$	$1.204+j0.$	$1.053+j0.$	$1.195+j0.$	$1.065+j0.$
α_3 [m]	$0.047+j0.$	$-0.696+j0.$	$-0.090+j0.$	$-0.018+j0.$	$-0.024+j0.$
β_3 [m]	$-0.050+j0.$	$0.617+j0.$	$0.176+j0.$	$-0.048+j0.$	$-0.040+j0.$
γ_3 [V]	$0.998+j0.$	$-0.366+j0.$	$-0.980+j0.$	$0.999+j0.$	$0.999+j0.$
λ_4	$1.042+j0.$	$1.093+j0.$	$1.005+j0.$	$1.035+j0.$	$1.013+j0.$

and
$$\begin{cases} \sum_{i=1}^{3} B_i t_i \cosh(\lambda_i h/2) = l p^o, \\ \sum_{i=1}^{3} B_i r_i \sinh(\lambda_i h/2) = l p_3^e \\ \sum_{i=1}^{3} B_i z_i \sinh(\lambda_i h/2) = -l\omega^e, \end{cases} \tag{42}$$

where

$$\begin{cases} t_i = C_{44}^{e}\alpha_i\lambda_i - C_{44}^{e}\beta_i - L_{15}\gamma_i \\ r_i = C_{13}^{e}\alpha_i + C_{33}^{e}\beta_i\lambda_i + L_{33}\gamma_i\lambda_i \\ z_i = L_{31}\alpha_i + L_{33}\beta_i\lambda_i - \epsilon_{33}^{E}\gamma_i\lambda_i \end{cases} \quad (i = 1 \ldots 3, \text{ unsummed}). \tag{43}$$

Hence, they allow to determine the constants A_1, A_2, A_3 and B_1, B_2, B_3, respectively. Analogously, Eqs. (38) and (39) can be, respectively, rewritten as

$$C_{44}^{e} A_4 \lambda_4 \sinh(\lambda_4 h/2) = l q^e \quad \text{and}$$
$$C_{44}^{e} B_4 \lambda_4 \cosh(\lambda_4 h/2) = l q^o, \tag{44}$$

and allow to determine the constants A_4 and B_4, respectively.

The functions V, U, S_3, Φ are determined by Eqs. (31) and (34); these functions turn out to be real, in spite of the fact that, in general, A_i, B_i are complex constants. Then, the displacement field is obtained by Eqs. (20) and (24):

$$\begin{cases} s_1 = (\nu_1 U - \nu_2 V)\sin q_1x_1 \cos q_2x_2 \\ s_2 = (\nu_2 U + \nu_1 V)\cos q_1x_1 \sin q_2x_2 \\ s_3 = S_3 \cos q_1x_1 \cos q_2x_2 \\ \phi = \Phi \cos q_1x_1 \cos q_2x_2; \end{cases} \tag{45}$$

the strain field is obtained by Eq. (10):

$$\begin{cases} E_1 = \nu_1/l(\nu_1 U - \nu_2 V)\cos q_1x_1 \cos q_2x_2 \\ E_2 = \nu_2/l(\nu_2 U + \nu_1 V)\cos q_1x_1 \cos q_2x_2 \\ E_3 = 1/lDS_3 \cos q_1x_1 \cos q_2x_2 \\ 2E_4 = 1/l[\nu_2(DU - S_3) + \nu_1 DV]\cos q_1x_1 \sin q_2x_2 \\ 2E_5 = 1/l[\nu_1(DU - S_3) - \nu_2 DV]\sin q_1x_1 \cos q_2x_2 \\ 2E_6 = 1/l[-2\nu_1\nu_2 U + (\nu_2^2 - \nu_1^2)V]\sin q_1x_1 \sin q_2x_2; \end{cases} \tag{46}$$

the electric field is obtained by Eq. (11):

$$\begin{cases} e_1 = \nu_1/l\Phi \sin q_1x_1 \cos q_2x_2 \\ e_2 = \nu_2/l\Phi \cos q_1x_1 \sin q_2x_2 \\ e_3 = -1/lD\Phi \cos q_1x_1 \cos q_2x_2; \end{cases} \tag{47}$$

the stress field is obtained by Eqs. (5) and (8):

$$\begin{cases} T_1 = 1/l[Q - 2\nu_2 C_{66}^{e}(\nu_2 U + \nu_1 V)]\cos q_1x_1 \cos q_2x_2 \\ T_2 = 1/l[Q - 2\nu_1 C_{66}^{e}(\nu_1 U - \nu_2 V)]\cos q_1x_1 \cos q_2x_2 \\ T_3 = R/l \cos q_1x_1 \cos q_2x_2 \\ T_4 = 1/l(\nu_2 W + \nu_1 C_{44}^{e} DV)\cos q_1x_1 \sin q_2x_2 \\ T_5 = 1/l(\nu_1 W - \nu_2 C_{44}^{e} DV)\sin q_1x_1 \cos q_2x_2 \\ T_6 = C_{66}^{e}/l[-2\nu_1\nu_2 U + (\nu_2^2 - \nu_1^2)V]\sin q_1x_1 \sin q_2x_2; \end{cases} \tag{48}$$

and the electric displacement field is obtained by Eqs. (5) and (8), too,

$$\begin{cases} d_1 = 1/l(\nu_1 Y - L_{15}\nu_2 DV)\sin q_1x_1 \cos q_2x_2 \\ d_2 = 1/l(\nu_2 Y + L_{15}\nu_1 DV)\cos q_1x_1 \sin q_2x_2 \\ d_3 = Z/l \cos q_1x_1 \cos q_2x_2, \end{cases} \tag{49}$$

where the following positions have been made:

$$\begin{cases} Q = C_{11}^{e} U + C_{13}^{e} DS_3 + L_{31} D\Phi \\ R = C_{13}^{e} U + C_{33}^{e} DS_3 + L_{33} D\Phi \\ W = C_{44}^{e} DU - C_{44}^{e} S_3 - L_{15}\Phi \\ Y = L_{15} DU - L_{15} S_3 + \epsilon_{11}^{E}\Phi \\ Z = L_{31} U + L_{33} DS_3 - \epsilon_{33}^{E} D\Phi. \end{cases} \tag{50}$$

Needless to say that if

$$\omega_{n_1,n_2}^{+} = \omega_{n_1,n_2}^{-}, \quad p_{3,n_1,n_2}^{+} = p_{3,n_1,n_2}^{-},$$
$$p_{1,n_2,n_2}^{+} = -p_{1,n_1,n_2}^{-}, \quad p_{2,n_1,n_2}^{+} = -p_{2,n_1,n_2}^{-}, \tag{51}$$

then $\omega^o = 0$, $p_3^o = 0$, $p^e = 0$, $q^e = 0$: hence the constants A_i turn out to be zero and $V^e = 0$, $U^e = 0$, $S_3^o = 0$, $\Phi^o = 0$. As a consequence, s_3, ϕ, E_4, E_5, e_1, e_2, T_4, T_5, d_1 and d_2 are even functions of the x_3 variable, while s_1, s_2, E_1, E_2, E_3, E_6, e_3, T_1, T_2, T_3, T_6 and d_3 are odd functions of the x_3 variable: this amounts to say that the plate has a bending behavior. Conversely, if

$$\omega_{n_1,n_2}^{+} = -\omega_{n_1,n_2}^{-}, \quad p_{3,n_1,n_2}^{+} = -p_{3,n_1,n_2}^{-},$$
$$p_{1,n_1,n_2}^{+} = p_{1,n_1,n_2}^{-}, \quad p_{2,n_1,n_2}^{+} = p_{2,n_1,n_2}^{-}, \tag{52}$$

Table 3 Auxiliary material constants

	CdS	PZT-4	PZT-5H	PZT-8	C-24
C [GPa]	63.1	120.	96.2	117.	143.
G [GPa]	59.7	153.	102.	163.	161.
K [GPa]	1.80	−7.66	9.49	4.33	0.236
F [C/m²]	0.458	25.1	36.3	20.4	0.622
M [C/m²]	0.529	36.3	46.5	29.9	10.9
N [C/m²]	−1.34	34.6	34.8	18.4	9.43
ϵ [nF/m]	79.8	12.8	27.6	11.4	2.09
μ	0.543	0.388	0.456	0.378	0.222
σ	0.967	0.924	0.814	0.870	0.821
ρ	1.00	0.793	0.767	0.841	0.992

then $\omega^e = 0$, $p_3^e = 0$, $p^o = 0$, $q^o = 0$. Hence, the constants B_i turn out to be zero and the previously stated evenness and oddness properties exchange: in this case, the plate exhibits a membrane behavior.

7 The Limit of the Three-Dimensional Solution as the Plate Thickness Aspect Ratio Approaches Zero

In this section the limit of the three-dimensional solution (45)–(49) as the plate thickness aspect ratio h approaches zero is explicitly carried out, by performing the expansion into a power series of h of the involved mechanical and electric quantities, and retaining only the lowest-order terms. The result obtained is denoted as *thin-plate limit* of the three-dimensional solution.

In order to preserve, in the limit, the dependence upon the variable in the thickness direction, the rescaled variable

$$\zeta := \frac{X_3}{h} = \frac{x_3}{H} \tag{53}$$

is introduced ($\zeta = 0$ on the middle plane of the plate, and $\zeta = \pm\frac{1}{2}$ on the upper or lower face).

Needless to say that, in spite of the fact that the mechanical and electric quantities are related by Eqs. (10), (11), and (12), the lowest-order terms of their expansions into a power series of h needn't be related by those equations.

The calculations are based on the expansion into a power series of h of the solutions of Eqs. (42) and (44). The constants λ_i, α_i, β_i, γ_i are irrational functions of the material constants introduced in Eqs. (8). However, by taking into account the fact that, for $i = 1 \ldots 3$, λ_i solves Eq. (29) and α_i, β_i, γ_i satisfy Eqs. (33), it is possible to prevent the constants λ_i, α_i, β_i, γ_i by appearing explicitly in the thin-plate limit. Some auxiliary material constants are used instead, depending rationally on the ones introduced in Eq. (8):

$$C := C_{11}^{e} - \frac{\epsilon_{33}^{E}(C_{13}^{e})^2 + 2L_{31}L_{33}C_{13}^{e} - L_{31}^2 C_{33}^{e}}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2}$$

$$\epsilon := \epsilon_{11}^{E} + \frac{L_{15}^2}{C_{44}^{e}}$$

$$G := \epsilon\,\frac{C_{11}^{e}C_{33}^{e} - (C_{13}^{e})^2}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2} \quad F := \epsilon\,\frac{L_{33}C_{13}^{e} - C_{33}^{e}L_{31}}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2}$$

$$M := \epsilon\,\frac{L_{33}C_{11}^{e} - C_{13}^{e}L_{31}}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2} \quad N := \frac{L_{15}C - C_{44}^{e}F}{C_{44}^{e}}$$

$$K := \frac{MC - NG}{M} \quad \mu := \frac{C_{13}^{e}\epsilon_{33}^{E} + L_{33}L_{31}}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2}$$

$$\sigma := \frac{C_{11}^{e}\epsilon_{33}^{E} + L_{31}^2}{C_{33}^{e}\epsilon_{33}^{E} + L_{33}^2} \quad \rho := 1 - \frac{L_{15}F}{\epsilon C}. \tag{54}$$

For the piezoelectric materials considered in Table 1, the values of these constants are given in Table 3, in the units of the International System.

Only one at a time of the eight quantities p_3^e, p^o, ω^e, q^o, p_3^o, p^e, ω^o, q^e is assumed to be different from zero: hence, eight different cases are considered. Any load condition can be covered by superposition of these eight basic cases. The results obtained if either one of the quantities $p_3^e, p^o, \omega^e, q^o$ is different from zero are reported in Table 4 (in this case, the plate has a bending behavior). The results obtained if either one of the quantities p_3^o, p^e, ω^o, q^e is different from zero are reported in Table 5 (in this case, the plate has a membrane behavior).

For a square plate of edge $L_1 = L_2 = A$ and thickness H, some useful formulas can be easily carried out by the results given in Tables 4 and 5. The limits, as the plate thickness aspect ratio approaches zero, of some mechanical and electric quantities of primary interest in technical applications are found. These results should be recovered by applying any consistent theory of thin piezoelectric plates.

Let $f(x_1, x_2) := \cos(\pi x_1/A)\cos(\pi x_2/A)$. In Table 6 there are reported the deflection w_m, the electric potential Φm, and the bending moment M_{1m} at the center of the plate:

$$w_m := s_3(A/2, A/2, 0)$$

$$\Phi_m := \phi(A/2, A/2, H/2)$$

$$M_{1m} := \int_{-H/2}^{H/2} x_3 T_1(A/2, A/2, x_3)\,dx_3 \tag{55}$$

when the plate is subjected only to a normal sinusoidal load of largest intensity p_0 ($p_3^{\pm} = p_0 f(x_1, x_2)/2$), or to a normal uniform load of intensity p_0 ($p_3^{\pm} = p_0/2$), or to a sinusoidal surface charge of largest density ω_0 ($\omega^{\pm} = \omega_0 f(x_1, x_2)/2$), or to a uniform surface charge of density ω_0 ($\omega^{\pm} = \omega_0/2$). In these cases the plate has a bending behavior. The following position has been made:

$$\xi = C_{66}^{e}/C. \tag{56}$$

It is pointed out that the results obtained for w_m when a sinusoidal or uniform load p_0 is applied, formally coincide with the results about an elastic (i.e., nonpiezoelectric) plate given by Lekhnitskii (1968), if the quantity $H^3C/12$ is replaced by $H^3/12[C_{11}^{e} - (C_{13}^{e})^2/C_{33}^{e}]$. The latter quantity is the bending stiffness of an elastic plate. Noting that, if no piezoelectric coupling exists, the expression for C given in Eqs. (54) reduces just to $C_{11}^{e} - (C_{13}^{e})^2/C_{33}^{e}$, it is natural to call the quantity $H^3C/12$ "bending stiffness" of a piezoelectric plate.

In Table 7 there are reported the variation of the thickness ΔH_m, the difference of electric potential $\Delta\Phi_m$, and the membrane force N_{1m} at the center of the plate:

$$\Delta H_m := s_3(A/2, A/2, H/2) - s_3(A/2, A/2, -H/2)$$

$$\Delta\Phi_m := \phi(A/2, A/2, H/2) - \phi(A/2, A/2, -H/2)$$

$$N_{1m} := \int_{-H/2}^{H/2} T_1(A/2, A/2, x_3)\,dx_3 \tag{57}$$

when the plate is subjected on the upper and lower face only to a normal sinusoidal load of largest intensity p_0 ($p_3^{\pm} = \pm p_0 f(x_1, x_2)$), or to a normal uniform load of intensity p_0 ($p_3^{\pm} = \pm p_0$), or to a sinusoidal surface charge of largest density ω_0 ($\omega^{\pm} = \pm\omega_0 f(x_1, x_2)$), or to a uniform surface charge of density ω_0 ($\omega^{\pm} = \pm\omega_0$). In these cases the plate has a membrane behavior. It is emphasized that both the sinusoidal and the uniform distribution produce the same results.

The results provided by the zeroth-order theory of Maugin and Attou (1990), based on the scaling theory of Ciarlet and Destuynder (1979) do not agree with the thin-plate limit obtained here. Indeed, the deflection of a plate according to the zeroth order theory of Maugin and Attou is governed by an equation (Eq. (7.10) in Maugin and Attou, 1990) which doesn't

Table 4 Thin-plate limit of the three-dimensional solution. Bending behavior.

	p_3^e	p^o	ω^e	q^o
U	$\frac{24\ell p_3^e}{h^2 C}\zeta$	$\frac{12\ell p^o}{hC}\zeta$	$2\ell\omega^e \frac{F}{\epsilon C}\zeta$	0
S_3	$\frac{24\ell p_3^e}{h^3 C}$	$\frac{12\ell p^o}{h^2 C}$	$\frac{-2\ell\omega^e}{h}\frac{N}{\epsilon C}$	0
Φ	$\frac{-2\ell p_3^e}{h}\frac{F}{\epsilon C}[-\frac{N}{F}-\frac{3}{2}+6\zeta^2]$	$-\ell p^o \frac{F}{\epsilon C}(6\zeta^2-\frac{1}{2})$	$\frac{2\ell\omega^e}{h\epsilon}$	0
DS_3	$\frac{-24\ell p_3^e}{h^2}\frac{\mu}{C}\zeta$	$\frac{-12\ell p^o}{h}\frac{\mu}{C}\zeta$	$-2\ell\omega^e \frac{M}{\epsilon C}\zeta$	0
$D\Phi$	$\frac{-24\ell p_3^e}{h^2}\frac{F}{\epsilon C}\zeta$	$\frac{-12\ell p^o}{h}\frac{F}{\epsilon C}\zeta$	$2\ell\omega^e \frac{G}{\epsilon C}\zeta$	0
$DU - S_3$	$\frac{2\ell p_3^e}{hC_{44}^e}[\frac{-\epsilon_{11}}{\epsilon}+\rho(6\zeta^2-\frac{1}{2})]$	$\frac{\ell p^o \rho}{C_{44}^e}(6\zeta^2-\frac{1}{2})$	$\frac{2\ell\omega^e}{h}\frac{L_{15}}{\epsilon C_{44}^e}$	0
Q	$\frac{24\ell p_3^e}{h^2}\zeta$	$\frac{12\ell p^o}{h}\zeta$	$\frac{-h^2\ell\omega^e}{60}\frac{MK}{\epsilon C}(-3\zeta+20\zeta^3)$	0
R	$\ell p_3^e(3\zeta-4\zeta^3)$	$\frac{h\ell p^o}{2}(\zeta-4\zeta^3)$	$\frac{h^4\ell\omega^e}{960}\frac{MK}{\epsilon C}(\zeta-8\zeta^3+16\zeta^5)$	0
W	$\frac{-3\ell p_3^e}{h}(1-4\zeta^2)$	$\ell p^o(6\zeta^2-\frac{1}{2})$	$\frac{-h^3\ell\omega^e}{960}\frac{MK}{\epsilon C}(1-24\zeta^2+80\zeta^4)$	0
Y	$\frac{2\ell p_3^e}{h}\frac{N}{C}(6\zeta^2-\frac{1}{2})$	$\ell p^o \frac{N}{C}(6\zeta^2-\frac{1}{2})$	$\frac{2\ell\omega^e}{h}$	0
Z	$\ell p_3^e \frac{N}{C}(\zeta-4\zeta^3)$	$\frac{h\ell p^o}{2}\frac{N}{C}(\zeta-4\zeta^3)$	$-2\ell\omega^e\zeta$	0
V	0	0	0	$\frac{h\ell q^o}{C_{44}^e}\zeta$
DV	0	0	0	$\frac{\ell q^o}{C_{44}^e}$

Table 5 Thin-plate limit of the three-dimensional solution. Membrane behavior.

	p_3^o	p^e	ω^o	q^e
U	$-\ell p_3^o \frac{\mu}{C}$	$\frac{2\ell p^e}{hC}$	$\ell\omega^o \frac{F}{\epsilon C}$	0
S_3	$\ell h p_3^o \frac{\sigma}{C}\zeta$	$-2\ell p^e \frac{\mu}{C}\zeta$	$-\ell h\omega^o \frac{M}{\epsilon C}\zeta$	0
Φ	$\ell h p_3^o \frac{M}{\epsilon C}\zeta$	$-2\ell p^e \frac{F}{\epsilon C}\zeta$	$\ell h\omega^o \frac{G}{\epsilon C}\zeta$	0
DS_3	$\ell p_3^o \frac{\sigma}{C}$	$\frac{-2\ell p^e}{h}\frac{\mu}{C}$	$-\ell\omega^o \frac{M}{\epsilon C}$	0
$D\Phi$	$\ell p_3^o \frac{M}{\epsilon C}$	$\frac{-2\ell p^e}{h}\frac{F}{\epsilon C}$	$\ell\omega^o \frac{G}{\epsilon C}$	0
$DU - S_3$	$\frac{\ell h p_3^o L_{15}}{C_{44}^e}\frac{M}{C\epsilon}\zeta$	$\frac{2\ell p^e \rho}{C_{44}^e}\zeta$	$\frac{\ell h\omega^o L_{15}}{C_{44}^e}\frac{G}{\epsilon C}\zeta$	0
Q	$\frac{p_3^o h^2 \ell}{12}[\sigma + \frac{NM}{\epsilon C}](6\zeta^2-\frac{1}{2})$	$\frac{2\ell p^e}{h}$	$\frac{-h^2\ell\omega^o}{12}\frac{MK}{\epsilon C}(6\zeta^2-\frac{1}{2})$	0
R	ℓp_3^o	$\frac{h\ell p^e}{4}(1-4\zeta^2)$	$\frac{h^4\ell\omega^o}{384}\frac{MK}{\epsilon C}(1-8\zeta^2+16\zeta^4)$	0
W	$\frac{-p_3^o \ell h^3}{24}[\sigma + \frac{NM}{\epsilon C}](\zeta-4\zeta^3)$	$2\ell p^e\zeta$	$\frac{h^3\ell\omega^o}{24}\frac{MK}{\epsilon C}(\zeta-4\zeta^3)$	0
Y	$\ell h p_3^o \frac{M}{C}\zeta$	$2\ell p^e \frac{N}{C}\zeta$	$h\ell\omega^o \frac{G}{C}\zeta$	0
Z	$\frac{h^2 p_3^o \ell}{8}\frac{M}{C}(1-4\zeta^2)$	$\frac{hp^e\ell}{4}\frac{N}{C}(1-4\zeta^2)$	$-\ell\omega^o$	0
V	0	0	0	$\frac{2\ell q^e}{hC_{66}^e}$
DV	0	0	0	$\frac{2\ell q^e}{C_{44}^e}\zeta$

Table 6 Some formulas for a square thin plate. Bending behavior.

	sinusoidal p_0	uniform p_0	sinusoidal ω_0	uniform ω_0
w_m	$\frac{3A^4}{\pi^4 H^3}\frac{p_0}{C}$	$\frac{4.7485A^4}{\pi^4 H^3}\frac{p_0}{C}$	$\frac{-A^2}{2\pi^2 H}\frac{\omega_0 N}{\epsilon C}$	$\frac{-0.7271A^2}{\pi^2 H}\frac{\omega_0 N}{\epsilon C}$
Φ_m	$\frac{A^2}{2\pi^2 H}\frac{p_0 N}{\epsilon C}$	$\frac{0.7271A^2}{\pi^2 H}\frac{p_0 N}{\epsilon C}$	$\frac{A^2}{2\pi^2 H}\frac{\omega_0}{\epsilon}$	$\frac{0.7271A^2}{\pi^2 H}\frac{\omega_0}{\epsilon}$
M_{1m}	$\frac{A^2 p_0(1-\xi)}{2\pi^2}$	$\frac{0.7271A^2 p_0(1-\xi)}{\pi^2}$	$\frac{-H^2}{12}\frac{\omega_0 F\xi}{\epsilon}$	$\frac{-H^2}{12}\frac{\omega_0 F\xi}{\epsilon}$

Table 7 Some formulas for a square thin plate. Membrane behavior.

	sinusoidal p_0	uniform p_0	sinusoidal ω_0	uniform ω_0
ΔH_m	$H\dfrac{p_0\sigma}{C}$	$H\dfrac{p_0\sigma}{C}$	$-H\dfrac{\omega_0 M}{\epsilon C}$	$-H\dfrac{\omega_0 M}{\epsilon C}$
$\Delta\Phi_m$	$H\dfrac{p_0 M}{\epsilon C}$	$H\dfrac{p_0 M}{\epsilon C}$	$H\dfrac{\omega_0 G}{\epsilon C}$	$H\dfrac{\omega_0 G}{\epsilon C}$
N_{1m}	$Hp_0\mu\xi$	$Hp_0\mu\xi$	$-H\dfrac{\omega_0\xi F}{\epsilon}$	$-H\dfrac{\omega_0\xi F}{\epsilon}$

take into account any piezoelectric coupling. In particular, the deflection given by this theory depends only upon the elastic constants and not upon the piezoelectric and dielectric constants, contradicting the results given in Tables 4 and 6.

A different theory of thin piezoelectric plates was proposed by Tzou (1993). He assumes, as a blanket hypothesis, that the only nonvanishing component of the electric field is e_3 (cf. Section 3.2.2 in Tzou, 1993). The thin-plate limit of the three-dimensional solution obtained in this paper, reported in Tables 4 and 5, shows that the ratio between the in-plane components e_1, e_2 and the transverse component e_3 of the electric field is of the order h, unless $\omega^e \neq 0$, when this ratio is of the order h^{-1}. Hence, noting that h is a small quantity for a thin plate, the hypothesis of Tzou turns out to be oversimplifying, at least in the latter case.

A first-order shear deformation theory of piezoelectric plates was proposed by Mindlin (1972). It is based upon the assumptions that the in-plane displacement components s_1 and s_2 and the potential ϕ are linearly variable with respect to x_3, while the out-of-plane displacement component s_3 is constant with respect to x_3. But, the assumption concerning ϕ is not completely satisfactory, since from Table 4 it turns out that even in the case of a thin plate the potential ϕ has a quadratic law of variation with respect to x_3, when $p_3^e \neq 0$ or $p^o \neq 0$.

Therefore, the interest for a more satisfactory theory of piezoelectric plates arises. The results reported in Tables 4 and 5 suggest which ones of the mechanical and electric quantities may be neglected in the formulation of a consistent thin-plate theory: hence, they may represent a useful guide to develop such a theory.

8 A Numerical Investigation

A numerical investigation in the case of a square plate of edge $L_1 = L_2 = A$ and thickness H is performed, in order to evaluate the influence of the thickness-to-side ratio on the three-dimensional solution of the plate problem. Customarily, in the literature the load is applied according to a sinusoidal law (Pagano, 1970; Srinivas and Rao, 1970; Ray et al., 1992). Here a uniform distribution is adopted, more frequent in technical applications. Really, the load is distributed according to the function:

$$\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\frac{16}{\pi^2}\frac{(-1)^{n_1+n_2}}{(2n_1+1)(2n_2+1)} \times \cos\frac{(2n_1+1)\pi x_1}{A}\cos\frac{(2n_2+1)\pi x_2}{A}, \quad (58)$$

which converges uniformly to the constant function 1 in compact subsets of the open square $]-A/2, A/2[\times]-A/2, A/2[$

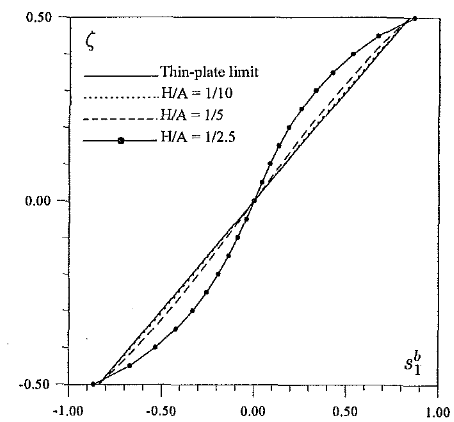

Fig. 1(*a*) Dimensionless side in-plane displacement s_1^b

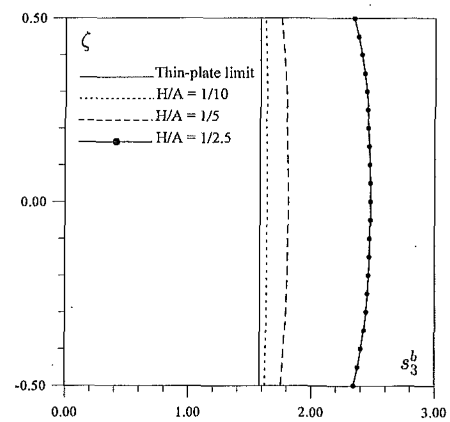

Fig. 1(*b*) Dimensionless central deflection s_3^b

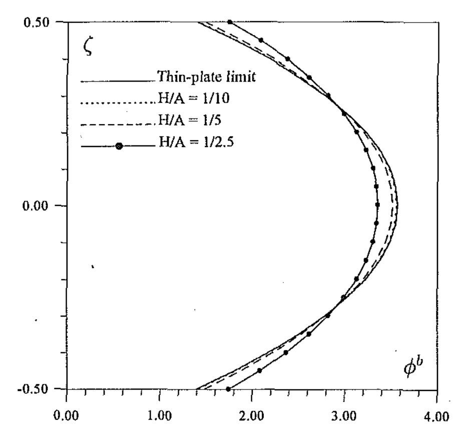

Fig. 1(*c*) Dimensionless central potential ϕ^b

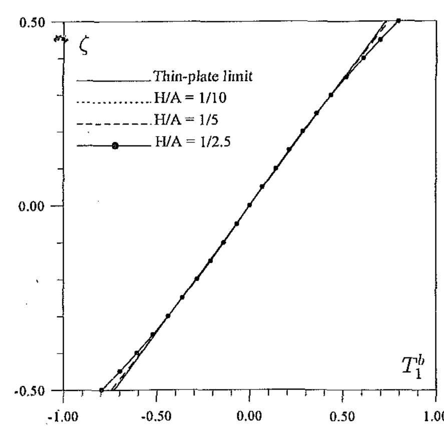

Fig. 1(*d*) Dimensionless central in-plane stress T_1^b

Fig. 1 Displacements, potential, and stress induced by q_0

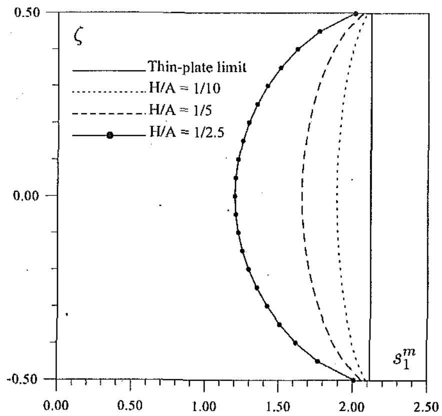

Fig. 2(a) Dimensionless side in-plane displacement s_1^m

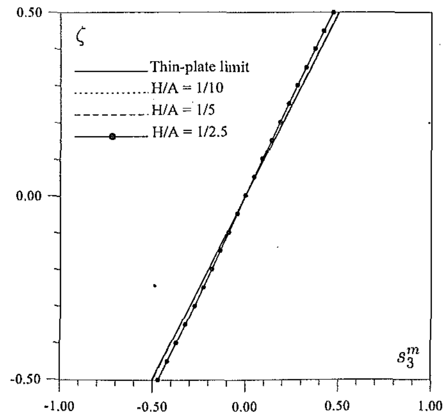

Fig. 2(b) Dimensionless central deflection s_3^m

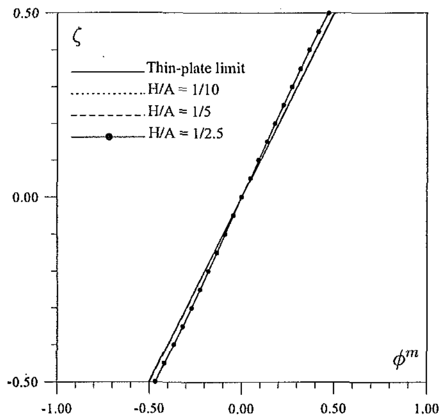

Fig. 2(c) Dimensionless central potential ϕ^m

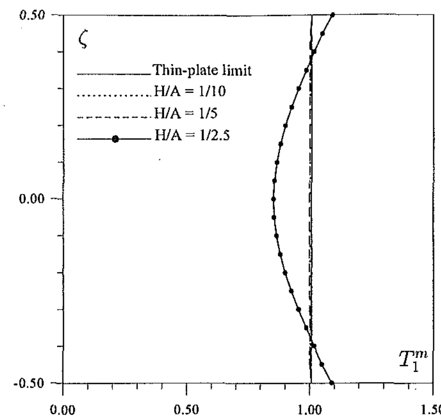

Fig. 2(d) Dimensionless central in-plane stress T_1^m

Fig. 2 Displacements, potential, and stress induced by ω_0

as $N_1, N_2 \to +\infty$. In the calculations it has been taken $N_1 = N_2 = 100$.

The results obtained are presented under the form of diagrams. The dimensionless variable ζ is reported on the ordinates axis. A dimensionless mechanical or electric quantity is reported on the abscissae axis. Four different lines are plotted in each diagram: three of them correspond to the values $H/A = 1/2.5$, $1/5$, $1/10$ of the thickness-to-side ratio of the plate; the fourth one represents the thin-plate limit. The material of the plate is the ceramic PZT-5H, presented in Table 1.

In Fig. 1 the case of a plate subjected to a uniform normal load of intensity p_0 ($p_0/2$ both on the upper and on the lower face: $p_3^+ = p_3^- = p_0/2$) is considered. In this case the plate has a bending behavior. The following dimensionless functions are plotted:

$$(a) \quad s_1^b(\zeta) := \frac{\pi^3 H^2 C}{3A^3 p_0}\, s_1\left(\frac{A}{2},\, 0,\, H\zeta\right)$$

$$(b) \quad s_3^b(\zeta) := \frac{\pi^4 H^3 C}{3A^4 p_0}\, s_3(0,\, 0,\, H\zeta)$$

$$(c) \quad \phi^b(\zeta) := \frac{2\pi^2 H \epsilon C}{A^2 F p_0}\, \phi(0,\, 0,\, H\zeta)$$

$$(d) \quad T_1^b(\zeta) := \frac{\pi^2 H^2}{6(1-\xi)A^2 p_0}\, T_1(0,\, 0,\, H\zeta).$$

In Fig. 2 the case of a plate subjected to a uniform surface charge of density ω_0 on the upper face and of density $-\omega_0$ on the lower face ($\omega^+ = \omega_0$, $\omega^- = -\omega_0$) is considered. In this case the plate has a membrane behavior. The following dimensionless functions are plotted:

$$(a) \quad s_1^m(\zeta) := \frac{2\pi\epsilon C}{AF\omega_0}\, s_1\left(\frac{A}{2},\, 0,\, H\zeta\right)$$

$$(b) \quad s_3^m(\zeta) := -\frac{\epsilon C}{HM\omega_0}\, s_3(0,\, 0,\, H\zeta)$$

$$(c) \quad \phi^m(\zeta) := \frac{\epsilon C}{HG\omega_0}\, \phi(0,\, 0,\, H\zeta)$$

$$(d) \quad T_1^m(\zeta) := -\frac{\epsilon}{F\xi\omega_0}\, T_1(0,\, 0,\, H\zeta).$$

It turns out that when the thickness-to-side ratio of the plate is less than or equal to $1/5$, the thin-plate limit provides results which agree with the exact ones within an error of 20 percent, for the evaluation of displacements, electric potential, and in-plane stress components; moreover, when that ratio is less than or equal to $1/10$, the maximum error reduces to 10 percent.

9 Conclusions

In this paper an exact three-dimensional solution for the problem of a simply supported transversely isotropic rectangular homogeneous piezoelectric plate is obtained in the framework of the linear theory of piezoelectricity.

The limit of this solution as the plate thickness aspect ratio approaches zero (thin-plate limit) is explicitly carried out. It represents a result to be agreed by a satisfactory theory of thin piezoelectric plates. This target doesn't seem to be reached by existing theories. The results obtained in the present paper may represent a useful guide to develop a consistent theory of thin piezoelectric plates.

A numerical investigation in the case of a square plate is also performed in order to evaluate the influence of the thickness-to-side ratio on the three-dimensional solution of the plate problem. Electric and mechanical uniformly distributed loads are applied on the plate. This corresponds to a very frequent case in technical applications. It turns out that, when the thickness-to-side ratio of the plate is less than or equal to 1/5 (1/10), the thin-plate limit provides results which agree with the exact ones within an error of 20 percent (10 percent), for the evaluation of displacements, electric potential and in-plane stress components.

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