Proceedings of the ASME 2009 28th International Conference on Ocean, Offshore and Arctic Engineering
OMAE2009
May 31 - June 5, 2009, Honolulu, Hawaii, USA

OMAE2009-80187

WAVE ELEVATION BETWEEN ELLIPTICAL STRUCTURES

Ioannis K. Chatjigeorgiou **Spyros A. Mavrakos**
National Technical University of Athens, Greece
School of Naval Architecture and Marine Engineering

ABSTRACT
The hydrodynamic interaction of waves with arrays of vertical elliptical cylinders is considered. The present paper aims at developing of an efficient calculation method for predicting the extreme elevation of the free surface, in the fluid domain between ship-shaped structures in close proximity. Linear potential theory is employed and the solution method is based on the semi-analytical formulation of the various velocity potentials in elliptical coordinates, using series expansions of Mathieu functions and the so-called addition theorem for Mathieu functions.

INTRODUCTION
The present paper deals with an important subject which is of both practical and academic interest. Namely, the extreme surface elevation, that is often observed by operators, between ship shaped structures in offshore applications into the open sea, which is apparently caused due to the resonant motion of the fluid confined between the vessels. It has been reported that the concerned impact is more pronounced in specific areas, the location of which with regard to length of the vessel, depends on the wave frequency and evidently, on the angle of heading.

Relevant hydrodynamic problems can be treated by numerical models which commonly implement panel methods. Nevertheless, panel methods are admittedly time consuming tools and strongly depend on the density of the grid. An alternative procedure is to approximate the required solutions analytically in cases where this is possible, which usually results to more robust, accurate and faster solution methodologies. In order to achieve this, several difficulties must be surmounted.

In the past, several researchers adopted analytical methods as tools for solving the hydrodynamic interaction problem among multiple bodies, usually for arrays of vertical cylinders (for example Mavrakos & Koumoutsakos (1987) for arbitrarily shaped multiple bodies of revolution with vertical symmetry axes, Spring & Monkmeyer (1974) and Linton & Evans (1990) for bottom fixed, free-surface piercing vertical cylinders). An extended review of the developed methods for analyzing the hydrodynamic interactions among multiple bodies has been presented by Newnan (2001), Linton & McIver (2001) and McIver (2002). With regard to elliptical cylinders, the implementation of analytical methods involves several challenges which mainly originate from the geometric complexity. The incident and the diffracted waves should be expressed with respect to elliptical coordinate systems and from the mathematical point of view, as series expansions of Mathieu functions. This can be traced back to the fact that the solutions of the Laplace equation (the governing field equation for inviscid, irrotational and incompressible flows), are the periodic and radial Mathieu functions (Moon and Spencer, 1971). Admittedly, Mathieu functions are not as popular as, say Bessel functions. In addition, different notations exist in the literature for representing these functions. Mathieu functions are referred as periodic and radial Mathieu functions (Meixner and Schäfke, 1954; Særmark, 1959; Nigsch, 2007), or even and odd periodic and radial Mathieu functions (Moon and Spencer, 1971; McLachlan, 1947; Abramowitz and Stegun, 1970). It appears however that the notation that has prevailed relies on the even and the odd periodic and radial Mathieu functions.

The works reported in the literature that treat the hydrodynamic interaction problem of incident wave fields with elliptical bodies using semi-analytical formulations, limit the investigation to single bodies (Williams, 1985a and 1985b; Williams and Darwiche, 1988). In order to consider arrays of elliptical cylinders and to provide a solution for the associated hydrodynamic problem, which apparently will involve the effect of the hydrodynamic interactions among the elliptical bodies, it is necessary to apply an addition theorem for Mathieu functions which should be similar in concept, with the well-known Graf's addition theorem for Bessel functions. The existence of an addition theorem for Mathieu functions in the notation Meixner and Schäfke (1954) was shown by Særmark (1959). Here, the addition theorem is properly treated in order to be expressed in terms of the even and the odd Mathieu functions, which in turn allows the use of the even and odd periodic and radial Mathieu functions for representing the velocity potentials of the incident and the diffracted waves.

FORMULATION OF THE HYDRODYNAMIC PROBLEM

The arrangement of the elliptical cylinders depicted in Fig. 1 (overview) is investigated. All bodies are considered fixed on the bottom. The bodies are exposed to the action of monochromatic incident waves of frequency ω and linear amplitude $H/2$, propagating at angle α to the positive x direction. The bodies are fixed in water of depth h. The large and the small radii of the kth body are denoted by a_k and b_k respectively.

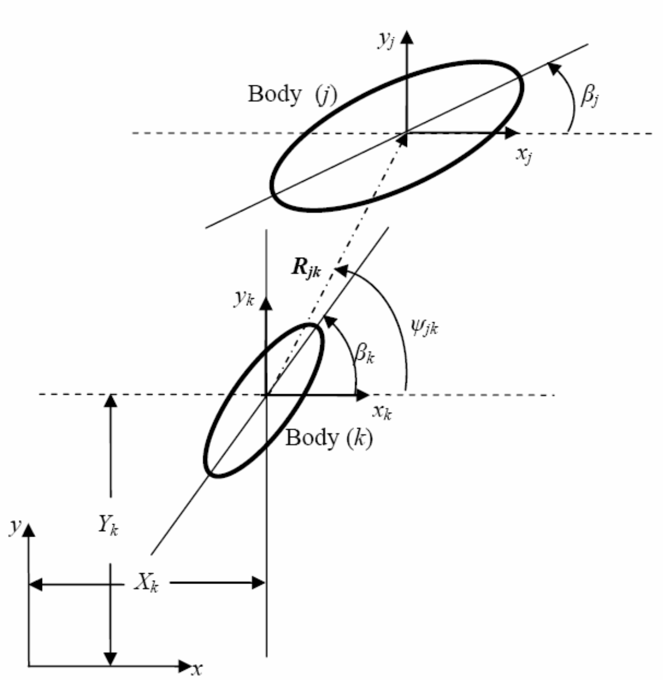

Figure 1: General arrangement. Coordinate systems and geometrical definitions

Elliptical cylindrical coordinates (u,v,z) are employed, u=constant, v=constant being orthogonaly intersecting families of confocal ellipses and hyperbolae, respectively. The z-axis is fixed on the bottom, pointing vertically upwards. The transformation from Cartesian to elliptical coordinates is

$$x = c\cosh u \cos v \tag{1}$$

$$y = c\sinh u \sin v \tag{2}$$

where $c=(a^2-b^2)^{1/2}=a\varepsilon$, ε being the elliptic eccentricity given by $\varepsilon^2=1-(b/a)^2$.

Higher order effects are neglected and the assumption is made that the fluid is inviscid, incompressible and irrotational. Thus, linear potential theory can be employed, meaning that the fluid's motion can be described by the first-order velocity potential, which in elliptical coordinates is expressed as

$$\phi(u,v,z;t) = \mathrm{Re}\left(-i\omega\frac{H}{2}\varphi(u,v,z)\mathrm{e}^{-\mathrm{i}\omega t}\right) \tag{3}$$

It follows that the velocity potential must satisfy the Laplace equation

$$\nabla^2\varphi = 0 \tag{4}$$

anywhere in the fluid domain, the kinematical condition on the bottom

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = 0 \tag{5}$$

and the linearized condition on the free surface

$$\left(-\frac{\omega^2}{g}\varphi + \frac{\partial\varphi}{\partial z}\right)_{z=h} = 0 \tag{6}$$

where g is the gravitational acceleration.

The velocity potential must also satisfy the kinematical conditions on the wetted surface of all bodies

$$\left(\frac{\partial\varphi}{\partial u}\right)_{u=u_0} = 0, \quad 0 \le z \le h \tag{7}$$

where u_0 stands for the radial boundary of any body with respect to its local elliptical coordinate system.

In the context of the linear theory, the velocity potential is decomposed into the incident wave potential φ_I and the total scattered potential that involves the scattering of waves by all bodies φ_S. Thus

$$\varphi = \varphi_I + \varphi_S \tag{8}$$

In addition to Eqs. (4)-(6), the total scattered potential must satisfy an appropriate radiation condition which allows only outgoing waves at infinity. In elliptical coordinates it is expressed as

$$\lim_{u\to\infty}(c\cosh u)^{1/2}\left\{\frac{1}{c\sinh u}\frac{\partial}{\partial u} - \mathrm{i}k_0\right\}\varphi_S = 0 \tag{9}$$

where k_0 is the wave number given by the dispersion relation

$$\frac{\omega^2}{g} = k_0\tanh(k_0 h) \tag{10}$$

INCIDENT WAVE POTENTIAL

When dealing with a multibody arrangement, all velocity potentials which are involved in the associated hydrodynamic problem should be expressed with respect to the local coordinates of each constituent body.

Let (x_k,y_k,z) be the Cartesian coordinates of any point in the reference field with respect to the local Cartesian coordinate system of body *k*. Then, the incident wave potential will be given by

$$\varphi_I = \frac{g}{\omega^2}\frac{Z_0(z)}{Z_0(h)}\Lambda_k \mathrm{e}^{\mathrm{i}k_0(x_k\cos\alpha + y_k\sin\alpha)} \quad (11)$$

where

$$Z_0(z) = N_0^{-1/2}\cosh(k_0 z)\,, \quad (12)$$

$$N_0 = \frac{1}{2}\left[1 + \frac{\sinh(k_0 h)}{2k_0 h}\right] \quad (13)$$

and

$$\Lambda_k = \mathrm{e}^{\mathrm{i}k_0(X_k\cos\alpha + Y_k\sin\alpha)} \quad (14)$$

where X_k, Y_k are the Cartesian coordinates of the center of body *k* with respect to the global Cartesian coordinate system (Fig.1).

The incident wave potential is now expressed in terms of the local elliptical coordinate system of body *k* (u_k,v_k,z) (Meixner and Schäfke, 1954).

$$\frac{1}{h}\varphi_I = \frac{1}{(\omega^2/g)h}\frac{Z_0(z)}{Z_0(h)}\Lambda_k \sum_{m=-\infty}^{\infty} e_m(\alpha;q_k)\mathrm{M}_m^{(1)}(u_k;q_k)\mathrm{me}_m(v_k;q_k) \quad (15)$$

$$e_m(\alpha;q_k) = \begin{cases} \mathrm{i}^m \mathrm{me}_m(\alpha;q_k) & m \ge 0 \\ -\mathrm{i}^m \mathrm{me}_m(\alpha;q_k) & m < 0 \end{cases} \quad (16)$$

In Eqs. (15) and (16) $q_k = (k_0 c_k/2)^2$ is the Mathieu parameter, $\mathrm{me}_m(v_k;q_k)$ is the periodic Mathieu function and $\mathrm{M}_m^{(1)}(u_k;q_k)$ is the radial Mathieu function (also referred as modified Mathieu function) of the first kind, in the notation of Meixner and Schäfke (1954).

The notation adopted in the present work is that of Abramowitz and Stegun (1970). This requires the transformation of the Mathieu functions and the Modified Mathieu functions to even and odd periodic Mathieu functions $\mathrm{ce}_m(v_k;q_k)$, $\mathrm{se}_m(v_k;q_k)$ and to even and odd radial Mathieu functions $\mathrm{Mc}_m^{(j)}(u_k;q_k)$, $\mathrm{Ms}_m^{(j)}(u_k;q_k)$ respectively. To this end, the following transformation formulas are used (Meixner and Schäfke, 1954)

$$\mathrm{ce}_m(v,q) = 2^{-1/2}\mathrm{me}_m(v,q) \qquad (m = 0,1,2,...) \quad (17)$$

$$\mathrm{se}_m(v,q) = \mathrm{i}2^{-1/2}\mathrm{me}_{-m}(v,q) \qquad (m = 1,2,3,...) \quad (18)$$

$$\mathrm{M}_m^{(j)}(u;q) = \mathrm{Mc}_m^{(j)}(u;q) \qquad (m = 0,1,2,...) \quad (19)$$

$$(-1)^m \mathrm{M}_{-m}^{(j)}(u;q) = \mathrm{Ms}_m^{(j)}(u;q) \qquad (m = 1,2,3,...) \quad (20)$$

In the Eqs. (19) and (20) the index (*j*) denotes the kind of the radial Mathieu functions.

Using Eqs. (17)-(20), the incident wave potential, Eq. (15), is recast to

$$\frac{1}{h}\varphi_I = \frac{2}{(\omega^2/g)h}\frac{Z_0(z)}{Z_0(h)}\Lambda_k \left\{ \sum_{m=0}^{\infty} \mathrm{i}^m \mathrm{Mc}_m^{(1)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k)\mathrm{ce}_m(\alpha;q_k) + \sum_{m=1}^{\infty} \mathrm{i}^m \mathrm{Ms}_m^{(1)}(u_k;q_k)\mathrm{se}_m(v_k;q_k)\mathrm{se}_m(\alpha;q_k) \right\} \quad (21)$$

Eq. (21) represents the incident wave potential with respect to the local elliptical coordinate system of the arbitrarily selected body *k*. In order to apply the zero velocity condition on the wetted surface of body *k*, the total scattered potential due to the scattering of waves by all bodies, should be expressed with respect to the same coordinate system and in particular with respect to the local elliptical system of body *k*. This requirement is achieved with the analysis outlined in the following section.

TOTAL SCATTERED POTENTIAL

The total scattered potential around the *k*th body should include the wave scattering components from all bodies of the arrangement. As a result, it holds that

$$\varphi_S = \sum_{k=1}^{N} \varphi_S^{(k)} \quad (22)$$

where *N* is the number of bodies being considered.

The scattered wave field around the *k*th body of the multi-body arrangement, $\varphi_S^{(k)}$, that satisfies Eqs. (4) - (6) and (9), can be expressed in the elliptical coordinate system of body *k* in terms of the Mathieu functions and the Modified Mathieu functions as follows:

$$\frac{1}{h}\varphi_S^{(k)}(u_k,v_k,z) = Z_0(z)\sum_{m=-\infty}^{\infty} F_m^{(k)}\mathrm{M}_m^{(3)}(u_k;q_k)\mathrm{me}_m(v_k;q_k) \quad (23)$$

Here, $\mathrm{M}_m^{(3)}(u_k;q_k)$ is the modified Mathieu function of the third kind, $\mathrm{M}_m^{(3)}(u_k;q_k) = \mathrm{M}_m^{(1)}(u_k;q_k) + \mathrm{i}\mathrm{M}_m^{(2)}(u_k;q_k)$ and $F_m^{(k)}$ denote unknown coefficients, which will be obtained by applying the zero velocity condition on the wetted surfaces of all bodies when the total wave field around the body *k*, see Eq. (22), is properly formulated.

Next, using again the expressions (17) - (20), Eq. (23) is transformed into

$$\frac{1}{h}\varphi_S^{(k)} = \sum_{m=0}^{\infty} A_m^{(k)} \mathrm{Mc}_m^{(3)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k) + Z_0(z)\sum_{m=1}^{\infty} B_m^{(k)} \mathrm{Ms}_m^{(3)}(u_k;q_k)\mathrm{se}_m(v_k;q_k) \tag{24}$$

Apparently, the Fourier coefficients $F_m^{(k)}$ in Eq. (23), have been replaced in Eq. (24) by $A_m^{(k)}$ and $B_m^{(k)}$.

For convenience, we choose to work with the following form of the scattered potential:

$$\frac{1}{h}\varphi_S^{(k)} = Z_0(z)\sum_{m=0}^{\infty} \mathrm{i}^m A_m^{(k)} \mathrm{Kc}_m^{(k)} \mathrm{Mc}_m^{(3)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k) + Z_0(z)\sum_{m=1}^{\infty} \mathrm{i}^m B_m^{(k)} \mathrm{Ks}_m^{(k)} \mathrm{Ms}_m^{(3)}(u_k;q_k)\mathrm{se}_m(v_k;q_k) \tag{25}$$

where the artificially introduced terms $\mathrm{Kc}_m^{(k)}$ and $\mathrm{Ks}_m^{(k)}$ are given by

$$\mathrm{Kc}_m^{(k)} = \frac{\mathrm{Mc}_m^{(1)\prime}(u_{k_0};q_k)}{\mathrm{Mc}_m^{(3)\prime}(u_{k_0};q_k)} \tag{26}$$

$$\mathrm{Ks}_m^{(k)} = \frac{\mathrm{Ms}_m^{(1)\prime}(u_{k_0};q_k)}{\mathrm{Ms}_m^{(3)\prime}(u_{k_0};q_k)} \tag{27}$$

The primes herein, denote differentiation with respect to the argument of the associated Mathieu function and u_{k0} is the radial boundary of body k with respect to its local elliptical coordinate system. Now, in accordance to (22) by superposing the individual scattered wave fields around each body, Eq. (25), the total scattered wave field around the body k due to all bodies of the arrangement can be formulated as:

$$\begin{aligned}\frac{1}{h}\varphi_S &= Z_0(z)\sum_{m=0}^{\infty} \mathrm{i}^m A_m^{(k)} \mathrm{Kc}_m^{(k)} \mathrm{Mc}_m^{(3)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k) \\ &+ Z_0(z)\sum_{m=1}^{\infty} \mathrm{i}^m B_m^{(k)} \mathrm{Ks}_m^{(k)} \mathrm{Ms}_m^{(3)}(u_k;q_k)\mathrm{se}_m(v_k;q_k) \\ &+ Z_0(z)\sum_{j\neq k}^{N}\sum_{m=0}^{\infty} \mathrm{i}^m A_m^{(j)} \mathrm{Kc}_m^{(j)} \mathrm{Mc}_m^{(3)}(u_j;q_j)\mathrm{ce}_m(v_j;q_j) \\ &+ Z_0(z)\sum_{j\neq k}^{N}\sum_{m=0}^{\infty} \mathrm{i}^m B_m^{(j)} \mathrm{Ks}_m^{(j)} \mathrm{Ms}_m^{(3)}(u_j;q_j)\mathrm{se}_m(v_j;q_j)\end{aligned} \tag{28}$$

The goal herein is to express the total scattered wave field with respect to the local elliptical coordinate system of body k, as was done previously for the incident wave potential. Thus, the products of Mathieu functions in the last two terms of the right hand side of Eq. (28), expressed with respect to the local elliptical coordinate systems (u_j,v_j,z) of each body of the arrangement, should be reduced with respect to (u_k,v_k,z) of body k. This will be performed in the following section using the so-called addition theorem for Mathieu functions.

ADDITION THEOREM FOR MATHIEU FUNCTIONS

The existence of an addition theorem for Mathieu functions was shown by Særmark (1959) who extended the formulas reported in Meixner and Schäfke (1954) in terms of the Bessel functions. In particular, Særmark (1959) showed that the addition theorem for Mathieu functions is described through the following relation (for the geometrical definitions, see Fig. 1):

$$\mathrm{M}_m^{(l)}(u_j;q_j)\mathrm{me}_m(v_j;q_j) = \sum_{n=-\infty}^{\infty} \mathrm{B}_{n,m}^{(l)} \mathrm{M}_n^{(1)}(u_k;q_k)\mathrm{me}_n(v_k;q_k) \tag{29}$$

where

$$\mathrm{B}_{n,m}^{(l)} = \sum_{s=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\left\{d'_{n-p,p}(q_k) Z_{p-s}^{(l)}(k_0 R_{jk}) d_{s-m,m}(q_j)\right\} \mathrm{e}^{\mathrm{i}(s-p)\psi_{jk}}\,\mathrm{e}^{-\mathrm{i}s(\beta_j-\beta_k)} \tag{30}$$

$$Z_m^{(l)}(k_0R_{jk}) = \begin{cases} \mathrm{J}_m(k_0R_{jk}) & l=1 \\ \mathrm{Y}_m(k_0R_{jk}) & l=2 \\ \mathrm{H}_m^{(1)}(k_0R_{jk}) & l=3 \\ \mathrm{H}_m^{(2)}(k_0R_{jk}) & l=4 \end{cases} \tag{31}$$

where J_m and Y_m are the Bessel functions of the first and the second kind respectively and $\mathrm{H}_m^{(1)}=\mathrm{J}_m+\mathrm{iY}_m$, $\mathrm{H}_m^{(2)}=\mathrm{J}_m-\mathrm{iY}_m$ are the Hankel functions

In Eq. (30) the coefficients $d'_{n-p,p}(q)$ and $d_{s-m,m}(q)$ are given in terms of the complex expansion coefficients C, of the periodic Mathieu functions (Meixner and Schäfke, 1954; Særmark, 1959), namely

$$\mathrm{me}_m(v;q) = \sum_{s=-\infty}^{\infty} C_{2s}^m(q)\mathrm{e}^{\mathrm{i}(m+2s)v} \tag{32}$$

According to Særmark (1959)

$$d_{2s,m}(q)=(-1)^s C_{2s}^m(q),\quad d_{2s+1,m}(q)=0 \tag{33}$$

$$d'_{2n,p}(q)=(-1)^n C_{-2n}^{p+2n}(q),\quad d'_{2n+1,p}(q)=0 \tag{34}$$

Since the scattered wave field are expressed in terms of the even and odd periodic and radial Mathieu functions, it is more appropriate for calculation purposes, to replace the *d* and *d’* coefficients with the *A* and *B* expansion coefficients, which are related to the even and odd periodic Mathieu functions according to

$$\mathrm{ce}_{2m}(v;q)=\sum_{r=0}^{\infty} A_{2r}^{2m}(q)\cos 2rv \tag{35}$$

$$\mathrm{ce}_{2m+1}(v;q)=\sum_{r=0}^{\infty} A_{2r+1}^{2m+1}(q)\cos(2r+1)v \tag{36}$$

$$\mathrm{se}_{2m+1}(v;q)=\sum_{r=0}^{\infty} B_{2r+1}^{2m+1}(q)\sin(2r+1)v \tag{37}$$

$$\mathrm{se}_{2m+2}(v;q)=\sum_{r=0}^{\infty} B_{2r+2}^{2m+2}(q)\sin(2r+2)v \tag{38}$$

To this end, the associated relations that can be found in Meixner and Schäfke (1954) will be employed. In particular, after short mathematical processing it can be shown that the *d* and *d’* coefficients which are involved in Eq. (30) can be given by

$$\begin{aligned}
d'_{n-p,p}(q)&=2^{-1/2}(-1)^{(n-p)/2}A_p^n(q) && (n=0,1,2,...)\\
d'_{-n-p,p}(q)&=-2^{-1/2}(-1)^{(-n-p)/2}B_p^n(q) && (n=1,2,3,...)\\
d'_{2n,0}(q)&=2^{1/2}(-1)^n A_0^{2n}(q) && p=0
\end{aligned} \tag{39}$$

$$\begin{aligned}
d_{s-m,m}(q)&=2^{-1/2}(-1)^{(s-m)/2}A_s^m(q) && (m=0,1,2,...)\\
d_{s+m,-m}(q)&=-2^{-1/2}(-1)^{(s+m)/2}B_s^m(q) && (m=1,2,3,...)\\
d_{-2m,2m}(q)&=2^{1/2}(-1)^{-m}A_0^{2m}(q) && s=0
\end{aligned} \tag{40}$$

for *n-p* and *s-m*, both even, otherwise *d* and *d’* are zero.

The transformation formulas (17)-(20) enable introducing Eq. (29) into Eq. (28), which after extensive mathematical manipulations will obtain the following form

$$\begin{aligned}
\frac{1}{h}\varphi_S &= Z_0(z)\sum_{m=0}^{\infty}\mathrm{i}^m A_m^{(k)}\mathrm{Kc}_m^{(k)}\mathrm{Mc}_m^{(3)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k)\\
&+Z_0(z)\sum_{m=1}^{\infty}\mathrm{i}^m B_m^{(k)}\mathrm{Ks}_m^{(k)}\mathrm{Ms}_m^{(3)}(u_k;q_k)\mathrm{se}_m(v_k;q_k)\\
&+Z_0(z)\sum_{j\neq k}^{N}\sum_{m=0}^{\infty}\mathrm{i}^m A_m^{(j)}\mathrm{Kc}_m^{(j)}\\
&\left[\sum_{r=0}^{\infty}\mathrm{B}_{r,m}^{(3)}\mathrm{Mc}_r^{(1)}(u_k;q_k)\mathrm{ce}_m(v_k;q_k)\right.\\
&\left.-\mathrm{i}\sum_{r=1}^{\infty}(-1)^{-r}\mathrm{B}_{-r,m}^{(3)}\mathrm{Ms}_r^{(1)}(u_k;q_k)\mathrm{se}_m(v_k;q_k)\right]\\
&+Z_0(z)\sum_{j\neq k}^{N}\sum_{m=1}^{\infty}\mathrm{i}^m B_m^{(j)}\mathrm{Ks}_m^{(j)}\\
&\left[\sum_{r=1}^{\infty}(-1)^{m-r}\mathrm{B}_{-r,-m}^{(3)}\mathrm{Ms}_r^{(1)}(u_k;q_k)\mathrm{se}_m(v_k;q_k)\right.\\
&\left.+\mathrm{i}\sum_{r=0}^{\infty}(-1)^m\mathrm{B}_{r,-m}^{(3)}\mathrm{Ms}_r^{(1)}(u_k;q_k)\mathrm{se}_m(v_k;q_k)\right]
\end{aligned} \tag{41}$$

Eq. (41) represents the velocity potential of the total scattered wave field with respect to the local elliptical coordinate system of the arbitrarily selected body *k*. The zero velocity condition on the wetted surface of body *k*, requires that

$$\frac{\partial\varphi_S}{\partial u_k}+\frac{\partial\varphi_I}{\partial u_k}=0 \text{ at } u_k=u_{k_0} \tag{42}$$

After introducing Eqs. (21) and (41) into Eq. (42), separating even and odd terms and equating the same orders of $\mathrm{ce}_m(v_k;q_k)$ and $\mathrm{se}_m(v_k;q_k)$ in the even and odd products respectively, the following is derived:

$$\begin{aligned}
&A_m^{(k)}+\sum_{j\neq k}^{N}\sum_{r=0}^{\infty}\mathrm{i}^{r-m}A_r^{(j)}\mathrm{Kc}_r^{(j)}\mathrm{B}_{m,r}^{(3)}\\
&+\mathrm{i}\sum_{j\neq k}^{N}\sum_{r=1}^{\infty}\mathrm{i}^{r-m}(-1)^r B_r^{(j)}\mathrm{Ks}_r^{(j)}\mathrm{B}_{m,-r}^{(3)}\\
&=-\frac{2}{vhZ_0(h)}\Lambda_k\mathrm{ce}_m(\alpha;q_k)
\end{aligned} \tag{43}$$

$$B_m^{(k)}-\mathrm{i}\sum_{j\neq k}^{N}\sum_{r=0}^{\infty}\mathrm{i}^{r-m}(-1)^{-m}A_r^{(j)}\mathrm{Kc}_r^{(j)}\mathrm{B}_{-m,r}^{(3)}$$

$$+\sum_{j\neq k}^{N}\sum_{r=1}^{\infty}\mathrm{i}^{r-m}(-1)^{r-m}B_r^{(j)}\mathrm{Ks}_r^{(j)}\mathrm{B}_{-m,-r}^{(3)}$$

$$=-\frac{2}{\nu hZ_0(h)}\Lambda_k\,\mathrm{se}_m(\alpha;q_k) \tag{44}$$

Eqs. (43) and (44) represent a $2\times(M+1)\times N$ complex truncated linear system, where M is the number of Fourier coefficients being considered and N is the number of bodies. This system can be solved using efficient methods of linear algebra. Finally, the total velocity potential is given by superposing the incident waves and the scattered waves by all bodies, both expressed with respect to the local elliptical coordinate system of body k. Thus, using Eqs. (21) and (41) to calculate $\varphi_I+\varphi_S$ and substituting Eqs. (43) and (44) into the resulting product, the following simple formula is derived:

$$\frac{1}{h}\varphi(u_k,v_k,z)=Z_0(z)$$

$$\left\{\sum_{m=0}^{\infty}\mathrm{i}^m A_m^{(k)}\mathrm{ce}_m(v_k;q_k)\left[\mathrm{Kc}_m^{(k)}\mathrm{Mc}_m^{(3)}(u_k;q_k)-\mathrm{Mc}_m^{(1)}(u_k;q_k)\right]\right.$$

$$\left.+\sum_{m=1}^{\infty}\mathrm{i}^m B_m^{(k)}\mathrm{se}_m(v_k;q_k)\left[\mathrm{Ks}_m^{(k)}\mathrm{Ms}_m^{(3)}(u_k;q_k)-\mathrm{Ms}_m^{(1)}(u_k;q_k)\right]\right\} \tag{45}$$

RESULTS AND DISCUSSION

The derivation of the total velocity potential allows the numerical calculation of all quantities that determine the hydrodynamic behavior of the multibody arrangement, namely, the hydrodynamic loading, the hydrodynamic pressure distribution and finally, the free surface elevation, which constitutes the scope of interest of the present contribution. The non-dimensional spatial dependence of the free surface elevation around any body with respect to its elliptical coordinate system is given by

$$\frac{\eta(u_k,v_k,h)}{H/2}=\nu h\varphi(u_k,v_k,h) \tag{46}$$

A single characteristic test case is examined, that could simulate the approach of two ship-shaped structures. The two elliptical cylinders have the same dimensions and they have been placed parallel, with a normal imaginable connection arm (Fig. 2). Thus, $a_1=a_2=a$, $b_1=b_2=b$. In addition the following dimensions are considered: b/a=0.25, h/a=1.5 and R/a=2. The numerical results examine two angles of wave heading, namely, 0^o and 90^o, (Figs 3-6 and 7-10 respectively) with respect to the global Cartesian coordinate system shown in Fig. 2. Here the global Cartesian coordinate system has been placed in the center of body 1.

For the 0^o of heading, the investigated frequencies of the incident waves correspond to k_0a=0.8 and 3.0 (Figs. 3-4 and 5-6, respectively), while for the 90^o of heading the values of k_0a were taken equal to 1.5 and 2.5 (Figs. 7-8 and 9-10, respectively). For all frequencies and wave headings, the wave elevation is plotted using local elliptical coordinate systems of bodies 1 and 2. The plots must be seen separately as they correspond to different systems and apparently the locations where the contours are given do not coincide. As expected, for the 0^o heading, according to which the waves propagate from negative to positive y-axis, the contours of the wave elevation exhibit apparent similarities. In fact, the contours around body 1 are nearly a specular reflection of the contours around body 2. Also, it is very important to note that the flow in the intermediate region between the bodies is not uniform. In addition, the flow is completely different for different frequencies (k_0a=0.8 and 3.0 in the present case). Figs. 3 and 4 show that notable elevations occur, to the right of the bow of body 1 and to the left of the bow of body 2. A respective phenomenon is observed for the higher wave frequency (Figs. 5 and 6). Here the flow has been completely disturbed, while the maxima of the wave elevation are detected just in front of the sterns and behind the bows of the bodies. Finally, it is immediately apparent that the waves do not propagate freely between the bodies, even in the low frequency case, as the diffraction phenomena are dominant.

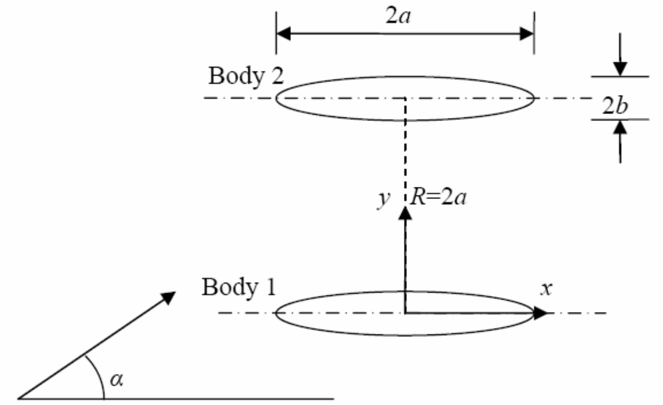

Figure 2: Test case. Two similar parallel elliptical cylinders

For the 90^o of heading (Figs. 7-10) the contours are obviously non catoptrical. The diffraction phenomena are again dominant and cause pronounced elevations in both the outside areas and between the bodies. The principal hydrodynamic action is exerted on body 1 which faces the front of the incoming waves (Figs. 7 and 9). The waves here propagate from negative to positive x-axis. The left sides of Figs. 7 and 9 show the initiation of the diffraction phenomena which are followed by pronounced disturbances in the area between the bodies. The flow starts to be normalized after it has passed the protected body 2 (right sides of Figs. 8 and 10). The diffraction phenomena are more influential for the higher wave frequency case (Figs. 9 and 10) where the simulacrum of symmetry with respect to y-axis, that can be observed for k_0a=1.5, vanishes completely. Finally, the plots for the 90^o of wave heading, demonstrate that the second elliptical cylinder (body 2) is not fully protected as the wave elevation between the bodies obtains significant values.

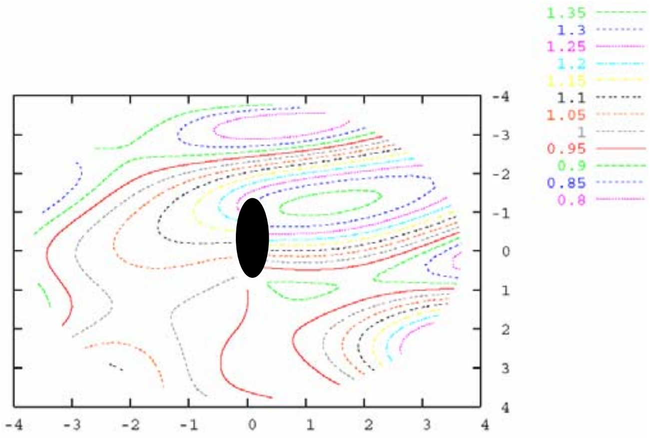

Figure 3: Body 1, 0^o heading, k_0a=0.8

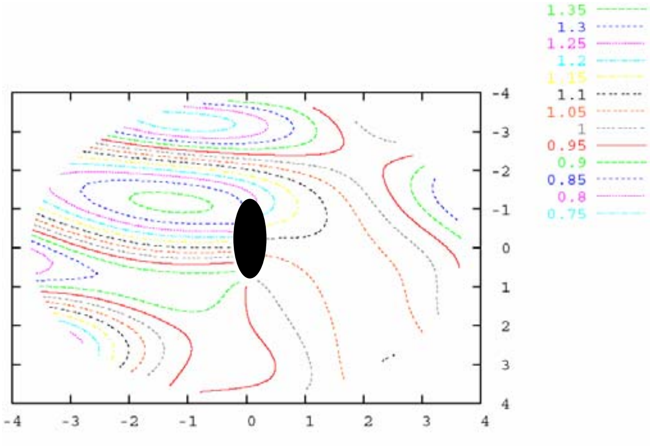

Figure 4: Body 2, 0^o heading, k_0a=0.8

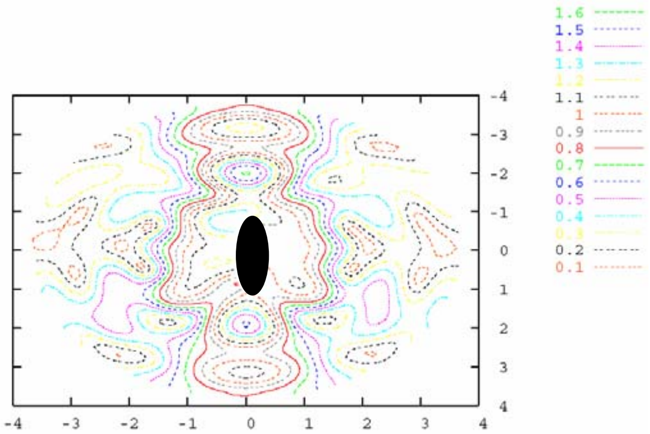

Figure 5: Body 1, 0^o heading, k_0a=3.0

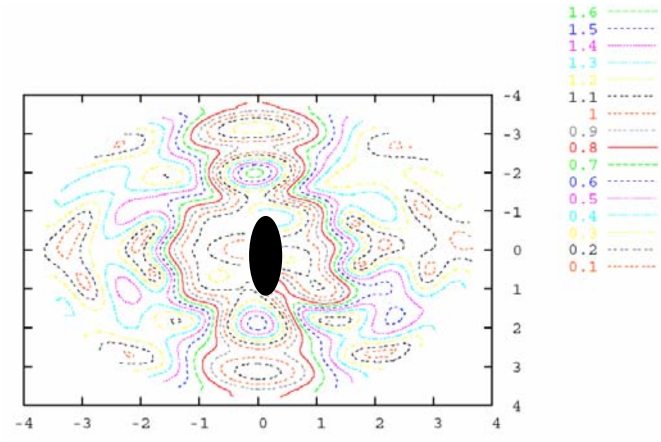

Figure 6: Body 2, 0^o heading, k_0a=3.0

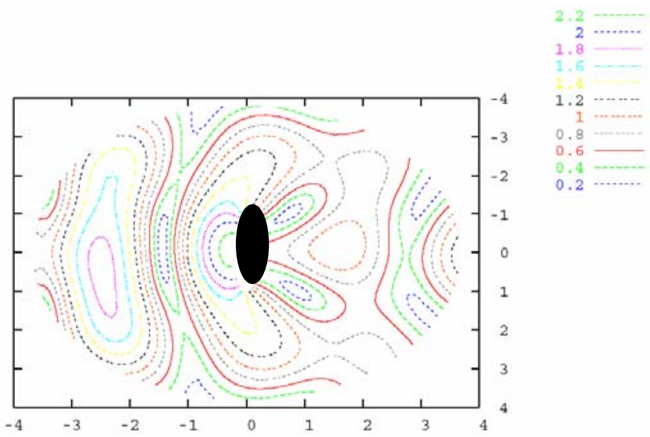

Figure 7: Body 1, 90^o heading, k_0a=1.5

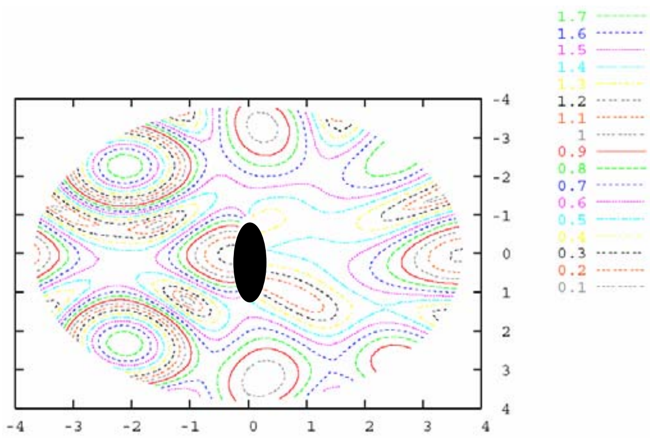

Figure 8: Body 2, 90^o heading, k_0a=1.5

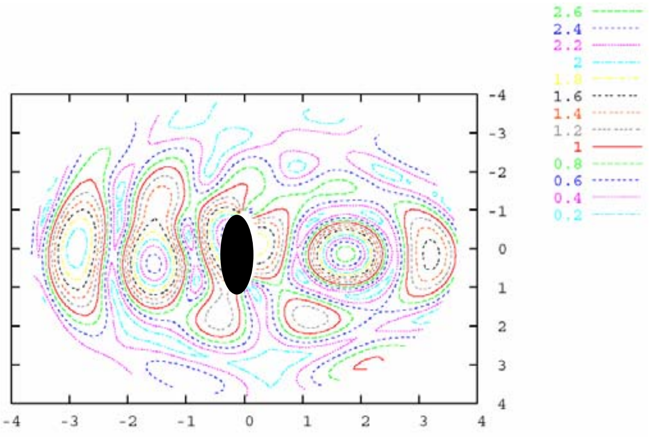

Figure 9: Body 1, 90° heading, k_0a=2.5

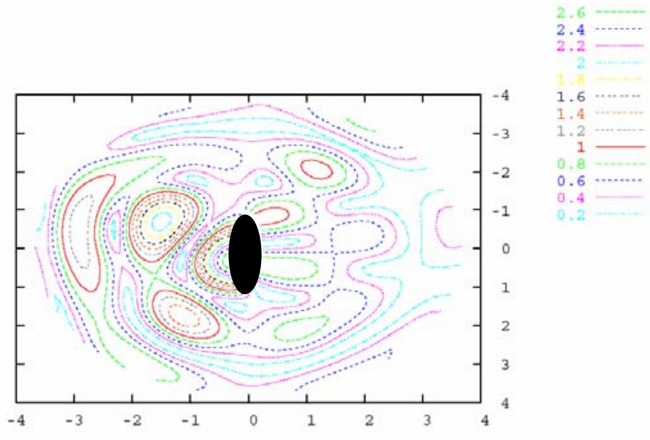

Figure 10: Body 2, 90° heading, k_0a=2.5

CONCLUSIONS

The hydrodynamic diffraction by arrays of elliptical cylinders was considered. The solution method was based on the semi-analytical formulation of the incident wave and the scattered wave field that were expressed in elliptical coordinate systems. The final solution was achieved using the addition theorem for Mathieu functions which for the purposes of the present contribution was expressed in terms of the even and odd periodic and radial Mathieu functions.

The goal of the present paper was to estimate the basic characteristics of the wave elevation in the fluid domain that encompasses ship-shaped structures in close proximity. The numerical results show that, depending on the wave frequency and the angle of incidence, the diffraction phenomena may cause pronounced elevations of the free surface, both in the area outside the bodies and in the intermediate domain. It should be stated that the proper evaluation of the elevation in the intermediate area is of paramount importance for practical applications as it can lead to several unwanted wave-structure interaction impacts which are related to the resonance of the water field between the bodies.

REFERENCES

Abramowitz, M., Stegun, I.A., 1970, *Handbook of Mathematical Functions*, Dover Publications Inc, New York.

Kagemoto, H., Yue, D.K.P., 1986, "Interactions among multiple three-dimensional bodies in water waves: an exact algebraic method", *Journal of Fluid Mechanics*, 166, 189–209.

Linton, C.M., Evans, D.V., 1990, "Interaction of Waves with Arrays of Vertical Circular Cylinders", *Journal of Fluid Mechanics* 215, 549-569.

Linton, C.M., McIver, P., 2001, *Handbook of Mathematical Techniques for Wave/structure Interactions*, CRC Press, ISBN 1584881321, 9781584881322.

Mavrakos, S.A., Koumoutsakos, P., 1987, "Hydrodynamic Interaction Among Vertical Axisymmetric Bodies Restrained in Waves", *Applied Ocean Research*, 9, 128-140.

McIver, P., 2002, "Wave Interaction with Arrays of Structures", *Applied Ocean Research*, 24, 121-126.

McLachlan, N.W., 1947, *Theory and Applications of Mathieu Functions*, Dover Publications, New York..

Meixner, J., Schäfke, F.W., 1954, *Mathieusche Funktionen und Sphäroidfunktionen*, Springer, Berlin

Moon, P., Spencer, D.E., 1971, *Field Theory Handbook*, Springer-Verlag, Berlin, 1971

Newman, J.N., 2001, "Wave Effects on Multiple Bodies. In: Hydrodynamics in Ship and Ocean Engineering", Research Institute for Applied Mechanics, Kyushu University, Japan, 3-26.

Nigsch, M., 2007, "Numerical Studies of Time-independent and Time-dependent Scattering by Several Elliptical Cylinders", *Journal of Computational and Applied Mathematics*, 204, 231-241.

Særmark, K., 1959, A Note on Addition Theorems for Mathieu Functions, *ZAMP* 10, 426-428

Spring, B.H. and Monkmeyer, P.L., 1974, "Interaction of Plane Waves with Vertical Cylinders, Proceedings, *14th Int. Conf. on Coastal Engineering*, 107, 1828 – 1849.

Williams, A.N., 1985a, "Wave Forces on an Elliptic Cylinder", *Journal of Waterways, Port, Coastal Ocean* Div. American Society of Civil Engineers, 111, 433-449.

Williams, A.N., 1985b, "Wave Diffraction by Elliptical Breakwaters in Shallow Water", *Ocean Engineering*, 12, 25-43.

Williams, A.N., Darwiche, M.K., 1988, "Three Dimensional Wave Scattering by Elliptical Breakwaters", *Ocean Engineering*, 15, 103-118.

