South Asian Journal of Mathematics 2012 , Vol. 2 (1) : 73 ∼ 77

www.sajm-online.com ISSN 2251-1512

RESEARCH ARTICLE

A note on generalized absolute Cesàro summability

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Received: 11-02-2011; Accepted: 12-20-2011 *Corresponding author

Abstract In the present paper , we give several improvements to the result of [2] concerning absolute Cesàro summablity of infinite series.

Key Words Cesàro summability, summability factor

MSC 2010 40D15

1 Introduction

Let (φ_n) be a sequence of positive real numbers, $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n), and let t_n^α we denote the n-th Cesàro means of order $\alpha > -1$ of the sequence (na_n), that is

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.1}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_{-n}^\alpha = 1, \quad n > 0. \tag{1.2}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geqslant 1$, $\alpha > -1$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty \tag{1.3}$$

and it is summable $\varphi - |C, \alpha|_k$, $k \geqslant 1$, $\alpha > -1$, if (see [2])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n^\alpha|^k < \infty. \tag{1.4}$$

$\varphi - |C, \alpha|_k$ summability reduces $\varphi - |C, 1|_k$ by taking $\alpha = 1$ and $\varphi - |C, 1|_k$ reduces $|C, 1|_k$ summability by taking $\varphi_n = n$.

The following result has been proved by Özarslan [2]

Citation: K. R. Gandhi, A note on generalized absolute Cesàro summability, South Asian J Math, 2012, 2(1), 73-77.

Theorem 1.1. *Let* (φ_n) *be a sequence of positive real numbers. If the conditions*

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{1.5}$$

$$\sum_{n=1}^{\infty} n \log n \left|\Delta^2 \lambda_n\right| = O(1), \tag{1.6}$$

$$\sum_{v=1}^{m} \frac{\varphi_v^{k-1}}{v^k} \left|t_v^\alpha\right|^k = O(\log m), \;\; as \; m \to \infty, \tag{1.7}$$

$$\sum_{n=v}^{m} \frac{\varphi_n^{k-1}}{n^{k+\alpha}} = O\left(\frac{\varphi_v^{k-1}}{v^{k+\alpha-1}}\right) \tag{1.8}$$

then the series $\sum a_n \lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $0 < \alpha \leqslant 1$, $k \geqslant 1$.

The object of the present paper is that to give three improvement to the theorem 1.1 as follows

1. Extending the scope by replacing *log*m in (1.7) by an almost increasing sequence X_m,

2. Weakening the condition (1.7).

3. Weakening the condition (1.8).

2 Lemmas

The following lemmas are needed for our aim

Lemma 2.1. *Let* (X_n) *be an almost increasing sequence, then the condition*

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k X_n^{k-1}} = O(X_m), \tag{2.1}$$

is weaker than the condition

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k} = O(X_m). \tag{2.2}$$

Proof. Let (2.2) satisfied, then

$$\sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k} = O(1) \sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k} = O(X_m)$$

On the other hand if (2.1) is satisfied, then

$$\begin{aligned}
\sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k} &= \sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k X_n^{k-1}} X_n^{k-1} \\
&= \sum_{n=1}^{m-1} \left(\sum_{v=1}^{n} \frac{\varphi_v^{k-1} \left|t_v^\alpha\right|^k}{v^k}\right) \Delta X_n^{k-1} + \left(\sum_{n=1}^{m} \frac{\varphi_n^{k-1} \left|t_n^\alpha\right|^k}{n^k}\right) X_m^{k-1} \\
&= O(1) \sum_{n=1}^{m-1} X_n \left|\Delta X_n^{k-1}\right| + O(1) X_m^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)X_{m-1}\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1}-X_1^{k-1}\right)+O(1)X_m^k \\
&= O(1)X_{m-1}X_m^{k-1}+O(1)X_m^k=O\left(X_m^k\right).
\end{aligned}
$$

Therefore (2.2) implies (2.1) but not conversely. □

Lemma 2.2. *Let (φ_n) be a sequence of positive real numbers. Then the condition*

$$\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}}=O\left(\frac{\varphi_v^{k-1}}{v^{\alpha k+k-1}}\right) \tag{2.3}$$

is weaker than (1.8).

Proof. Suppose that (1.8) is satisfied, then

$$\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}}=\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha+\alpha(k-1)}}=O\left(\frac{1}{v^{\alpha(k-1)}}\right)\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha}}=O\left(\frac{\varphi_v^{k-1}}{v^{\alpha k+k-1}}\right).$$

Now suppose that (2.3) is satisfied, then

$$\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha}}=\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+\alpha k+\alpha-k\alpha}}=O\left(m^{\alpha(k-1)}\right)\sum_{n=v}^{m+1}\frac{\varphi_n^{k-1}}{n^{k+k\alpha}}=O\left(m^{\alpha(k-1)}\frac{\varphi_v^{k-1}}{v^{\alpha k+k-1}}\right)\neq O\left(\frac{\varphi_v^{k-1}}{v^{k+\alpha-1}}\right).$$

□

Lemma 2.3. *Let (X_n) be an almost increasing sequence such that the conditions (1.5) and*

$$\sum_{n=1}^{\infty}nX_n\left|\Delta^2\lambda_n\right|<\infty, \tag{2.4}$$

are satisfied, then

$$X_n\left|\lambda_n\right|=O(1),\quad \text{as } n\to\infty \tag{2.5}$$

$$\sum_{n=1}^{\infty}X_n\left|\Delta\lambda_n\right|<\infty \tag{2.6}$$

$$nX_n\left|\Delta\lambda_n\right|=O(1)\quad \text{as } n\to\infty. \tag{2.7}$$

Proof. As $\Delta\lambda_n\to 0$, therefore we have

$$
\begin{aligned}
nX_n\left|\Delta\lambda_n\right| &= nX_n\sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right|=O(1)\sum_{v=n}^{\infty}vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\sum_{v=n}^{\infty}vX_v\left|\Delta^2\lambda_v\right|=O(1).
\end{aligned}
$$

This proves (2.7). To prove (2.6), we observe that

$$\sum_{v=1}^{m}X_v\left|\Delta\lambda_v\right| = \sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}X_r\right)\Delta\left|\Delta\lambda_v\right|+\left(\sum_{v=1}^{m}X_v\right)\left|\Delta\lambda_m\right|$$

$$= \quad O(1)\sum_{v=1}^{m-1} vX_v \left|\Delta^2\lambda_v\right| + O(1)mX_m\left|\Delta\lambda_m\right| = O(1).$$

Finally,

$$X_n\left|\lambda_n\right| = X_n\sum_{v=n}^{\infty}\Delta\left|\lambda_v\right| = O(1)\sum_{v=n}^{\infty}X_v\left|\Delta\lambda_v\right| = O(1).$$

□

3 Main Result

We prove the following

Theorem 3.1. *Let* (φ_n) *be a sequence of positive real numbers. If the conditions (1.5), (2.1), (2.3) and (2.4) are all satisfied, then the series* $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $0<\alpha\leqslant 1$, $k\geqslant 1$.

Proof. Let T_n^α be the n-th (C,α) means of the sequence $(na_n\lambda_n)$. Then by (1.1),we have for $0<\alpha\leqslant 1$,

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\lambda_v. \tag{3.1}$$

Abel transformation gives

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{r=1}^{v}A_{n-r}^{\alpha-1} r a_r + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1} v a_v \\ &= \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha t_v^\alpha\Delta\lambda_v + \lambda_n t_n^\alpha \\ &: \ = T_{n1}^\alpha + T_{n2}^\alpha. \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|T_{nr}^\alpha\right|^k < \infty, \quad j=1,2.$$

Now, applying Hölder's inequality, we have by Lemma 2.3

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|T_{n1}^\alpha\right|^k &= \sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^k}\left|\frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha t_v^\alpha\Delta\lambda_v\frac{X_v}{X_v}\right|^k \\ &\leqslant \sum_{n=1}^{\infty}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}}\sum_{v=1}^{n-1}v^{\alpha k}\left|t_v^\alpha\right|^k\left|\Delta\lambda_v\right|X_v^{1-k}\left(\sum_{v=1}^{n-1}X_v\left|\Delta\lambda_v\right|\right)^{k-1} \\ &= O(1)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{X_v^{1-k}}v^{\alpha k}\left|t_v^\alpha\right|^k\left|\Delta\lambda_v\right|^k\sum_{n=v}^{m}\frac{\varphi_n^{k-1}}{n^{k+\alpha k}} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\frac{v\left|\Delta\lambda_v\right|\varphi_v^{k-1}}{v^kX_v^{k-1}}\left|t_v^{\alpha}\right|^k\\
&= O(1)\sum_{v=1}^{m}\left(\left(v\left|\Delta\lambda_v\right|\right)\right)\sum_{r=1}^{m-1}\frac{\varphi_r^{k-1}}{r^kX_r^{k-1}}\left|t_r^{\alpha}\right|^k+O(1)\left(m\left|\Delta\lambda_m\right|\right)\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^kX_v^{k-1}}\left|t_v^{\alpha}\right|^k\\
&= O(1)\sum_{v=1}^{m}vX_v\left|\Delta^2\lambda_m\right|+O(1)\sum_{v=1}^{m}X_v\left|\Delta\lambda_v\right|+O(1)mX_m\left|\Delta\lambda_m\right|\\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|T_{n2}^{\alpha}\right|^k &= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}}{n^k}\left|\lambda_nt_n^{\alpha}\right|^k\\
&= \sum_{n=1}^{m}\frac{\varphi_n^{k-1}\left|t_n^{\alpha}\right|^k}{n^kX_n^{k-1}}\left(X_n\left|\lambda_n\right|\right)^{k-1}\left|\lambda_n\right|\\
&= O(1)\sum_{n=1}^{m}\frac{\varphi_n^{k-1}\left|t_n^{\alpha}\right|^k}{n^kX_n^{k-1}}\left|\lambda_n\right|\\
&= O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{v=1}^{n}\frac{\varphi_v^{k-1}\left|t_v^{\alpha}\right|^k}{v^kX_v^{k-1}}+O(1)\left|\lambda_m\right|\sum_{n=1}^{m}\frac{\varphi_n^{k-1}\left|t_n^{\alpha}\right|^k}{n^kX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}\left|\Delta\lambda_n\right|X_n+O(1)\left|\lambda_m\right|X_m\\
&= O(1),
\end{aligned}
$$

in view of Lemma 2.3. This completes the proof of the theorem. □

References

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