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FISHER INFORMATION MATRIX FOR MODELS OF SYSTEMS BASED ON FACE-SPLITTING MATRIX PRODUCTS

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Expressions for blocks of the information Fisher matrix are presented based on factorization of the Neudecker derivative of a transposed face-splitting matrix product.

Keywords: *matrix product, Neudecker derivative, Fisher matrix, face-splitting matrix product.*

As is generally known, the measurement accuracy of the parameters of signals can be analyzed using different approaches [1]. Among a variety of factors used in this case, the low Cramer–Rao bound of the variance of an unbiased estimate is most popular. Calculation of this bound involves inversion of the Fisher information matrix; therefore, the formation of this matrix can be considered as the key problem in the entire process of determination of the low bound. The inversion of an information matrix is a trivial operation and can be automatically executed even by the spreadsheet processor MS Excel, let alone specialized software packages such as Mathcad 6.0 (7.0).

This paper presents a detailed review of the technique of Fisher-matrix formation for analytical models of radar-tracking systems (RTS). These models have been formalized on the basis of transposed face-splitting matrix product [2]. Recall that the transposed face-splitting product (TFP) of a $g \times p$ matrix $A = [a_{ij}]$ and an $s \times p$ matrix B, represented in the form of a block-matrix of columns $[B_j] (B = [B_j],\ j = 1, \ldots, p)$, is a $gs \times p$ matrix $A \blacksquare B$ defined by the equality

$$A \blacksquare B = [a_{ij} B_j] \tag{1}$$

($\blacksquare$ is the symbol of face-splitting product [2].)

For example, for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ we have

$$A \blacksquare B = \begin{bmatrix} a_{11} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & a_{12} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} & a_{13} \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \\ \cdots & \cdots & \cdots \\ a_{21} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & a_{22} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} & a_{23} \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} \end{bmatrix}.$$

As was mentioned in [3], the Fisher information matrix can be obtained based on the the Neudecker matrix derivative [4]. The general form of the Fisher matrix according [2, 3] is

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$$I = \frac{1}{\sigma^2}\left[\begin{array}{c:c} P^{\mathrm{T}}P & (A^{*}\otimes P^{\mathrm{T}})\dfrac{\partial P}{\partial Y} \\ \hdashline \left(\dfrac{\partial P}{\partial Y}\right)^{\mathrm{T}}(A\otimes P) & \left(\dfrac{\partial P}{\partial Y}\right)^{\mathrm{T}}(AA^{*}\otimes 1_{RR})\dfrac{\partial P}{\partial Y} \end{array}\right] \tag{2}$$

Here, $\frac{\partial P}{\partial Y}$ is the Neudecker derivative of the matrix P with respect to the vector Y, which is composed of unknown parameters of the signals of M sources, 1_{RR} is a unit $(R\times R)$-matrix, and $\otimes$ is the symbol of Kronecker product. The Fisher matrix in general form can be used to form the Cramer–Rao low bound, achievable by a multicoordinate single- or multiposition radar-tracking system in a multsignal situation. As was mentioned in [2], it is sufficient to use in expression (2) instead of the matrix P its value represented by a face-splitting matrix product, and to appropriately adjust the dimension of the matrix 1_{RR}.

A further simplification of the procedure of the Fisher matrix formation in the class of problems being considered relates to the property of factorization of the Neudecker derivative of TFP. To write it in the general form, we first turn to the Neudecker differentiation of a matrix product

$$P = S\blacksquare F, \tag{3}$$

where S is a $T\times M$ matrix, and F is an $R\times M$ matrix.

By analogy with [3], the matrix S can be treated as a $T\times M$ matrix of responses T of synthesized frequency filters on the frequencies of M signals, and F as an $R\times M$ matrix of the directional characteristics R of the reception channels of a digital antenna array (DAA) in the directions of M sources. In this case, if noise is absent, the set of response voltages of the DAA reception channels can be writen as $U = PA$, where $A = [\dot{a}_1, \dot{a}_2 \ldots \dot{a}_M]^T$ is the vector of the complex amplitudes of the signals.

When two signal sources are acting, we can write

$$S = \begin{bmatrix} S_1(\omega_1) & S_1(\omega_2) \\ S_2(\omega_1) & S_2(\omega_2) \\ \vdots & \vdots \\ S_T(\omega_1) & S_T(\omega_2) \end{bmatrix}, \qquad F = \begin{bmatrix} F_1(x_1) & F_1(x_2) \\ F_2(x_1) & F_2(x_2) \\ \vdots & \vdots \\ F_R(x_1) & F_R(x_2) \end{bmatrix}$$

$$P = S\blacksquare F = \left[\begin{array}{c:c} S_1(\omega_1)\begin{bmatrix} F_1(x_1) \\ \vdots \\ F_R(x_1) \end{bmatrix} & S_1(\omega_2)\begin{bmatrix} F_1(x_2) \\ \vdots \\ F_R(x_2) \end{bmatrix} \\ \hdashline \vdots & \vdots \\ \hdashline S_T(\omega_1)\begin{bmatrix} F_1(x_1) \\ \vdots \\ F_R(x_1) \end{bmatrix} & S_T(\omega_2)\begin{bmatrix} F_1(x_2) \\ \vdots \\ F_R(x_2) \end{bmatrix} \end{array}\right]. \tag{4}$$

Let us consider $Y = [x_1\ x_2\ \omega_1\ \omega_2]^T$ as a vector of unknowns. Then the Neudecker derivative of P will take the form

$$
\frac{\partial P}{\partial Y} =
$$

$$
= \left[\begin{array}{c|c|c|c}
S_1(\omega_1)\begin{bmatrix}\frac{\partial F_1(x_1)}{\partial x_1}\\ \vdots\\ \frac{\partial F_R(x_1)}{\partial x_1}\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \frac{\partial S_1(\omega_1)}{\partial \omega_1}\begin{bmatrix}F_1(x_1)\\ \vdots\\ F_R(x_1)\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
S_T(\omega_1)\begin{bmatrix}\frac{\partial F_1(x_1)}{\partial x_1}\\ \vdots\\ \frac{\partial F_R(x_1)}{\partial x_1}\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \frac{\partial S_T(\omega_1)}{\partial \omega_1}\begin{bmatrix}F_1(x_1)\\ \vdots\\ F_R(x_1)\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\
\hline
\begin{matrix}0\\ \vdots\\ 0\end{matrix} & S_1(\omega_2)\begin{bmatrix}\frac{\partial F_1(x_2)}{\partial x_2}\\ \vdots\\ \frac{\partial F_R(x_2)}{\partial x_2}\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \frac{\partial S_1(\omega_2)}{\partial \omega_2}\begin{bmatrix}F_1(x_2)\\ \vdots\\ F_R(x_2)\end{bmatrix} \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
\begin{matrix}0\\ \vdots\\ 0\end{matrix} & S_T(\omega_2)\begin{bmatrix}\frac{\partial F_1(x_2)}{\partial x_2}\\ \vdots\\ \frac{\partial F_R(x_2)}{\partial x_2}\end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \frac{\partial S_T(\omega_2)}{\partial \omega_2}\begin{bmatrix}F_1(x_2)\\ \vdots\\ F_R(x_2)\end{bmatrix}
\end{array}\right]. \tag{5}
$$

Analyzing expression (5) one can easily arrive at the idea of factorization of $\frac{\partial P}{\partial Y}$. Indeed, on closer look, it turns out that (5) can be obtained by transposing the block face-splitting product (BFP) [2] of the following two matrices:

$$
\partial S_y = \left[\begin{array}{cc|cc}
S_1(\omega_1) & 0 & \frac{\partial S_1(\omega_1)}{\partial \omega_1} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
S_T(\omega_1) & 0 & \frac{\partial S_T(\omega_1)}{\partial \omega_1} & 0 \\
\hline
0 & S_1(\omega_2) & 0 & \frac{\partial S_1(\omega_2)}{\partial \omega_2} \\
\vdots & \vdots & \vdots & \vdots \\
0 & S_T(\omega_2) & 0 & \frac{\partial S_T(\omega_2)}{\partial \omega_2}
\end{array}\right], \quad
\partial F_y = \left[\begin{array}{cc|cc}
\frac{\partial F_1(x_1)}{\partial x_1} & 0 & F_1(x_1) & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial F_R(x_1)}{\partial x_1} & 0 & F_R(x_1) & 0 \\
\hline
0 & \frac{\partial F_1(x_2)}{\partial x_2} & 0 & F_1(x_2) \\
\vdots & \vdots & \vdots & \vdots \\
0 & \frac{\partial F_R(x_2)}{\partial x_2} & 0 & F_R(x_2)
\end{array}\right]. \tag{6}
$$

Denoting

$$\partial S_y = \left[\begin{array}{c|c} S_1 & \dfrac{\partial S_1}{\partial \omega_1} \\ \hline S_2 & \dfrac{\partial S_2}{\partial \omega_2} \end{array}\right], \quad \partial F_y = \left[\begin{array}{c|c} \dfrac{\partial F_1}{\partial x_1} & F_1 \\ \hline \dfrac{\partial F_2}{\partial x_2} & F_2 \end{array}\right], \tag{7}$$

where

$$S_1 = \begin{bmatrix} S_1(\omega_1) & 0 \\ \vdots & \vdots \\ S_T(\omega_1) & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & S_1(\omega_2) \\ \vdots & \vdots \\ 0 & S_T(\omega_2) \end{bmatrix}, \quad \frac{\partial S_1}{\partial \omega_1} = \begin{bmatrix} \dfrac{\partial S_1(\omega_1)}{\partial \omega_1} & 0 \\ \vdots & \vdots \\ \dfrac{\partial S_T(\omega_1)}{\partial \omega_1} & 0 \end{bmatrix},$$

$$\frac{\partial S_2}{\partial \omega_2} = \begin{bmatrix} 0 & \dfrac{\partial S_1(\omega_2)}{\partial \omega_2} \\ \vdots & \vdots \\ 0 & \dfrac{\partial S_T(\omega_2)}{\partial \omega_2} \end{bmatrix}, \quad F_1 = \begin{bmatrix} F_1(x_1) & 0 \\ \vdots & \vdots \\ F_R(x_1) & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & F_1(x_2) \\ \vdots & \vdots \\ 0 & F_R(x_2) \end{bmatrix},$$

$$\frac{\partial F_1}{\partial x_1} = \begin{bmatrix} \dfrac{\partial F_1(x_1)}{\partial x_1} & 0 \\ \vdots & \vdots \\ \dfrac{\partial F_R(x_1)}{\partial x_1} & 0 \end{bmatrix}, \quad \frac{\partial F_2}{\partial x_2} = \begin{bmatrix} 0 & \dfrac{\partial F_1(x_2)}{\partial x_2} \\ \vdots & \vdots \\ 0 & \dfrac{\partial F_R(x_2)}{\partial x_2} \end{bmatrix},$$

according to the definition of transposed BFP [2], we obtain

$$\frac{\partial P}{\partial Y} = \partial S_y \circledast \partial F_y = \left[\begin{array}{c|c} S_1 & \dfrac{\partial S_1}{\partial \omega_1} \\ \hline S_2 & \dfrac{\partial S_2}{\partial \omega_2} \end{array}\right] \circledast \left[\begin{array}{c|c} \dfrac{\partial F_1}{\partial x_1} & F_1 \\ \hline \dfrac{\partial F_2}{\partial x_2} & F_2 \end{array}\right]$$

$$= \left[\begin{array}{c|c} S_1 \blacksquare \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial S_1}{\partial \omega_1} \blacksquare F_1 \\ \hline S_2 \blacksquare \dfrac{\partial F_2}{\partial x_2} & \dfrac{\partial S_2}{\partial \omega_2} \blacksquare F_2 \end{array}\right]. \tag{8}$$

It is noteworthy that not only the result of Neudecker differentiation can be subject to factorization but also the product $\left(\dfrac{\partial P}{\partial Y}\right)^T \dfrac{\partial P}{\partial Y}$. Thus, for relation (5) it is easy to obtain

$$\left(\frac{\partial P}{\partial Y}\right)^T \frac{\partial P}{\partial Y} = [P_1 \mid P_2], \tag{9}$$

where

$$P_1=\begin{bmatrix} \sum_{t=1}^{T} S_t^2(\omega_1)\left\{\sum_{r=1}^{R}\left(\frac{\partial F_r(x_1)}{\partial x_1}\right)^2\right\} & 0 \\ 0 & \sum_{t=1}^{T} S_t^2(\omega_2)\left\{\sum_{r=1}^{R}\left(\frac{\partial F_r(x_2)}{\partial x_2}\right)^2\right\} \\ \left\{\sum_{t=1}^{T} S_t(\omega_1)\frac{\partial S_t(\omega_1)}{\partial \omega_1}\right\}\sum_{r=1}^{R} F_r(x_1)\frac{\partial F_r(x_1)}{\partial x_1} & 0 \\ 0 & \left\{\sum_{t=1}^{T} S_t(\omega_2)\frac{\partial S_t(\omega_2)}{\partial \omega_2}\right\}\sum_{r=1}^{R} F_r(x_2)\frac{\partial F_r(x_2)}{\partial x_2} \end{bmatrix},$$

$$P_2=\begin{bmatrix} \left\{\sum_{t=1}^{T} S_t(\omega_1)\frac{\partial S_t(\omega_1)}{\partial \omega_1}\right\}\sum_{r=1}^{R} F_r(x_1)\frac{\partial F_r(x_1)}{\partial x_1} & 0 \\ 0 & \left\{\sum_{t=1}^{T} S_t(\omega_2)\frac{\partial S_t(\omega_2)}{\partial \omega_2}\right\}\sum_{r=1}^{R} F_r(x_2)\frac{\partial F_r(x_2)}{\partial x_2} \\ \left\{\sum_{t=1}^{T}\left(\frac{\partial S_t(\omega_1)}{\partial \omega_1}\right)^2\right\}\left\{\sum_{r=1}^{R} F_r^2(x_1)\right\} & 0 \\ 0 & \left\{\sum_{t=1}^{T}\left(\frac{\partial S_t(\omega_2)}{\partial \omega_2}\right)^2\right\}\left\{\sum_{r=1}^{R} F_r^2(x_2)\right\} \end{bmatrix}.$$

It is easy to note the obvious identity

$$\left(\frac{\partial P}{\partial Y}\right)^T \frac{\partial P}{\partial Y} = Z \circ V, \tag{10}$$

in which

$$Z=\begin{bmatrix} \sum_{t=1}^{T} S_t^2(\omega_1) & 0 & \sum_{t=1}^{T} S_t(\omega_1)\frac{\partial S_t(\omega_1)}{\partial \omega_1} & 0 \\ 0 & \sum_{t=1}^{T} S_t^2(\omega_2) & 0 & \sum_{t=1}^{T} S_t(\omega_2)\frac{\partial S_t(\omega_2)}{\partial \omega_2} \\ \sum_{t=1}^{T} S_t(\omega_1)\frac{\partial S_t(\omega_1)}{\partial \omega_1} & 0 & \sum_{t=1}^{T}\left(\frac{\partial S_t(\omega_1)}{\partial \omega_1}\right)^2 & 0 \\ 0 & \sum_{t=1}^{T} S_t(\omega_2)\frac{\partial S_t(\omega_2)}{\partial \omega_2} & 0 & \sum_{t=1}^{T}\left(\frac{\partial S_t(\omega_2)}{\partial \omega_2}\right)^2 \end{bmatrix},$$

$$V=\begin{bmatrix} \sum_{r=1}^{R}\left(\frac{\partial F_r(x_1)}{\partial x_1}\right)^2 & 0 & \sum_{r=1}^{R} F_r(x_1)\frac{\partial F_r(x_1)}{\partial x_1} & 0 \\ 0 & \sum_{r=1}^{R}\left(\frac{\partial F_r(x_2)}{\partial x_2}\right)^2 & 0 & \sum_{r=1}^{R} F_r(x_2)\frac{\partial F_r(x_2)}{\partial x_2} \\ \sum_{r=1}^{R} F_r(x_1)\frac{\partial F_\kappa(x_1)}{\partial x_1} & 0 & \sum_{r=1}^{R} F_r^2(x_1) & 0 \\ 0 & \sum_{r=1}^{R} F_r(x_2)\frac{\partial F_r(x_2)}{\partial x_2} & 0 & \sum_{r=1}^{R} F_r^2(x_2) \end{bmatrix},$$

and in terms of notation (7) we have

$$Z = \partial S_y^T \, \partial S_y, \qquad V = \partial F_y^T \, \partial F_y . \tag{11}$$

Thus, the following property is characteristic of the Neudecker derivative:

$$[\partial S_y^T \circledcirc \partial F_y^T][\partial S_y \circledast \partial F_y] = [\partial S_y^T \, \partial S_y] \circ [\partial F_y^T \, \partial F_y]. \tag{12}$$

This property can be expressed for block matrices of matched dimensions A, B, K, and M, whose structures are similar to those of ∂S_y and ∂F_y, in the form

$$(A \circledcirc B)(K \circledast M) = (A\ K) \circ (B\ M). \tag{13}$$

Here $\circledcirc$ is the symbol of BFP [2].

Result (13) is rather significant, despite its validity only for a narrow class of block matrices, since it allows us to simplify significantly the analysis of the accuracy of radar-tracking systems, whose models are formed based on TFP. Since transposed BFP naturally follows from the operation of differentiation of TFP, the property of factorization of derivative (8) can be also generalized to the case of TFP of greater dimension. Applying calculations similar to those presented above, but more cumbersome, it is possible, for example, to generalize relation (8) to the case of M signals and five-component matrix product P.

Assume that $P = Q \blacksquare F \blacksquare S \blacksquare D \blacksquare L$, i.e., the angular coordinates, velocities, distances, and, in particular, accelerations of M sources are to be measured. Then

$$\frac{\partial P}{\partial Y} = \partial Q_y \circledast \partial F_y \circledast \partial S_y \circledast \partial D_y \circledast \partial L_y = \left[\begin{array}{c:c:c:c:c} Q_1 & Q_1 & Q_1 & Q_1 & \frac{\partial Q_1}{\partial q_1} \\ \hdashline Q_2 & Q_2 & Q_2 & Q_2 & \frac{\partial Q_2}{\partial q_2} \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline Q_M & Q_M & Q_M & Q_M & \frac{\partial Q_M}{\partial q_M} \end{array}\right] \circledast$$

$$\circledast \left[\begin{array}{c:c:c:c:c} F_1 & F_1 & F_1 & \frac{\partial F_1}{\partial x_1} & F_1 \\ \hdashline F_2 & F_2 & F_2 & \frac{\partial F_2}{\partial x_2} & F_2 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline F_M & F_M & F_M & \frac{\partial F_M}{\partial x_M} & F_M \end{array}\right] \circledast \left[\begin{array}{c:c:c:c:c} S_1 & S_1 & \frac{\partial S_1}{\partial \omega_1} & S_1 & S_1 \\ \hdashline S_2 & S_2 & \frac{\partial S_2}{\partial \omega_2} & S_2 & S_2 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline S_M & S_M & \frac{\partial S_M}{\partial \omega_M} & S_M & S_M \end{array}\right] \circledast$$

$$\circledast \left[\begin{array}{c:c:c:c:c} D_1 & \frac{\partial D_1}{\partial d_1} & D_1 & D_1 & D_1 \\ \hdashline D_2 & \frac{\partial D_2}{\partial d_2} & D_2 & D_2 & D_2 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline D_M & \frac{\partial D_M}{\partial d_M} & D_M & D_M & D_M \end{array}\right] \circledast \left[\begin{array}{c:c:c:c:c} \frac{\partial L_1}{\partial z_1} & L_1 & L_1 & L_1 & L_1 \\ \hdashline \frac{\partial L_2}{\partial z_2} & L_2 & L_2 & L_2 & L_2 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \frac{\partial L_M}{\partial z_M} & L_M & L_M & L_M & L_M \end{array}\right] \tag{14}$$

Expression (14) clearly demonstrates the following features of the factorization of the derivative $\frac{\partial P}{\partial Y}$, which can be taken into account in the automatic formation of this derivative:

— The index of a block in all of the block matrices corresponds to the number of a nonzero column of the block's elements (all the rest of the elements are zero) and to the number of the source whose coordinate is used as the argument of the corresponding characteristic; in particular, for S_4 in the matrix ∂S_y we have

$$S_4 = \begin{bmatrix} 0 & 0 & 0 & S_1(\omega_4) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & S_T(\omega_4) & 0 \end{bmatrix};$$

— The block-column of derivatives for the inverse order of formation of the vector of unknowns Y used here moves from the right to the left in the block matrices (in the case being considered, $Y = [z_1 \dots z_M \mid d_1 \dots d_M \mid \omega_1 \dots \omega_M \mid x_1 \dots x_M \mid q_1 \dots q_M]^T$);

— The number of block-rows in the block matrices is determined by the number of sources (if there are M sources, then we have M block-rows), and the number of block-columns is equal to the number of matrices participating in the formation of TFP;

— All of the block-columns, excluding the block-column of derivatives, are identical within one block matrix.

Such a detailed consideration of the properties of the factorization of the Neudecker derivative of TFP plays an important role in subsequent derivation of the final expressions for the blocks of the Fisher matrix, which include the derivative $\frac{\partial P}{\partial Y}$. In particular, for a two-coordinate DAA model with the matrix P in the form (3), (4), the blocks of the Fisher matrix that are not located in the principal diagonal can be expressed as

$$\left(\frac{\partial P}{\partial Y}\right)^T (A \otimes P) = [\partial S_y^T \mathbin{\bigcirc\mkern-15mu\square} \partial F_y^T][A \otimes (S \blacksquare F)], \tag{15}$$

$$(A^* \otimes P^T)\frac{\partial P}{\partial Y} = [(A^* \otimes (S^T \square F^T)]\partial S_y \mathbin{\bigcirc\mkern-15mu\blacksquare} \partial F_y]. \tag{16}$$

Investigations prove that relations (15), (16) can be reduced to Hadamard products (all intermediate calculations for the relations presented below are omitted due to their extreme awkwardness)

$$\left(\frac{\partial P}{\partial Y}\right)^T (A \otimes P) = [\partial S_y^T (A \otimes S)] \circ [\partial F_y^T \mathbf{1}_M \otimes F)], \tag{17}$$

$$(A^* \otimes P^T)\frac{\partial P}{\partial Y} = [(A^* \otimes S^T)\partial S_y] \circ [(\mathbf{1}_M \otimes F^T)\partial F_y]. \tag{18}$$

Here, $\mathbf{1}_M$ is the vector or matrix of unit amplitudes whose dimension coincides with that of the vector (matrix) of amplitudes A. In this case, for M signals we have an M-vector of units $\mathbf{1}_M = [1 \dots 1]^T$.

Reduction of dimension of block expressions by reducing them to Hadamard products allows us to considerably simplify both the readability of the resultant writing of the blocks of the Fisher matrix and the derivation of final relations suitable for the use in numerical computations. The efficiency of such an approach manifests itself especially in problems of analysis of the accuracy of multicoordinate RTSs.

For the right lower block of the Fisher matrix, there exists an identity that employs the factorability of the derivative $\frac{\partial P}{\partial Y}$. It should be noted that the derivation of the final expression for this block was difficult first of all due to the awkwardness of the necessary analytical calculations. Now this procedure will be simpler when it is considered that

$$\left(\frac{\partial P}{\partial Y}\right)^T (AA^* \otimes 1_{RT})\frac{\partial P}{\partial Y} = [\partial S_y^T \mathbin{\bigcirc\mkern-15mu\square} \partial F_y^T][AA^* \otimes 1_{RT}][\partial S_y \mathbin{\bigcirc\mkern-15mu\blacksquare} \partial F_y], \tag{19}$$

whence the desired identity, corresponding to (3), has the form

$$\left(\frac{\partial P}{\partial Y}\right)^T (AA^* \otimes 1_{RT}) \frac{\partial P}{\partial Y} = [\partial S_y^T (AA^* \otimes 1_T) \partial S_y] \circ [\partial F_y^T (\mathbf{1}_M \otimes 1_R) \partial F_y]; \tag{20}$$

here $\mathbf{1}_M$ is a matrix of units.

In the case of large dimension of the matrix P (for example, when the number of coordinates measured by a RTS exceeds four), we should use the property (which is valid for the type of matrices being considered) of "absorption" of Kronecker products by face-splitting ones

$$[G^Y \circledcirc N^T \circledcirc D^T \circledcirc L^T \ldots][AA^* \otimes 1_G \otimes 1_N \otimes 1_D \otimes 1_L \ldots][G \circledbullet N \circledbullet D \circledbullet L \ldots]$$

$$= [G^T (AA^* \otimes 1_G) G] \circ [N^T (\mathbf{1}_M \otimes 1_N) N]$$

$$\circ [D^T (\mathbf{1}_M \otimes 1_D) D] \circ [L^T (\mathbf{1}_M \otimes 1_L) L] \ldots \tag{21}$$

Finally, it remains only to write the left upper block of the Fisher matrix, which is the simplest one. Applying the well-known identity $(A \square B)(C \blacksquare D) = (AC) \circ (BD)$ [5], it is easy to obtain

$$P^T P = (S^T \square F^T)(S \blacksquare F) = (S^T S) \circ (F^T F). \tag{22}$$

The results presented here can be treated as an original basis for a subsequent simplification of the analysis of Fisher information matrices corresponding to models of systems based on BFP, in particular, multiposition RTSs based on conformal (multisectional) DAAs. Since their review deserves an individual publication, it is necessary to note that the Neudecker derivative of the transposed BFP is also factorable. This allows us to also simplify the notation of the corresponding Fisher matrices, in sizes similar to those considered above, for multiposition RTSs with DAAs.

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