Arithmetic completely regular codes

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Abstract

In this paper, we explore completely regular codes in the Hamming graphs and related graphs. We focus on cartesian products of completely regular codes and products of their corresponding coset graphs in the additive case. Connections to the theory of distance-regular graphs are explored and several open questions are posed.

1 Introduction

In this paper, we present the theory of completely regular codes in the Hamming graph enjoying the property that the eigenvalues of the code are in arithmetic progression. We call these *arithmetic completely regular codes* and propose their full classification, modulo the classification of perfect codes of Hamming type. Our results are strongest when the Hamming graph admits a completely regular partition into such codes (e.g., the partition into cosets of some additive completely regular code), since it is known that the quotient graph obtained from any such partition is distance-regular. Using Leonard's Theorem, the list of possible quotients is determined, with a few special cases left as open problems. In the case of linear arithmetic completely regular codes, more can be said.

The techniques employed are mainly combinatorial and products of codes as well as decompositions of codes into "reduced" codes play a fundamental

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role. The results have impact in algebraic coding theory, but more so in the theory of distance-regular graphs. Since, for large minimum distance, the coset graph of any such linear code is a distance-regular graph locally isomorphic to a Hamming graph, the analysis can be viewed as a special case of a fairly difficult open classification problem in the theory of distance-regular graphs. What is interesting here is how the combinatorics of the Hamming graph gives leverage in the combinatorial analysis of its quotients, a tool not available in the unrestricted problem.

It is important here to draw a connection between the present paper and a companion paper, currently in preparation. In [8], we will introduce the concept of *Leonard completely regular codes* and we will develop the basic theory for them. We will also give several important families of Leonard completely regular codes in the classical distance-regular graphs, demonstrating their fundamental role in structural questions about these graphs. We conjecture that all completely regular codes in the Hamming graphs with large enough covering radius are in fact Leonard. The class of arithmetic codes we introduce in this paper is perhaps the most important subclass of the Leonard completely regular codes in the Hamming graphs and something similar is likely true for the other classical families, but this investigation is left as an open problem.

The layout of the paper is as follows. After an introductory section outlining the required background, we explore products of completely regular codes in Hamming graphs. First noting (Prop. 3.1) that a completely regular product must arise from completely regular constituents, we determine in Proposition 3.4 exactly when the product of two completely regular codes in two Hamming graphs is completely regular. At this point, the role of the arithmetic property becomes clear and we present a generic form for the quotient matrix of such a code in Lemma 3.5. With this preparatory material out of the way, we are then ready to present the main results in Section 3.2. Applying the celebrated theorem of Leonard, Theorem 3.7 provides powerful limitations on the combinatorial structure of a quotient of a Hamming graph when the underlying completely regular partition is composed of arithmetic codes. Stronger results are obtained in Proposition 3.9, Theorem 3.11 and Proposition 3.13 when one makes additional assumptions about the minimum distance of the codes or the specific structure of the quotient. When C is a linear completely regular code with the arithmetic property and C has minimum distance at least three and covering radius at most two, we show that C is closely related to some Hamming code. These results are summarized in Theorem 3.16, which gives a full classification of possible codes and quotients in the linear case (always assuming the arithmetic and completely regular properties) and Corollary 3.17 which characterizes Hamming quotients of Hamming graphs.

2 Preliminaries and definitions

In this section, we summarize the background material necessary to understand our results. Most of what is covered here is based on Chapter 11 in the monograph [3] by Brouwer, Cohen and Neumaier. The theory of codes in distance-regular graphs began with Delsarte [5]. The theory of association schemes is also introduced in the book [1] of Bannai and Ito while connections between these and related material (especially equitable partitions) can be found in Godsil [7].

2.1 Distance-regular graphs

Suppose that Γ is a finite, undirected, connected graph with vertex set $V\Gamma$. For vertices x and y in $V\Gamma$, let $d(x, y)$ denote the distance between x and y, i.e., the length of a shortest path connecting x and y in Γ. Let D denote the diameter of Γ; i.e., the maximal distance between any two vertices in $V\Gamma$. For $0 \leq i \leq D$ and $x \in V\Gamma$, let $\Gamma_i(x) := \{y \in V\Gamma \mid d(x, y) = i\}$ and put $\Gamma_{-1}(x) := \emptyset$, $\Gamma_{D+1}(x) := \emptyset$. The graph Γ is called *distance-regular* whenever it is regular of valency k, and there are integers b_i, c_i $(0 \leq i \leq D)$ so that for any two vertices x and y in $V\Gamma$ at distance i, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. It follows that there are exactly $a_i = k - b_i - c_i$ neighbors of y in $\Gamma_i(x)$. The numbers c_i, b_i and a_i are called the *intersection numbers* of Γ and we observe that $c_0 = 0$, $b_D = 0$, $a_0 = 0$, $c_1 = 1$ and $b_0 = k$. The array $\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ.

From now on, assume Γ is a distance-regular graph of valency $k \geq 2$ and diameter $D \geq 2$. Define A_i to be the square matrix of size $|V\Gamma|$ whose rows and columns are indexed by $V\Gamma$ with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq i \leq D,\ x, y \in V\Gamma).$$

We refer to A_i as the i^{th} *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Since Γ is distance-regular, we have for $2 \le i \le D$

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i$$

so that $A_i = p_i(A)$ for some polynomial $p_i(t)$ of degree i. Let $\mathbb{A}$ be the *Bose-Mesner algebra*, the matrix algebra over $\mathbb{C}$ generated by A. Then $\dim \mathbb{A} = D+1$ and $\{A_i \mid 0 \le i \le D\}$ is a basis for $\mathbb{A}$. As $\mathbb{A}$ is semi-simple and commutative, $\mathbb{A}$ has also a basis of pairwise orthogonal idempotents $\left\{E_0 = \frac{1}{|V\Gamma|}J, E_1, \ldots, E_D\right\}$. We call these matrices the *primitive idempotents* of Γ. As $\mathbb{A}$ is closed under the entry-wise (or Hadamard) product $\circ$, there exist real numbers q_{ij}^{ℓ}, called the *Krein parameters*, such that

$$E_i \circ E_j = \frac{1}{|V\Gamma|}\sum_{\ell=0}^{D} q_{ij}^{\ell} E_\ell \quad (0 \le i, j \le D) \tag{1}$$

The graph Γ is called *Q-polynomial* if there exists an ordering $E_0, \ldots, E_D$ of the primitive idempotents and there exist polynomials q_i of degree i such that $E_i = q_i(E_1)$, where the polynomial q_i is applied entrywise to E_1. We recall that the distance-regular graph Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of its primitive idempotents provided its Krein parameters satisfy

- $q_{ij}^{\ell} = 0$ unless $|j-i| \le \ell \le i+j$;
- $q_{ij}^{\ell} \ne 0$ whenever $\ell = |j-i|$ or $\ell = i+j \le D$.

By an *eigenvalue* of Γ, we mean an eigenvalue of $A = A_1$. Since Γ has diameter D, it has at least $D+1$ eigenvalues; but since Γ is distance-regular, it has exactly $D+1$ eigenvalues[1]. We denote these eigenvalues by $\theta_0, \ldots, \theta_D$ and, aside from the convention that $\theta_0 = k$, the valency of Γ, we make no further assumptions at this point about the eigenvalues except that they are distinct.

2.2 Codes in distance-regular graphs

Let Γ be a distance-regular graph with distinct eigenvalues $\theta_0 = k, \theta_1, \ldots, \theta_D$. By a *code* in Γ, we simply mean any nonempty subset C of $V\Gamma$. We call C

[1] See, for example, Lemma 11.4.1 in [7].

trivial if $|C| \leq 1$ or $C = V\Gamma$ and *non-trivial* otherwise. For $|C| > 1$, the *minimum distance* of C, $\delta(C)$, is defined as

$$\delta(C) := \min\{\, d(x,y) \mid x, y \in C, x \neq y \,\}$$

and for any $x \in V\Gamma$ the distance $d(x, C)$ from x to C is defined as

$$d(x, C) := \min\{\, d(x,y) \mid y \in C \,\}.$$

The number

$$\rho(C) := \max\{\, d(x, C) \mid x \in V\Gamma \,\}$$

is called the *covering radius* of C.

For C a nonempty subset of $V\Gamma$ and for $0 \leq i \leq \rho$, define

$$C_i = \{\, x \in V\Gamma \mid d(x, C) = i \,\}.$$

Then $\Pi(C) = \{C_0 = C, C_1, \ldots, C_\rho\}$ is the *distance partition* of $V\Gamma$ with respect to code C.

A partition $\Pi = \{P_0, P_1, \ldots, P_k\}$ of $V\Gamma$ is called *equitable* if, for all i and j, the number of neighbors a vertex in P_i has in P_j is independent of the choice of vertex in P_i. Following Neumaier [10], we say a code C in Γ is *completely regular* if this distance partition $\Pi(C)$ is equitable[2]. In this case the following quantities are well-defined:

$$\gamma_i = \left|\{y \in C_{i-1} \mid d(x,y) = 1\}\right|, \tag{2}$$

$$\alpha_i = \left|\{y \in C_{i} \mid d(x,y) = 1\}\right|, \tag{3}$$

$$\beta_i = \left|\{y \in C_{i+1} \mid d(x,y) = 1\}\right| \tag{4}$$

where x is chosen from C_i. The numbers $\gamma_i, \alpha_i, \beta_i$ are called the *intersection numbers* of code C. Observe that a graph Γ is distance-regular if and only if each vertex is a completely regular code and these $|V\Gamma|$ codes all have the same intersection numbers. Set the tridiagonal matrix

$$U := U(C) = \begin{pmatrix} \alpha_0 & \beta_0 & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ & \gamma_2 & \alpha_2 & \beta_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_\rho & \alpha_\rho \end{pmatrix}.$$

[2] This definition of a completely regular code is equivalent to the original definition, due to Delsarte [5].

For C a completely regular code in Γ, we say that η is an *eigenvalue of* C if η is an eigenvalue of the quotient matrix U defined above. By $\mathrm{Spec}\,(C)$, we denote the set of eigenvalues of C. Note that, since $\gamma_i + \alpha_i + \beta_i = k$ for all i, $\theta_0 = k$ belongs to $\mathrm{Spec}\,(C)$.

2.3 Completely regular partitions

Given a partition Π of the vertex set of a graph Γ (into nonempty sets), we define the *quotient graph* Γ/Π on the classes of Π by calling two classes $C, C' \in \Pi$ adjacent if $C \neq C'$ and Γ contains an edge joining some vertex of C to some vertex of C'. A partition Π of $V\Gamma$ is *completely regular* if it is an equitable partition of Γ and all $C \in \Pi$ are completely regular codes with the same intersection numbers. If $\Pi = \{C^{(1)}, C^{(2)}, \ldots, C^{(t)}\}$ is a completely regular partition we write $\mathrm{Spec}\,(\Pi) = \mathrm{Spec}\,(C^{(1)})$ and we say ρ is the covering radius of Π if it is the covering radius of $C^{(1)}$.

Proposition 2.1 (Cf. [3, p. 352-3]) *Let $\Pi = \{C^{(1)}, C^{(2)}, \ldots, C^{(t)}\}$ be a completely regular partition of any distance-regular graph Γ such that each $C^{(i)}$ has intersection numbers γ_i, α_i and β_i $(0 \leq i \leq \rho)$. Then Γ/Π is a distance-regular graph with intersection array*

$$\iota(\Gamma/\Pi) = \left\{ \frac{\beta_0}{\gamma_1}, \frac{\beta_1}{\gamma_1}, \frac{\beta_{\rho-1}}{\gamma_1};\ 1, \frac{\gamma_2}{\gamma_1}, \ldots \frac{\gamma_\rho}{\gamma_1} \right\},$$

remaining intersection numbers $a_i = \frac{\alpha_i - \alpha_0}{\gamma_1}$, and eigenvalues $\frac{\theta_j - \alpha_0}{\gamma_1}$ for $\theta_j \in$ $\mathrm{Spec}\,(C)$. *All of these lie among the eigenvalues of the matrix $\frac{1}{\gamma_1}(A - \alpha_0 I)$.* ∎

Proposition 2.2 *Let Π be a non-trivial completely regular partition of a distance-regular graph Γ and assume that* $\mathrm{Spec}\,(\Pi) = \{\eta_0 \geq \eta_1 \geq \cdots \geq \eta_\rho\}$. *Then $\eta_\rho \leq \alpha_0 - \gamma_1$.*

Proof: By Proposition 2.1, the eigenvalues of Γ/Π are $\frac{\eta_0 - \alpha_0}{\gamma_1}, \frac{\eta_1 - \alpha_0}{\gamma_1}, \ldots, \frac{\eta_\rho - \alpha_0}{\gamma_1}$. As Γ/Π has at least one edge, it follows that its smallest eigenvalue is at most -1. Hence $\frac{\eta_\rho - \alpha_0}{\gamma_1} \leq -1$. ∎

Question: Let C be a non-trivial completely regular code in a distance-regular graph Γ. Let $\theta := \min\{\eta \mid \eta \in \mathrm{Spec}\,(C)\}$. Is it true that $\theta \leq \alpha_0 - \gamma_1$?

3 Codes in the Hamming graph

Let X be a finite abelian group. A *translation distance-regular graph* on X is a distance-regular graph Γ with vertex set X such that if x and y are adjacent then $x+z$ and $y+z$ are adjacent for all $x, y, z \in X$. A code $C \subseteq X$ is called *additive* if for all $x, y \in C$, also $x-y \in C$; i.e., C is a subgroup of X. If C is an additive code in a translation distance-regular graph on X, then we obtain the usual coset partition $\Delta(C) := \{C+x \mid x \in X\}$ of X; whenever C is a completely regular code, it is easy to see that $\Delta(C)$ is a completely regular partition. For any additive code C in a translation distance-regular graph Γ on vertex set X, the *coset graph* of C in Γ is the graph with vertex set X/C and an edge joining coset C' to coset C'' whenever Γ has an edge with one end in C' and the other in C''. An important result of Brouwer, Cohen and Neumaier [3, p353] states that every translation distance-regular graph of diameter at least three defined on an elementary abelian group X is necessarily a coset graph of some additive completely regular code in some Hamming graph. Of course, the Hamming graph itself is a translation graph.

Let Q be an abelian group with $|Q| = q$. Then we may identify the vertex set of the Hamming graph $H(n, q)$ with the group $X = Q^n$; so the Hamming graph can be viewed as a translation distance-regular graph in a variety of ways. For the remainder of this paper, we will consider q-ary codes of length n as subsets of the vertex set of the Hamming graph $H(n, q)$. In this section we will focus on q-ary completely regular codes, i.e. q-ary codes of length n which are completely regular in $H(n, q)$.

3.1 Products of completely regular codes

We now recall the cartesian product of two graphs Γ and Σ. This new graph has vertex set $V\Gamma \times V\Sigma$ and adjacency $\sim$ defined by $(x, y) \sim (u, v)$ precisely when either $x = u$ and $y \sim v$ in Σ or $x \sim u$ in Γ and $y = v$. Now let C be a nonempty subset of $V\Gamma$ and let C' be a nonempty subset of $V\Sigma$. The cartesian product of C and C' is the code defined by

$$C \times C' := \{(c, c') \in V\Gamma \times V\Sigma \mid c \in C \text{ and } c' \in C'\}.$$

We are interested in the cartesian product of codes in the Hamming graphs $H(n, q)$ and $H(n', q')$. Note that if C and C' are additive codes then $C \times C'$ is also additive. In particular, if C is a completely regular code in $H(n, q)$ with vertex set Q^n, then $Q \times C$ is a completely regular code in $H(n+1, q)$.

Also if $\Pi = \{P_1, \ldots, P_t\}$ is a completely regular partition of $H(n,q)$ then $Q \times \Pi := \{Q \times P_1, \ldots, Q \times P_t\}$ is a completely regular partition of $H(n+1,q)$. We say a completely regular code $C \subseteq H(n,q)$ is *non-reduced* if $C \cong C' \times Q$ for some $C' \subseteq H(n-1,q)$, and *reduced* otherwise. In similar fashion we say that a completely regular partition is *non-reduced* or *reduced*.

The next three results will determine exactly when the cartesian product of two arbitrary codes in two Hamming graphs is completely regular.

Proposition 3.1 *Let C and C' be non-trivial codes in the Hamming graphs $H(n,q)$ and $H(n',q')$ respectively. If the cartesian product $C \times C'$ is completely regular in $H(n,q) \times H(n',q')$, then both C and C' themselves must be completely regular codes in their respective graphs.*

Proof: If the distance partition of $H(n,q)$ with respect to C is $\{C_0 = C, C_1, \ldots, C_\rho\}$, then we easily see that every vertex (x,y) of $C_i \times C'$ is at distance i from $C \times C'$ in the product graph. Moreover, the neighbors of this vertex which lie at distance $i-1$ from the product code are precisely $\{(u,y) \,|\, u \sim x,\ u \in C_{i-1}\}$; so the size of the set $\{u \in Q^n \,|\, u \sim x,\ u \in C_{i-1}\}$ must be independent of the choice of $x \in C_i$. This shows that the intersection numbers γ_i are well-defined for C. An almost identical argument gives us the intersection numbers β_i. Swapping the roles of C and C', we find that each of these is a completely regular code. ∎

Remark 3.2 *Note that, if $q \neq q'$, then the product graph is not distance-regular. Although this case is not of primary interest, there are examples where $C \times C'$ can still be a completely regular code (in the sense of Neumaier) in such a graph. For instance, suppose C is a perfect code of covering radius one in $H(n,q)$ and C' is a perfect code of covering radius one in $H(n',q')$. If we happen to have $n(q-1) = n'(q'-1)$, then, by the above proposition, $C \times C'$ is a completely regular code in $H(n,q) \times H(n',q')$ with intersection numbers $\gamma_1 = 1$ and $\gamma_2 = 2$, $\beta_0 = 2n(q-1)$ and $\beta_1 = n(q-1)$.*

It is well-known that equitable partitions are preserved under products. If Π is an equitable partition in any graph Γ and Δ is an equitable partition in another graph Σ, then $\{P \times P' \mid P \in \Pi,\ P' \in \Delta\}$ is an equitable partition in the Cartesian product graph $\Gamma \times \Sigma$. The following special case will prove useful in our next proposition.

Lemma 3.3 *Consider completely regular codes C in $H(n,q)$ and C' in $H(n',q')$ with distance partitions $\Pi = \{C_0 = C, \ldots, C_\rho\}$ and $\Pi' = \{C'_0 = C', \ldots, C'_{\rho'}\}$ respectively.*

(a) The partition
$$\left\{C_i \times C'_j \,|\, 0 \le i \le \rho,\ 0 \le j \le \rho'\right\}$$
is an equitable partition in the product graph $H(n,q) \times H(n',q')$.

(b) A vertex in $C_i \times C'_j$ has all its neighbors in
$$\left[(C_{i-1} \cup C_i \cup C_{i+1}) \times C'_j\right] \cup \left[C_i \times \left(C'_{j-1} \cup C'_j \cup C'_{j+1}\right)\right].$$

(c) If C has intersection numbers $\gamma_i, \alpha_i, \beta_i$ and C' has intersection numbers $\gamma'_i, \alpha'_i, \beta'_i$, then in the product graph, a vertex in $C_i \times C'_j$ has: γ_i neighbors in $C_{i-1} \times C'_j$; β_i neighbors in $C_{i+1} \times C'_j$; γ'_j neighbors in $C_i \times C'_{j-1}$; β'_j neighbors in $C_i \times C'_{j+1}$; and $\alpha_i + \alpha'_j$ neighbors in $C_i \times C'_j$.

Proof: Straightforward. ∎

Proposition 3.4 *Let C be a non-trivial completely regular code in $H(n,q)$ with $\rho(C) := \rho \ge 1$ and intersection numbers α_i, β_i and γ_i ($0 \le i \le \rho$). Let C' be a non-trivial completely regular code in $H(n',q')$ with $\rho(C') := \rho' \ge 1$ and intersection numbers α'_i, β'_i and γ'_i ($0 \le i \le \rho'$). Assume, without loss, that $\rho \le \rho'$. Then $C \times C'$ is a completely regular code in $H(n,q) \times H(n',q')$ if and only if there exist integers n_1 and n_2 satisfying (a) and (b):*

(a) $\gamma_i = n_1 i$ for $0 \le i \le \rho$ and $\gamma'_i = n_1 i$ for $0 \le i \le \rho'$;

(b) $\beta_{\rho-i} = n_2 i$ for $0 \le i \le \rho$ and $\beta'_{\rho'-i} = n_2 i$ for $0 \le i \le \rho'$.

In this case, $C \times C'$ has covering radius $\bar{\rho} := \rho + \rho'$ and intersection numbers $\bar{\gamma}_i = n_1 i$ and $\bar{\beta}_i = n_2(\bar{\rho} - i)$ for $0 \le i \le \bar{\rho}$.

Proof: ($\Rightarrow$) Suppose first that $C \times C'$ is a completely regular code. From Lemma 3.3, we see that $C \times C'$ has covering radius $\bar{\rho} := \rho' + \rho''$. For $0 \le j \le \bar{\rho}$, let
$$S_j = \{(x,y) \,|\, d((x,y), C \times C') = j\}.$$

Then it follows easily from the second statement in Lemma 3.3 that

$$S_j = \bigcup_{h+i=j} C_h \times C'_i.$$

Moreover a vertex in $C_h \times C'_i$ has $\gamma_h + \gamma'_i$ neighbors in S_{j-1} and $\beta_h + \beta'_i$ neighbors in S_{j+1}. For $j = 1$, this forces $\gamma_1 = \gamma'_1$. Assume inductively that $\gamma_i = i\gamma_1$ and $\gamma'_i = i\gamma'_1$ for $i < j$. Then, considering a vertex in

$$S_j = C_r \times C'_{j-r} \cup \cdots \cup C_{j-s} \times C'_s$$

(where $r = \max(0, j - \rho')$ and $s = \max(0, j - \rho)$), we find

$$\gamma_r + \gamma'_{j-r} = \gamma_{r+1} + \gamma'_{j-r-1} = \cdots = \gamma_{j-s} + \gamma'_s.$$

For $j \leq \rho$, this gives $\gamma_j = j\gamma_1 = \gamma'_j$. For $\rho < j \leq \rho'$, we deduce, $\gamma'_j = j\gamma'_1$. So we have (a) by induction. A symmetrical argument establishes part (b).

($\Leftarrow$) Considering the same partition of S_j into cells of the form $C_i \times C'_{j-i}$, we obtain the converse in a straightforward manner: if C and C' have intersection numbers given by (a) and (b), then their cartesian product $C \times C'$ is completely regular in the product graph. ∎

Lemma 3.5 *Let k, γ, β and ρ be positive integers. The tridiagonal matrix*

$$L = \begin{pmatrix} \alpha_0 & \rho\beta & & & & \\ \gamma & \alpha_1 & (\rho-1)\beta & & & \\ & 2\gamma & \alpha_2 & (\rho-2)\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & (\rho-1)\gamma & \alpha_{\rho-1} & \beta \\ & & & & \rho\gamma & \alpha_\rho \end{pmatrix}$$

where $\alpha_i = k - i\gamma - (\rho - i)\beta$ $(0 \leq i \leq \rho)$, has eigenvalues $\mathrm{Spec}\,(L) = \{k - ti \mid 0 \leq i \leq \rho\}$ *where $t = \gamma + \beta$.*

Proof: By direct verification. (See, for example, Terwilliger [12, Example 5.13].) ∎

From now on we will look at a completely regular partition Π with $\mathrm{Spec}\,(\Pi) = \{n(q-1), n(q-1) - qt, \ldots, n(q-1) - q\rho t\}$ in $H(n, q)$. As a direct consequence of Proposition 2.2, we have the following proposition.

Proposition 3.6 *Let* Π *be a non-trivial completely regular partition of the Hamming graph* $H(n,q)$ *such that* Π *has covering radius* ρ *and* $\mathrm{Spec}\,(\Pi) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ *for some* t. *Then* $n(q-1)-\alpha_0 \leq q\rho t$.
∎

3.2 Classification

Now we will classify the possible quotients Γ/Π where Π is a completely regular partition of $\Gamma := H(n,q)$ with $\mathrm{Spec}\,(\Pi) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ for some t.

Theorem 3.7 *Let* Γ *be the Hamming graph* $H(n,q)$ *and let* Π *be a completely regular partition of its vertices. Assume that* Π *has covering radius* $\rho \geq 3$ *and eigenvalues* $\mathrm{Spec}\,(\Pi) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ *for some* t. *Then* Γ/Π *is has diameter* ρ *and is isomorphic to one of the following:*

(a) a folded cube;

(b) a Hamming graph;

(c) a Doob graph (i.e., the cartesian product of some number of 4-cliques and at least one Shrikhande graph);

(d) a distance-regular graph with intersection array $\{6,5,4;1,2,6\}$.

Proof: Let Π be a completely regular partition with covering radius $\rho \geq 3$ and eigenvalues $\mathrm{Spec}\,(\Pi) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ for some t. Denote $\eta_i := k - i\tau$ $(0 \leq i \leq \rho)$ where, $k := \frac{n(q-1)-\alpha_0}{\gamma_1}$ and $\tau := \frac{qt}{\gamma_1}$. Then by Proposition 2.1, Γ/Π is a distance-regular graph with eigenvalues $\{\eta_i \mid 0 \leq i \leq \rho\}$ and diameter ρ. By [8, Proposition 3.3, Lemma 4.3], Γ/Π is Q-polynomial with respect to $E_0, E_1, \cdots, E_\rho$ where E_i is the primitive idempotent corresponding to η_i.

Leonard (1982) showed that Q-polynomial distance regular-graphs with diameter D at least 3, fall into nine types, namely, (I), (IA), (IB), (II), (IIA), (IIB), (IIC), (IID) and (III), where we follow the notation of Bannai and Ito [1, p263]. Type (IID) is not possible as it only occurs if $D = \infty$. As the eigenvalues of Γ/Π are in arithmetic progression and $\rho \geq 3$, neither type (I), (IA), (IB), (II), (IIA) nor (III) can occur. The only remaining possibilities are types (IIB) and (IIC). Terwilliger [11] showed that a Q-polynomial distance-regular graph Σ of type (IIB) with diameter $D \geq 3$ is either the antipodal quotient of the $(2D+1)$-cube, or has the same intersection numbers as the antipodal quotient of the $2D$-cube. By [3, Theorem 9.2.7], it follows that Γ/Π is either the folded graph of a n-cube for $n \geq 7$ or has intersection array $\{6,5,4;1,2,6\}$. For type (IIC), we have (in the notation of Bannai and Ito [1, p270])

$$\eta_i = \eta_0 + s'i,$$

$$r_1 = -d - 1 \text{ since } D < \infty,$$

$$b_i = (i+1+r_1)r/s^{*\prime} = (i-D)r/s^{*\prime} \Rightarrow b_0 = -Dr/s^{*\prime},$$

$$c_i = i(r - s's^{*\prime})/s^{*\prime}.$$

Since $c_1 = (r - s's^{*\prime})/s^{*\prime} = 1$,

$$c_i = i$$

and

$$a_i = k - b_i - c_i = -(1 + \frac{r}{s^{*\prime}}) \cdot i = a_1 \cdot i.$$

Hence Γ/Π has the same parameters as a Hamming graph. But Egawa [6] showed that a distance-regular graph with same parameters as a Hamming graph must be a Hamming graph or a Doob graph. Hence the theorem is proved. ∎

Remark 3.8 A. *Theorem 3.7 also holds when Γ is a Doob graph. There are completely regular partitions of Doob graphs that are Hamming graphs with $q \neq 4$. For example, let $s \geq 1$ be an integer. In any Doob graph Γ of diameter $\frac{4^s-1}{3}$, there exist an additive completely regular code, say C, with covering radius* 1 [9]. *Let $n \geq 1$ be an integer. Then C^n is an additive completely regular code with covering radius n in Γ^n. Recall $\Delta(C^n) := \{C^n + x \mid x \in V(\Gamma^n)\}$. Then $\Gamma^n/\Delta(C^n) \cong H(n, 4^s)$.*

B. *There are 3 nonisomorphic graphs with the same intersection array, $\{6,5,4;1,2,6\}$, as the folded 6-cube. They are the point-block incident graphs of the 2-(16,6,2)-designs. We do not know any example in which the quotient graph has intersection array $\{6,5,4;1,2,6\}$, but is not the folded 6-cube.*

C. *If Γ is a Hamming graph and C is an additive completely regular code in Γ satisfying the eigenvalue conditions of the above theorem, then $\Gamma/\Delta(C)$ is always a Hamming graph, a Doob graph or a folded cube. We do not know any example where Γ is a Hamming graph and Π is a completely regular partition such that Γ/Π is a Doob graph. But we will prove below in Proposition 3.13 that this cannot occur when $\Pi = \Delta(C)$ for some linear completely regular code C.*

D. *We wonder whether it is true that if the quotient graph Γ/Π in the above Theorem 3.7 is isomorphic to a Hamming graph or a Doob graph, then each part in Π can be expressed as a cartesian product of completely regular codes with covering radius at most 2. We will show (Theorem 3.16, below) that this holds when Π is the coset partition of a linear completely regular code satisfying the eigenvalue conditions of the above theorem.*

Proposition 3.9 *Let Γ be the Hamming graph $H(n,q)$. Let Π be any completely regular partition of Γ where each code C in Π has minimum distance $\delta(C) \geq 2$.*

(a) If $\Gamma/\Pi \cong H(m,q')$ then $q' \geq q$.

(b) If $q \geq 4$ then Γ/Π is not isomorphic to any Doob graph.

(c) If $q \geq 3$ then Γ/Π can not have the same intersection array as any folded cube of diameter at least two (including the array $\{6,5,4;1,2,6\}$).

(d) Suppose further that $\Pi = \Delta(C)$ for some additive code C and Γ/Π has intersection array $\{6,5,4;1,2,6\}$. Then $q = 2$ and Γ/Π is the folded 6-cube.

Proof: Since each C in Π satisfies $\delta(C) \geq 2$, the vertices in any clique in Γ belong to pairwise distinct classes in Π. So the quotient Γ/Π has a clique of size at least q, the clique number of $H(n,q)$. This immediately implies (a)-(c) as: $H(m,q')$ has no clique of size larger than q'; any Doob graph has maximum clique size three or four; a folded cube other than K_4 has maximum clique size two, as does any graph with intersection array $\{6,5,4;1,2,6\}$.

For part (d), we clearly need only consider the case where $q = 2$. As $\delta(C) \geq 2$, it is easy to see that every claw $K_{1,3}$ in Γ/Π belongs to a unique 3-cube so that Γ/Π must be the folded 6-cube [3, Corollary 4.3.8]. ∎

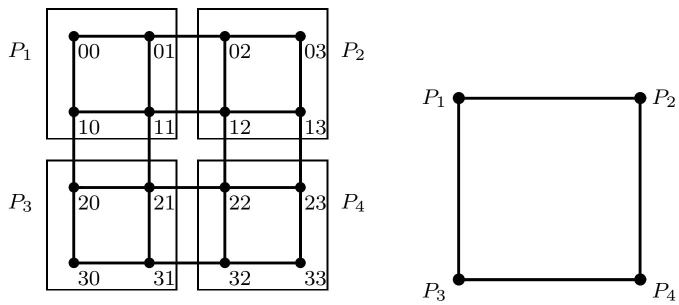

Figure 1: Hamming graph H(2,4) and its quotient graph H(2,2)

Example 3.10 *Let Γ be the Hamming graph $H(2,4)$ and let $\Pi := \Pi(\Gamma) = \{P_1, P_2, P_3, P_4\}$, where $P_1 = \{00, 01, 10, 11\}$, $P_2 = \{02, 03, 12, 13\}$, $P_3 = \{20, 21, 30, 31\}$, and $P_4 = \{22, 23, 32, 33\}$. Then the quotient graph Γ/Π is the Hamming $H(2,2)$ as depicted in Figure 1. Observe that each class in partition Π is expressible as a cartesian product:*

$$P_1 = \{0,1\} \times \{0,1\} \quad P_2 = \{0,1\} \times \{2,3\}$$

$$P_3 = \{2,3\} \times \{0,1\} \quad P_4 = \{2,3\} \times \{2,3\}.$$

Finally note that this example can be easily extended to give $H(n,q)$ as a quotient of $H(n,sq)$, $n,s \geq 1$.

From Example 3.10, we see that the n-cube and the folded n-cube can be quotients of $H(n,4)$. So Proposition 3.9 does not hold when $\delta(C) = 1$, even for additive codes. We next show that, if Π is the coset partition of $H(n,q)$ with respect to a linear code with a quotient of Hamming type, then each class in Π may be expressed as a cartesian product of completely regular codes with covering radius one.

Theorem 3.11 *Let Γ be the Hamming graph $H(n,q)$. Let C be an additive completely regular code with minimum distance $\delta(C) \geq 2$ in Γ and let $\Delta(C)$ be the coset partition of Γ with respect to C. Suppose that $\Gamma/\Delta(C) \cong H(m,q')$. Then m divides n and $C = \prod_{i=1}^{m} C^{(i)}$, where each $C^{(i)}$ is a q-ary completely regular code with covering radius 1 and length $\frac{n}{m}$.*

Proof: Let $\mathbf{e}_i$ denote the codeword with i^{th} position 1 and all other positions 0. Since C is additive, the relation $\approx$ defined by $i \approx j$ if and only if $\mathbf{e}_i - \mathbf{e}_j \in C$ is an equivalence relation on $\{1, \ldots, n\}$. Let R_i denote the equivalence class containing i. Note that

$$\gamma_1 = |\{(j, h) : \mathbf{e}_i - h\mathbf{e}_j \in C\}| = |\{j : \exists h(\mathbf{e}_i - h\mathbf{e}_j \in C)\}|$$

since $\mathbf{e}_i \in C_1$ and $\delta(C) \geq 2$. Since $\mathbf{0}$ and $\mathbf{e}_i$ have $q-1$ common neighbors in Γ while C and $\mathbf{e}_i + C$ have $q'-1$ common neighbors in $\Gamma/\Delta(C)$, we find that $|R_i|(q-1) = \gamma_1(q'-1)$. This gives $|R_i| = \frac{\gamma_1(q'-1)}{q-1} = \frac{n}{m}$ for $1 \leq i \leq n$. So there are exactly m equivalence classes and they all have the same size.

Let $D_i := Q^n|_{R_i}$ denote the set of vertices $\mathbf{a}$ of $H(n, q)$ satisfying $\mathbf{a}_h = 0$ for all $h \notin R_i$ and let Σ be the subgraph of Γ induced by D_i so $\Sigma \cong H(\frac{n}{m}, q)$. Note that Σ has the property $\mathcal{P}$: if $\mathbf{a}, \mathbf{b} \in D_i$ and $\mathbf{a} \neq \mathbf{b}$ then the common neighbors of $\mathbf{a}$ and $\mathbf{b}$ in Γ are also in D_i. Let $C^{(i)} := D_i \cap C$.

Claim: $C^{(i)}$ is a q-ary completely regular code with covering radius 1 in Σ with $U(C^{(i)}) = \begin{pmatrix} 0 & \beta \\ \gamma_1 & \beta - \gamma_1 \end{pmatrix}$ where $\beta = \frac{n(q-1)}{m}$.

Proof of Claim: We show the following two statements (i) and (ii).

(i) For any $\mathbf{d} \in D_i$, distance $d(\mathbf{d}, C) \leq 1$:

This is obvious by induction on the weight of $\mathbf{d}$, as Σ has property $\mathcal{P}$.

(ii) $C^{(i)}$ is a q-ary completely regular code of length $\frac{n}{m}$:

Assume that for $\mathbf{d} \in D_i$, $d(\mathbf{d}, C) = 1$. Then clearly $\Gamma(\mathbf{d}) \cap C = \Gamma(\mathbf{d}) \cap C^{(i)}$ by (i) and $C^{(i)} := D_i \cap C$.

This gives us the claim. ∎

To finish the proof of the theorem note that $C^{(i)} \subseteq C$ so that $\prod_{i=1}^m C^{(i)} \subseteq C$ as C is additive. By Proposition 3.4, $\prod_{i=1}^m C^{(i)}$ is a completely regular code with the same quotient matrix as C. So $C = \prod_{i=1}^m C^{(i)}$. This shows the theorem. ∎

Lemma 3.12 *Assume that q is a prime power. Let C be a non-trivial reduced linear completely regular code over $GF(q)$ of length n. Then the minimum distance of C, $\delta(C)$, is at least 2.* ∎

Proposition 3.13 *If C is a linear q-ary completely regular code of length n, then $H(n, q)/\Delta(C)$ is never isomorphic to a Doob graph.*

Proof: Without loss of generality, we may assume C is reduced. Since every Doob graph has the Shrikhande graph as a quotient, we may suppose that $H(n,q)/\Delta(C) \cong \Sigma$, the Shrikhande graph. Since C is non-trivial and reduced, by Lemma 3.12, it has minimum distance $\delta(C) \geq 2$ and by Proposition 3.9(b), we have $q \leq 3$. The Shrikhande graph is locally a 6-gon; denote by $C_1, \ldots, C_6$ the six vertices adjacent to vertex C in the coset graph, with the understanding that C_i is adjacent to C_{i+1} $(1 \leq i \leq 5)$ in Σ. Let $\mathbf{e}_i$ denote the codeword with i^{th} position 1 and all other positions 0.

Consider first the case $q = 2$. With an appropriate ordering of the coordinates, we may assume that $\mathbf{e}_1 \in C_1$, $\mathbf{e}_1 + \mathbf{e}_2 \in C_2$ and $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \in C$. But then $\mathbf{e}_2 + \mathbf{e}_3$ must lie in C_1 and a contradiction is obtained by considering which coset contains $\mathbf{e}_2$: this coset must be distinct from C, C_1 and C_2 (as $\delta(C) \geq 2$) and yet adjacent to all three. But Σ contains no 4-clique.

It remains to consider $q = 3$. We may suppose, without loss that $\mathbf{e}_1 \in C_1$ and $-\mathbf{e}_1 \in C_2$. Now some neighbor of the zero vector — we may call it $\mathbf{e}_2$ — lies in C_3. The word $-\mathbf{e}_2$ must lie in some coset adjacent to both C and C_3 since $\delta(C) \geq 2$. It cannot be C_2 since otherwise $\mathbf{e}_2 - \mathbf{e}_1 \in C$ forcing $\mathbf{e}_1$ and $\mathbf{e}_2$ into the same coset. So $-\mathbf{e}_2 \in C_4$. But now we obtain a contradiction by considering the coset containing $x = -\mathbf{e}_1 - \mathbf{e}_2$. In Σ, this coset must be adjacent to both C_2 and C_4. If $x \in C$, then $-\mathbf{e}_1$ and $\mathbf{e}_2$ must belong to the same coset, contrary to the way $\mathbf{e}_2$ was chosen. Likewise, $x \in C_3$ is not possible since $\mathbf{e}_1$ and $\mathbf{e}_2$ were chosen to belong to distinct cosets. ∎

Lemma 3.14 *Let C be a non-trivial, reduced, linear completely regular code over $GF(q)$ in the $H(n,q)$ with parity check matrix H. Let $\mathbf{h}_i$ $(i = 1, \ldots, n)$ denote columns of H and consider the relation $\equiv$ on $\{1, \cdots, n\}$ given by $i \equiv j$ if $\mathbf{h}_i$ and $\mathbf{h}_j$ are linearly dependent. Then $\equiv$ is an equivalence relation and its each equivalence class has size $\gamma_1 := \gamma_1(C)$.*

Proof: It is easy to check that the relation $\equiv$ is reflexive, symmetric and transitive and hence an equivalence relation. As $\mathbf{e}_1$ has γ_1 neighbors in C, there are $\gamma_1 - 1$ two-weight codewords in C and each of these codewords is orthogonal to every row of H. This means immediately that each equivalence class has size γ_1. ∎

Let C be a non-trivial, reduced, linear completely regular code over $GF(q)$ in $H(n,q)$. Let $P \subseteq \{1, \cdots, n\}$ such that each equivalence class of $\equiv$ defined in Lemma 3.14 contains exactly one element from P. Let $D := C|_P$. Then D is linear and has minimum distance $\delta(D) \geq 3$. Moreover, if C has $\text{Spec}\,(C) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ and D is completely regular, then D has $\text{Spec}\,(D) = \{\frac{n(q-1)}{\gamma_1}, \frac{n(q-1)-qt}{\gamma_1}, \ldots, \frac{n(q-1)-q\rho t}{\gamma_1}\}$, where $\gamma_1 = \gamma_1(C)$.

The following result is probably known but we could not find it in the literature.

Theorem 3.15 *Let C be a linear completely regular code over $GF(q)$ with minimum distance $\delta(C) \geq 3$ in $H(n,q)$.*

(a) If C has $\text{Spec}\,(C) = \{n(q-1), n(q-1)-qt\}$ *for some $t \geq 1$, then C is a Hamming code.*

(b) If C has $\text{Spec}\,(C) = \{n(q-1), n(q-1)-qt, n(q-1)-2qt\}$ *for some $t \geq 1$, then $C = D \times D$, where D is a Hamming code or $q = 2$ and C is an extended Hamming code.*

Proof: (a) Since C has covering radius one and minimum distance three, it follows immediately that C has to be perfect.

(b) Let G be the graph having as vertices the codewords of $C^\perp$ in which two codewords $\mathbf{c}$ and $\mathbf{d}$ are joined if and only if $wt(\mathbf{c}-\mathbf{d}) = t$. Then by [4, Thm 5.7], G is strongly regular. With μ denoting the number of common neighbors of two nonadjacent vertices in G, we first aim to show that $\mu = 2$ when $q \geq 3$. Let $\mathbf{a}, \mathbf{b}$ be codewords of $C^\perp$ such that $wt(\mathbf{a}) = 2t$, $wt(\mathbf{b}) = t$ and $wt(\mathbf{a}-\mathbf{b}) = t$. Let us consider the common neighbors of $\mathbf{0}$ and $\mathbf{a}$. If $\mathbf{c} \in C^\perp$ is a common neighbor of these two which is distinct from $\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$, then $wt(\mathbf{c}) = t$, $wt(\mathbf{a}-\mathbf{c}) = t$ and $wt(\mathbf{b}-\mathbf{c}) > 0$. If $supp(\mathbf{b}) \cap supp(\mathbf{c}) = \emptyset$, then $\mathbf{c}$ must have form $\gamma(\mathbf{a}-\mathbf{b})$ for some $\gamma \in GF(q)$ since otherwise, there exist $\sigma \in GF(q)$ for which the codeword $\mathbf{a}-\mathbf{b}+\sigma\mathbf{c}$ has weight lying strictly between 0 and t. But $\mathbf{c} = \gamma(\mathbf{a}-\mathbf{b})$ is also impossible as $d(\mathbf{a},\mathbf{c}) = t$ and not $2t$.

If $0 < |supp(\mathbf{b}) \cap supp(\mathbf{c})| < t$, then for $\gamma \in GF(q) - \{0,1\}$, the codewords $\mathbf{b}-\mathbf{c}$ and $\mathbf{b}-\gamma\mathbf{c}$ have distinct weights both less than $2t$. So whenever $q \geq 3$, we must have $\mu = 2$. In this case, G must be an $m \times m$-grid graph, i.e, $H(2,m)$, where $m^2 = |C^\perp|$ and, with the same reasoning as in Theorem 3.11, the result follows.

Suppose now that $q = 2$ and $\mu > 2$. Let us consider the induced subgraph G' of G on the common neighbors of $\mathbf{0}$ and $\mathbf{a}$. Then G' is a cocktail party graph and by Lemma 3.5, $2t \leq n \leq 4t$. For $n > 2t$, let $\mathbf{a}$ and $\mathbf{c}$ be codewords in $C^\perp$ such that $wt(\mathbf{a}) = wt(\mathbf{c}) = 2t$. Then as $C^\perp$ is linear, $|supp(\mathbf{a}) \cap supp(\mathbf{c})|$ is either $\frac{3}{2}t$ or t. But in both cases, we must have $\mu = 2$, a contradiction. For $n = 2t$, there exist only one codeword $\mathbf{a}$ with $wt(\mathbf{a}) = 2t$. Since G is vertex transitive, G must be a cocktail party graph (i.e., the complement of a perfect matching). This implies $C^\perp$ is the dual of an extended Hamming code so C is an extended Hamming code. Hence the result follows. ∎

Theorem 3.16 *Let C be a non-trivial, reduced, linear completely regular code over $GF(q)$. Suppose further that C has covering radius ρ in $H(n,q)$ and* $\mathrm{Spec}\,(C) = \{n(q-1), n(q-1)-qt, \ldots, n(q-1)-q\rho t\}$ *for some t. Then one of the following holds:*

(a) $q = 2$ *and*

$$C \cong nullsp \underbrace{\begin{bmatrix} M & | & \cdots & | & M \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where $M = \begin{bmatrix} I & | & \mathbf{1} \end{bmatrix}$ *is a parity check matrix for a binary repetition code,* $\gamma_1 = \gamma_1(C)$ *and the quotient is the folded cube;*

(b) $\rho = 1$ *and*

$$C \cong nullsp \underbrace{\begin{bmatrix} H & | & \cdots & | & H \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where H is a parity check matrix for some Hamming code and $\gamma_1 = \gamma_1(C)$*;*

(c) $\rho = 2$, $q = 2$ *and*

$$C \cong nullsp \underbrace{\begin{bmatrix} E & | & \cdots & | & E \end{bmatrix}}_{\gamma_1 \text{ copies}},$$

where E is a parity check matrix for a fixed extended Hamming code and $\gamma_1 = \gamma_1(C)$*;*

(d) $\rho \geq 2$, $q \geq 2$ *and*

$$C \cong \underbrace{C_1 \times \cdots \times C_1}_{\rho},$$

where C_1 is a completely regular code with covering radius 1.

Proof: The reader can easily check that examples (a)–(d) are all completely regular. (See also Bier [2].) That they are the only completely regular codes possible follows directly from Proposition 3.9, Theorem 3.11, Proposition 3.13 and Theorem 3.15. ∎

Note that the above result includes the following generalization of the result of Bier [2] concerning coset graphs which are isomorphic to Hamming graphs (cf. [3, p354]):

Corollary 3.17 *Let C be a linear completely regular code in $H(n,q)$ whose coset graph is a Hamming graph $H(m,q')$. Then one of the following holds:*

(a) $\rho = 1$ *and*

$$C \cong \mathit{nullsp}\, \underbrace{\left[\begin{array}{c|c|c} H & \cdots & H \end{array}\right]}_{\gamma_1 \text{ copies}},$$

where H is a parity check matrix for some Hamming code and $\gamma_1 = \gamma_1(C)$;

(b) $\rho = 2$, $q = 2$ *and*

$$C \cong \mathit{nullsp}\, \underbrace{\left[\begin{array}{c|c|c} E & \cdots & E \end{array}\right]}_{\gamma_1 \text{ copies}},$$

where E is a parity check matrix for the extended Hamming code and $\gamma_1 = \gamma_1(C)$.

(c) $\rho \geq 2$, $q \geq 2$ *and*

$$C \cong \underbrace{C_1 \times \cdots \times C_1}_{\rho},$$

where C_1 is a completely regular code with covering radius 1.

Proof: Assume $\mathrm{Spec}\,(C) = \{\theta_0, \theta_{j_1}, \ldots, \theta_{j_m}\}$. The eigenvalues of the Hamming graph $\Gamma/\Delta(C)$ are $m(q'-1) - hq'$ $(0 \leq h \leq m)$ but are also obtained from Proposition 2.1 giving

$$m(q'-1) - hq' = \frac{\theta_{j_h} - \alpha_0}{\gamma_1} \qquad (0 \leq h \leq m).$$

This in turn gives $\theta_{j_h} = n(q-1) - h(\gamma_1 q')$ for $0 \leq h \leq m$. So Theorem 3.16 applies. ∎

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