

# Jacobian variety and integrable system — after Mumford, Beauville and Vanhaecke<sup>☆</sup>

Rei Inoue<sup>a</sup>, Yukiko Konishi<sup>b</sup>, Takao Yamazaki<sup>c,\*</sup>

<sup>a</sup>Department of Physics, Graduate School of Science, The University of Tokyo, 7-3-1 Hongo, Bunkyo, Tokyo 113-0033, Japan

<sup>b</sup>Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

<sup>c</sup>Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

Received 13 December 2005; received in revised form 10 May 2006; accepted 9 June 2006

Available online 27 July 2006

## Abstract

Beauville [A. Beauville, *Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables*, Acta. Math. 164 (1990) 211–235] introduced an integrable Hamiltonian system whose general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve. This can be regarded as a generalization of the Mumford system [D. Mumford, *Tata Lectures on Theta II*, Birkhäuser, 1984]. In this article, we construct a variant of Beauville's system whose general level set is isomorphic to the complement of the *intersection* of the translations of the theta divisor in the Jacobian. A suitable subsystem of our system can be regarded as a generalization of the even Mumford system introduced by Vanhaecke [P. Vanhaecke, *Linearising two-dimensional integrable systems and the construction of action-angle variables*, Math. Z. 211 (1992) 265–313; P. Vanhaecke, *Integrable systems in the realm of algebraic geometry*, in: *Lecture Notes in Mathematics*, vol. 1638, 2001].

© 2006 Elsevier B.V. All rights reserved.

MSC: primary 37J35; secondary 14H70; 14H40

Keywords: Completely integrable system; Mumford system; Jacobian variety; Spectral curve

## 1. Introduction

The Mumford system [13] is an integrable Hamiltonian system with the Lax matrix

$$A(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & -v(x) \end{pmatrix} \in M_2(\mathbb{C}[x]). \quad (1.1)$$

Here  $u(x)$  and  $w(x)$  are monic of degree  $d - 1$  and  $d$ , and  $v(x)$  is of degree  $\leq d - 2$  where  $d$  is a fixed positive integer. The space of Lax matrices  $A(x)$  is endowed with  $d - 1$  independent Hamiltonian vector fields, defining an

<sup>☆</sup> The previous version of this manuscript was entitled 'Integrable Hamiltonian system on the Jacobian of a spectral curve — after Beauville'.

\* Corresponding author.

E-mail addresses: [reiy@monet.phys.s.u-tokyo.ac.jp](mailto:reiy@monet.phys.s.u-tokyo.ac.jp) (R. Inoue), [konishi@ms.u-tokyo.ac.jp](mailto:konishi@ms.u-tokyo.ac.jp) (Y. Konishi), [ytakao@math.tohoku.ac.jp](mailto:ytakao@math.tohoku.ac.jp) (T. Yamazaki).

algebraically completely integrable dynamical system. Its general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve of the Lax matrix, which is a hyperelliptic curve of genus  $d - 1$ . See [4,5] for the definition of algebraically completely integrability.

A variant called the even Mumford system was introduced by Vanhaecke [20,21], whose Lax matrix has the same form as (1.1) but the polynomial  $w(x)$  is monic of degree  $d + 1$ . This small difference gives rise to another type of general level set, which is isomorphic to the complement of the *union* of two translates of the theta divisor in the Jacobian of a hyperelliptic curve.

On the other hand, Beauville [6] introduced a generalization of the Mumford system. The Lax matrix is given by  $A(x) \in M_r(\mathbb{C}[x])$  with a certain condition on the degree of each entry, where  $r \geq 2$  can be an arbitrary integer. He constructed a completely integrable Hamiltonian system on the space of (the gauge equivalence classes of) the Lax matrix  $A(x)$ . Its general level set is isomorphic to the complement of the theta divisor in the Jacobian of the spectral curve of the Lax matrix, which is not hyperelliptic in general. The Mumford system can be recovered as the case  $r = 2$  of Beauville's system.

In this paper, we employ Beauville's method to construct a system which generalizes the even Mumford system. The Lax matrix is again given by  $A(x) \in M_r(\mathbb{C}[x])$  with arbitrary  $r \geq 2$ , but we impose a condition, different from Beauville's, on the degree of each entry. (Hence the spectral curve is not hyperelliptic in general.) We construct a completely integrable Hamiltonian system on the space of (the gauge equivalence classes of) the Lax matrix  $A(x)$ . An interesting feature of this system is that the general level set is isomorphic to the complement of the *intersection* of  $r$  translates of the theta divisor (Theorems 2.8 and 3.11), which is not an affine variety. In addition, we construct a family of subsystems, which provides an open (finite) covering of our system. The level set of each subsystem is isomorphic to the complement of the *union* of  $r$  translates of the theta divisor in the Jacobian (Theorem 4.5). We also construct the spaces of representatives of the subsystems, and explicitly describe the Hamiltonian vector fields (Proposition 4.11) and the correspondence between the Lax matrix and the divisor (Proposition 4.9). The even Mumford system can be recovered as the case  $r = 2$  of a subsystem.

This paper is organized as follows: in Section 2 we study the Jacobian of the spectral curves for the Lax matrix. Section 3 is devoted to the construction of Hamiltonian vector fields, and to the proof of the integrability. In Section 4 we introduce a family of subsystems and show that each of them is algebraically completely integrable. Further we construct the spaces of representatives of the subsystems, and study the integrable structure. The proofs of many results in Section 2 and Section 3 are given by a modification of the argument of Beauville [6]; nevertheless we include a rather whole proof in the present paper for the sake of completeness, and for the importance of Beauville's argument.

### Remarks on related works

The theory of algebraic integrability on a Poisson manifold was considered by Adler and van Moerbeke [4,5]. Integrable systems described in terms of the Lax matrix with the (Laurent) polynomial entries were discussed by several authors [16,8,3,21]. In [15,1], the Mumford system was generalized to other directions. A new treatment of the Mumford system was developed in [18]. (See also [11].)

One of the reasons that make the Mumford system (and its variants) interesting is a connection to many models arising from physics, such as the Neumann system [13], the Moser system [2], the Toda lattice [9], the Lotka–Volterra lattice [9], and the Noumi–Yamada system [11]. We hope to find a physical model that realizes our system in a future study.

## 2. Jacobian of the spectral curve

### 2.1. Intersection of translations of the theta divisor

Let  $C$  be a smooth projective irreducible curve of genus  $g$  (over  $\mathbb{C}$ ). For each integer  $k$ , we write  $J^k$  for the space of invertible sheaves of degree  $k$ , which we regard as a principal homogeneous space under the Jacobian  $J^0$  of  $C$ . We define the *theta divisor*  $\Theta \subset J^{g-1}$  by

$$\begin{aligned} \Theta &= \{L \in J^{g-1} \mid H^0(C, L) \neq 0\} \\ &= \{\mathcal{O}_C(E) \mid E \text{ is an effective divisor of degree } g - 1\}. \end{aligned}$$

For each point  $q \in C$ , we write  $\Theta_q$  for the translation  $\Theta + q = \{L(q) = L \otimes \mathcal{O}_C(q) \mid L \in \Theta\}$  of  $\Theta$ . This is a divisor on  $J^g$ . Let  $\pi : C \rightarrow \mathbb{P}^1$  be a finite morphism of degree  $r$ . We define a subvariety  $J'$  of  $J^g$  by

$$J' = \{L \in J^g \mid \pi_*L \cong \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus r-1}\},$$

where we abbreviate  $\mathcal{O}_{\mathbb{P}^1}$  to  $\mathcal{O}$ . (In [6],  $J'$  is denoted by  $J(0, -1, \dots, -1)$ .) In this subsection, we prove the following.

**Proposition 2.1.** *For any point  $a \in \mathbb{P}^1$  unramified with respect to  $\pi$ , we have*

$$J' = J^g \setminus \left( \bigcap_{q \in C} \Theta_q \right) = J^g \setminus \left( \bigcap_{q \in \pi^{-1}(a)} \Theta_q \right).$$

It is enough to show the following two lemmas:

**Lemma 2.2.** *For any point  $q \in C$ , we have  $J^g \setminus \Theta_q \subset J'$ .*

**Lemma 2.3.** *For any point  $a \in \mathbb{P}^1$  unramified with respect to  $\pi$ , we have*

$$J' \subset J^g \setminus \left( \bigcap_{q \in \pi^{-1}(a)} \Theta_q \right).$$

We need some preliminaries to prove them. Let  $L$  be an arbitrary invertible sheaf on  $C$ . We can write  $\pi_*L \cong \bigoplus_{i=1}^r \mathcal{O}(d_i)$  for some integers  $d_1 \leq d_2 \leq \dots \leq d_r$  such that  $\deg L = g - 1 + r + \sum d_i$ . We have

$$h^0(C, L) = h^0(\mathbb{P}^1, \pi_*L) = \sum_i h^0(\mathbb{P}^1, \mathcal{O}(d_i)) = \sum_{i \in \{j \mid d_j \geq 0\}} (d_i + 1), \tag{2.1}$$

$$h^1(C, L) = h^1(\mathbb{P}^1, \pi_*L) = \sum_i h^0(\mathbb{P}^1, \mathcal{O}(-2 - d_i)) = - \sum_{i \in \{j \mid d_j \leq -2\}} (d_i + 1), \tag{2.2}$$

where we used the notation  $h^*(X, F) = \dim H^*(X, F)$ . This computation, together with the Riemann–Roch theorem, implies the following two lemmas:

**Lemma 2.4** (Cf. [6] 1.8). *For  $L \in J^{g-1}$ , the following conditions are equivalent:*

$$(1) L \in J^{g-1} \setminus \Theta, \quad (2) h^0(C, L) = 0, \quad (3) h^1(C, L) = 0, \quad (4) \pi_*L \cong \mathcal{O}(-1)^{\oplus r}.$$

**Lemma 2.5.** *For  $L \in J^g$ , the following conditions are equivalent:*

$$(1) L \in J' \text{ (i.e. } \pi_*L \cong \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus r-1}\text{)}, \quad (2) h^0(C, L) = 1, \quad (3) h^1(C, L) = 0.$$

**Proof of Lemma 2.2.** For an invertible sheaf  $L$  on  $C$ , we have the exact sequence

$$0 \rightarrow H^0(C, L(-q)) \rightarrow H^0(C, L) \xrightarrow{sq} \mathbb{C} \rightarrow H^1(C, L(-q)) \rightarrow H^1(C, L) \rightarrow 0 \tag{2.3}$$

deduced from the short exact sequence  $0 \rightarrow L(-q) \rightarrow L \rightarrow \mathbb{C}_q \rightarrow 0$ . Now we assume  $L \in J^g \setminus \Theta_q$ . This amounts to assuming  $L(-q) \in J^{g-1} \setminus \Theta$ , and Lemma 2.4 shows  $h^0(C, L(-q)) = h^1(C, L(-q)) = 0$ . Then the exact sequence (2.3) implies  $h^0(C, L) = 1$ , which means  $L \in J'$  by Lemma 2.5. This completes the proof.  $\square$

**Proof of Lemma 2.3.** We take  $L \in J'$ . By Lemma 2.5, we have  $h^0(C, L) = 1$ . For  $q \in C$ , we regard  $H^0(C, L(-q))$  as a subspace of  $H^0(C, L)$  by the injection appearing in Eq. (2.3).

Now we assume  $L \in \bigcap_{q \in \pi^{-1}(a)} \Theta_q$ . This amounts to assuming  $L(-q) \in \Theta$  for any  $q \in \pi^{-1}(a)$ . Then Lemma 2.4 shows that the inclusion  $H^0(C, L(-q)) \rightarrow H^0(C, L)$  is bijective for any  $q \in \pi^{-1}(a)$ . In other words, any non-zero global section of  $L$  must have a zero at  $q$  for any  $q \in \pi^{-1}(a)$ . Therefore  $H^0(C, L(-\pi^*a)) =$

$\cap_{q \in \pi^{-1}(a)} H^0(C, L(-q))$  is isomorphic to  $H^0(C, L)$ , and we have  $h^0(C, L(-\pi^*a)) = h^0(C, L) = 1$ . However, by the projection formula (and the assumption  $L \in J'$ ), we have

$$h^0(C, L(-\pi^*a)) = h^0(\mathbb{P}^1, \pi_*L \otimes O(-1)) = h^0(\mathbb{P}^1, O(-1) \oplus O(-2)^{\oplus r-1}) = 0.$$

This is a contradiction, and the proof is done.  $\square$

### 2.2. Jacobian of the spectral curve

We fix natural numbers  $r$  and  $d$ . Let us consider a polynomial of the form

$$P(x, y) = y^r + s_1(x)y^{r-1} + \dots + s_r(x)$$

with  $s_i(x) \in \mathbb{C}[x]$  is of degree  $\leq di$ . We regard  $x$  as a fixed coordinate function on  $\mathbb{P}^1$ , so that the equation  $P(x, y) = 0$  defines a finite map  $\pi : C_P \rightarrow \mathbb{P}^1$  of degree  $r$ , where  $C_P$  is the spectral curve of  $P$ . One can define  $C_P$  to be the closure of the affine curve defined by  $P(x, y) = 0$  in the Hirzebruch surface of degree  $d$ . More explicitly,  $C_P$  can be described by gluing two plane affine curves defined by the polynomials  $P(x, y)$  and  $z^{dr}P(z^{-1}, z^{-d}w) \in \mathbb{C}[z, w]$  by the relation  $x = z^{-1}$ ,  $y = z^{-d}w$ . The aim of this subsection is to give an explicit representation (the matrix realization) of the variety  $J'$  considered in Section 2.1 assuming  $C = C_P$  is smooth (hence irreducible). We remark that, under this assumption, the genus of  $C_P$  is  $g = \frac{1}{2}(r-1)(rd-2)$ .

We introduce some notations:

$$S_k(x) = \{s(x) \in \mathbb{C}[x] \mid \deg s(x) \leq k\},$$

$$M(r, d) = \left\{ A(x) \in M_r(\mathbb{C}[x]) \mid \begin{array}{l} A(x)_{11} \in S_d(x), \quad A(x)_{1j} \in S_{d+1}(x), \\ A(x)_{i1} \in S_{d-1}(x), \quad A(x)_{ij} \in S_d(x), \end{array} \quad (2 \leq i, j \leq r) \right\},$$

$$V(r, d) = \{P(x, y) = y^r + s_1(x)y^{r-1} + \dots + s_r(x) \in \mathbb{C}[x, y] \mid s_i(x) \in S_{di}(x)\},$$

$$G_r = \left\{ g(x) = \begin{pmatrix} 1 & {}^t\vec{b}_1 x + {}^t\vec{b}_0 \\ 0 & B \end{pmatrix} \mid B \in GL_{r-1}(\mathbb{C}), \quad \vec{b}_1, \vec{b}_0 \in \mathbb{C}^{r-1} \right\}.$$

In this article we denote column vectors using a notation such as  $\vec{b}$ . We write the adjoint action of  $G_r$  on  $M(r, d)$  as

$$g(A(x)) = g(x)^{-1}A(x)g(x) \quad \text{for } g(x) \in G_r, \quad A(x) \in M(r, d). \tag{2.4}$$

Further we introduce a map:

$$\psi : M(r, d) \rightarrow V(r, d); \quad A(x) \mapsto \det(y\mathbb{I}_r - A(x)),$$

and define subsets of  $V(r, d)$  or  $M(r, d)$  as follows:

$$M_P = \psi^{-1}(P(x, y)),$$

$$V_{ir}(r, d) = \{P(x, y) \in V(r, d) \mid C_P \text{ is irreducible}\},$$

$$V_{sm}(r, d) = \{P(x, y) \in V_{ir}(r, d) \mid C_P \text{ is smooth}\},$$

$$M_{ir}(r, d) = \psi^{-1}(V_{ir}(r, d)),$$

$$M_{sm}(r, d) = \psi^{-1}(V_{sm}(r, d)).$$

Then we have  $V(r, d) \supset V_{ir}(r, d) \supset V_{sm}(r, d)$  and  $M(r, d) \supset M_{ir}(r, d) \supset M_{sm}(r, d)$ . Note that each  $M_P, M_{ir}(r, d)$  and  $M_{sm}(r, d)$  is stable with respect to the action of  $G_r$  (2.4). For later use we introduce a lemma:

**Lemma 2.6.** *The action (2.4) of  $G_r$  on  $M_{ir}(r, d)$  is free.*

**Proof.** We have to show that the stabilizer is trivial for all  $A(x) \in M_{ir}(r, d)$ . Since any element of  $G_r$  has an eigenvalue 1, this follows from the following lemma on elementary linear algebra:  $\square$

**Lemma 2.7.** *Let  $K = \mathbb{C}(x)$  be the field of rational functions over  $\mathbb{C}$ . Let  $r \in \mathbb{N}$ , and suppose  $A, B \in M_r(K)$  satisfies the following conditions: (1)  $AB = BA$ , (2)  $B$  is not a scalar matrix, (3)  $B$  has an eigenvalue  $b$  in  $K$ . Then  $\det(y\mathbb{I}_r - A) \in K[y]$  is a reducible polynomial in  $y$ .*

**Proof.** This follows at once by noting that the eigenspace of  $B$  with respect to  $b$  is a non-trivial, proper subspace of  $K^{\oplus r}$  stable under  $A$ .  $\square$

We define a projection map  $\eta$ :

$$\eta : M_{ir}(r, d) \rightarrow M_{ir}(r, d)/G_r. \tag{2.5}$$

In the following, we respectively write  $J_P$  and  $J'_P$  for the variety  $J$  and  $J'$  defined in Section 2.1 associated to  $(C_P, \pi)$ . For  $k \in \mathbb{Z}$  and an invertible sheaf  $L$  on  $C_P$ , we use a notation  $L(k) = L \otimes \pi^* O(k)$ . The main result in this subsection is the following:

**Theorem 2.8** (Cf. [6] 1.4). *Let  $P(x, y) \in V_{sm}(r, d)$ , and let  $\pi : C_P \rightarrow \mathbb{P}^1$  be the finite map defined by  $x$ . Then,  $M_P$  is a principal fiber bundle under  $G_r$ , and the base space  $M_P/G_r$  is isomorphic to  $J'_P$ .*

**Proof.** The first part follows from Lemma 2.6. We construct a surjective map  $M_P \rightarrow J'_P$  and show that each fiber is a principal homogeneous space under  $G_r$ . We remark that a matrix  $A(x) \in M(r, d)$  defines an  $O$ -linear map  $O \oplus O(-1)^{\oplus r-1} \rightarrow O(d) \oplus O(d-1)^{\oplus r-1}$ . (Here we consider  $O(d) = O(d \cdot \infty)$ .) Due to [7] (see also [6] 1.4), the set

$$\{(L, v) \mid L \in J'_P, v : O \oplus O(-1)^{\oplus r-1} \cong \pi_* L\} \tag{2.6}$$

is in one-to-one correspondence with  $M_P$  in such a way that the diagram

$$\begin{array}{ccc} O \oplus O(-1)^{\oplus r-1} & \xrightarrow{A(x)} & O(d) \oplus O(d-1)^{\oplus r-1} \\ v \downarrow & & v^{(d)} \downarrow \\ \pi_* L & \xrightarrow{\pi_* y} & \pi_* L(d) \end{array} \tag{2.7}$$

commutes whenever  $(L, v)$  corresponds to  $A(x) \in M_P$ . (Note that  $A(x)$  must be in  $M_P$  because of the relation  $P(x, y) = 0$  in  $\mathcal{O}_C$ .) By composing this correspondence with the ‘forgetful’ map  $(L, v) \mapsto L$ , we obtain the desired surjection  $M_P \rightarrow J'_P$ . The fiber of this map over  $L \in J'_P$  is the set of isomorphisms  $O \oplus O(-1)^{\oplus r-1} \cong \pi_* L$  which is a principal homogeneous space under  $G_r$  where the action of  $g(x) \in G_r$  is given by  $v \mapsto g(x)^{-1} \circ v \circ g(x)$ . (Here we regard  $g(x)$  as an automorphism on  $O \oplus O(-1)^{\oplus r-1}$  as well as  $O(d) \oplus O(d-1)^{\oplus r-1}$ .) On the set  $M_P$ , this action corresponds to the conjugation. This completes the proof.  $\square$

**Remark 2.9.** Given an invertible sheaf  $L \in J'_P$ , a corresponding matrix  $A(x) \in M_P$  is constructed in the following way. We have to choose an isomorphism  $v : O \oplus O(-1)^{\oplus r-1} \rightarrow \pi_* L$ . This amounts to a choice of a basis of  $H^0(C_P, L(1))$  of the form  $(f_0, f_1, \dots, f_{r-1}, x f_0)$  with  $f_0 \in H^0(C_P, L)$ . The multiplication by  $y$  defines elements  $y f_0 \in H^0(C_P, L(d)) = (f_0 S_d(x)) \oplus (\oplus_{j=1}^{r-1} f_j S_{d-1}(x))$  and  $y f_1, \dots, y f_{r-1} \in H^0(C, L(d+1)) = (f_0 S_{d+1}(x)) \oplus (\oplus_{j=1}^{r-1} f_j S_d(x))$ . Now the matrix  $A(x)$  is characterized by

$$y(f_0, f_1, \dots, f_{r-1}) = (f_0, f_1, \dots, f_{r-1})A(x).$$

In other words, the set  $M_P$  is in one-to-one correspondence with the set of pairs  $(L, v)$  where  $L \in J'_P$  and  $v : S_1(x) \oplus \mathbb{C}^{\oplus r-1} \xrightarrow{\cong} H^0(C_P, L(1))$ . A matrix  $A(x) \in M_P$  corresponds to  $(L, v)$  iff

$$\begin{array}{ccc} S_1(x) \oplus \mathbb{C}^{\oplus r-1} & \xrightarrow{\cong} & H^0(C_P, L(1)) \\ \downarrow A(x) & & \downarrow y \\ S_{d+1}(x) \oplus S_d(x)^{\oplus r-1} & \xrightarrow{\cong} & H^0(C_P, L(d+1)) \end{array} \tag{2.8}$$

commutes.

### 2.3. Characterization of a translation of the theta divisor

We fix  $P \in V_{sm}(r, d)$ . Let  $A(x) \in M_P$ , and let  $L \in J'_P$  be the corresponding invertible sheaf. We take  $a \in \mathbb{P}^1 \setminus \{\infty\}$  unramified with respect to  $\pi$ , so that  $\pi^{-1}(a) = \{q_1, \dots, q_r\}$  consists of  $r$  distinct points. Then  $y(q_1), \dots, y(q_r)$  are

the distinct eigenvalues of the matrix  $A(a)$ . Let  $\rho_{q_i} : \mathbb{C}^r \rightarrow \mathbb{C}$  be the projection to the eigenspace associated with the eigenvalue  $y(q_i)$ . For each  $i = 1, \dots, r$ , we write  $s_{q_i} : H^0(C_P, L) \rightarrow \mathbb{C}$  for the map in the exact sequence (2.3) applied to  $q = q_i$ . In this subsection, we show the following.

**Proposition 2.10.** *For each  $i = 1, \dots, r$ , the following conditions are equivalent:*

- (1)  $\rho_{q_i}(1, 0, \dots, 0) \neq 0$ ,
- (2)  $\text{Im}(s_{q_i}) \neq 0$ ,
- (3)  $L \in J'_P \setminus \Theta_{q_i}$ .

**Proof.** The equivalence between (2) and (3) is a consequence of Lemma 2.4 and the exact sequence (2.3), as is shown in the same way as Lemma 2.2. We show the equivalence between (1) and (2). We recall that the map  $s_{q_i}$  is induced by the map  $\tilde{s}_{q_i}$  in the following short exact sequence of sheaves on  $C_P$

$$0 \rightarrow L(-q_i) \rightarrow L \xrightarrow{\tilde{s}_{q_i}} \mathbb{C}_{q_i} \rightarrow 0.$$

We then have a commutative diagram

$$\begin{array}{ccc} \pi_* L & \xrightarrow{\oplus \tilde{s}_{q_i}} & \bigoplus_{i=1}^r \pi_* \mathbb{C}_{q_i} \\ \pi_* y \downarrow & & \downarrow \oplus_i y(q_i) \\ \pi_* L(d) & \xrightarrow{\oplus \tilde{s}_{q_i}(d)} & \bigoplus_{i=1}^r \pi_* \mathbb{C}_{q_i}, \end{array}$$

where the right vertical map is defined as the multiplication by  $y(q_i)$  on the  $i$ -th component. Let  $v : O \oplus O(-1)^{\oplus r-1} \cong \pi_* L$  be the isomorphism corresponding to  $A(x)$ . The pull-back of this diagram by  $v$  is written as

$$\begin{array}{ccc} O \oplus O(-1)^{\oplus r-1} & \xrightarrow{l_1} & \mathbb{C}_a^{\oplus r} \\ \downarrow A(x) & & \downarrow A(a) \\ O(d) \oplus O(d-1)^{\oplus r-1} & \xrightarrow{l_2} & \mathbb{C}_a^{\oplus r}, \end{array}$$

where  $l_1$  and  $l_2$  are defined simply by the direct sum of  $O(k) \rightarrow \mathbb{C}_a$  for  $k \in \{0, -1, d, d-1\}$ . This means that  $\pi_* \mathbb{C}_{q_i}$  maps to the eigenspace of  $y(q_i)$  in  $\mathbb{C}^r$  under the isomorphism  $v_a : \mathbb{C}_a^r \cong \bigoplus_{i=1}^r \pi_* \mathbb{C}_{q_i}$ . The image of the map  $H^0(\mathbb{P}^1, O \oplus O(-1)^{\oplus r-1}) \rightarrow \mathbb{C}^r$  induced by  $l_1$  is generated by  $(1, 0, \dots, 0)$ . Therefore the image of  $s_{q_i}$  is non-trivial if and only if  $\rho_{q_i}(1, 0, \dots, 0) \neq 0$ . This shows the proposition.  $\square$

**Remark 2.11.** Let us consider the case  $a = \infty$  (still assuming that  $\pi$  is unramified at  $a = \infty$ ). The statement of Proposition 2.10 remains true if we replace  $A(a)$  by  $A(\infty)$ , where the  $(i, j)$ -component of  $A(\infty)$  is the coefficient of the leading term of  $A(x)_{ij}$ . Note that, if we set  $w = y/x^d$ , then  $w(q_1), \dots, w(q_r)$  are the distinct eigenvalues of  $A(\infty)$ .

### 3. Integrable system

#### 3.1. Vector fields

We identify the tangent space  $T_{A(x)}M(r, d)$  at  $A(x) \in M(r, d)$  with the affine space  $M(r, d)$  and write vector fields on  $M(r, d)$  in the matrix form. For a positive integer  $p$  and  $a \in \mathbb{C}$ , we define a vector field  $\Upsilon_a^{(p)}$  on  $M(r, d)$  by the Lax form

$$\Upsilon_a^{(p)}(A(x)) := \frac{1}{x-a} [A(a)^p, A(x)]. \tag{3.1}$$

If we let  $a \in \mathbb{C}$  vary,  $\Upsilon_a^{(p)}$  can be written as a polynomial in  $a$  of degree  $pd$ . For  $j = 0, \dots, pd$ , we define a vector field  $Y_j^{(p)}$  to be the coefficient of  $a^j$  in this polynomial, viz.

$$\Upsilon_a^{(p)} = \sum_{j=0}^{pd} a^j Y_j^{(p)}. \tag{3.2}$$

**Remark 3.1.** For each  $a \in \mathbb{C}$ , the sets of the vector fields  $\{\Upsilon_a^{(p)} | 1 \leq p \leq r - 1\}$  and  $\{\Upsilon_a^{(p)} | 1 \leq p\}$  generate the same vector space by Hamilton–Cayley’s formula for  $A(a)$ . Further for each  $p \geq 1$ , the sets  $\{\Upsilon_a^{(p)} | a \in \mathbb{C}\}$  and  $\{Y_j^{(p)} | 0 \leq j \leq pd\}$  generate the same vector space by Vandermond’s determinant formula.

**Lemma 3.2.** The projection map  $\eta$  (2.5) induces the equality  $\eta_* \Upsilon_a^{(p)}(A(x)) = \eta_* \Upsilon_a^{(p)}(g(A(x)))$  in  $T_{\eta(A(x))}(M_{ir}(r, d)/G_r)$  for all  $g(x) \in G_r$  and  $A(x) \in M_{ir}(r, d)$ .

**Proof.** A vector field  $X$  on  $M_{ir}(r, d)$  satisfies  $\eta_* X(A(x)) = \eta_* X(g(A(x)))$  in  $T_{\eta(A(x))}(M_{ir}(r, d)/G_r)$  if and only if  $X(A(x)) - g_* X(A(x))$  is tangent to  $G_r$ -orbits for any  $g(x) \in G_r$ . A direct calculation shows that  $\Upsilon_a^{(p)}(A(x)) - g_* \Upsilon_a^{(p)}(A(x))$  is a linear combination of the vector fields of Lie  $G_r$ :

$$X_E(A(x)) = [E, A(x)], \quad \text{for } E = E_{ij}, E_{1j}, E'_{1j} \ (2 \leq i, j \leq r). \tag{3.3}$$

Here  $E_{ij}$  is given by  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ , and  $E'_{1j} = xE_{1j}$ . Thus the claim follows.  $\square$

**Corollary 3.3.** For each  $a \in \mathbb{C}$ ,  $1 \leq p \leq r - 1$ ,  $0 \leq j \leq pd$ , we have well-defined vector fields  $\tilde{Y}_a^{(p)}$  and  $\tilde{Y}_j^{(p)}$  on  $M_{ir}(r, d)/G_r$  which satisfies at  $[A(x)] = \eta(A(x))$

$$\tilde{Y}_a^{(p)}([A(x)]) = \eta_* \Upsilon_a^{(p)}(A(x)), \quad \tilde{Y}_j^{(p)}([A(x)]) = \eta_* Y_j^{(p)}(A(x)).$$

We collect some properties of  $\tilde{Y}_j^{(p)}$ .

**Lemma 3.4.** (1) For each  $P \in V_{ir}(r, d)$ , the vector field  $Y_j^{(p)}$  is tangent to  $M_P$  and  $\tilde{Y}_j^{(p)}$  is tangent to  $M_P/G_r$ .

(2) For any  $i$  and  $j$ , the vector fields  $Y_i^{(p)}$  and  $Y_j^{(q)}$  commute. So do  $\tilde{Y}_i^{(p)}$  and  $\tilde{Y}_j^{(q)}$ .

(3) We have  $\tilde{Y}_{pd}^{(p)} = \tilde{Y}_{pd-1}^{(p)} = 0$ . The dimension of the vector space generated by  $\tilde{Y}_j^{(p)}$  with  $1 \leq p \leq r - 1$ ,  $0 \leq j \leq pd - 2$  is at most  $g$ .

**Proof.** 1: A vector field on  $M(r, d)$  is equivalently given as a derivation on the affine ring of  $M(r, d)$ . We write  $t_k(x) = \text{tr}A(x)^k$  and let  $s_k(x)$  be the coefficients of  $y^{r-k}$  in  $\det(y\mathbb{I}_r - A(x))$  for  $1 \leq k \leq r$ . By Newton’s formula, each  $s_k(x)$  is written as a function in  $\mathbb{Q}[t_1(x), \dots, t_k(x)]$ . Since  $\Upsilon_a^{(p)}$  is given by the Lax form (3.1), the associated derivation satisfies  $\Upsilon_a^{(p)}(t_k(x)) = 0$ . Thus we see  $\Upsilon_a^{(p)}(s_k(x)) = 0$ , and the claim follows.

2: This is shown by a direct computation.

3: Since  $Y_{pd}^{(p)}$  and  $Y_{pd-1}^{(p)}$  are tangent to  $G_r$ -orbits,  $\tilde{Y}_{pd}^{(p)}$  and  $\tilde{Y}_{pd-1}^{(p)}$  vanish. Therefore the space in question is generated by  $\tilde{Y}_j^{(p)}$  with  $1 \leq p \leq r - 1$ ,  $0 \leq j \leq pd - 2$ . The number of the members is  $\sum_{p=1}^{r-1} (pd - 1) = \frac{1}{2}(r - 1)(dr - 2) = g$ .

$\square$

### 3.2. Translation invariance

We have seen that  $M_P/G_r$  is isomorphic to an open subset  $J'_p$  of  $J_p^g$  for  $P(x, y) \in V_{sm}(r, d)$  (Theorem 2.8). We regard the restriction of the vector fields  $\tilde{Y}_a^{(p)}$  and  $\tilde{Y}_j^{(p)}$  as vector fields on  $J'_p$ . In this subsection, we show that  $\tilde{Y}_a^{(p)}|_{M_P/G_r}$  and  $\tilde{Y}_j^{(p)}|_{M_P/G_r}$  are translation invariant under the action of the Jacobian  $J_p^0$  on  $J_p^g$ .

The space of translation invariant (holomorphic) vector fields on  $J_p$  is canonically dual to  $H^0(C_P, \Omega_{C_P}^1)$ . Let  $C_P^0$  be the set of points  $q \in C_P$  such that  $\pi : C_P \rightarrow \mathbb{P}^1$  is unramified at  $q$  and  $\pi(q) \neq \infty$ . For  $q \in C_P^0$ , we write  $X_q$  for the vector field corresponding to the linear form  $\omega \mapsto \frac{\omega}{d(x-x(q))}(q)$  on  $H^0(C_P, \Omega_{C_P}^1)$ . (Recall we have fixed a coordinate  $x$  on  $\mathbb{P}^1$ .) Equivalently,  $X_q$  is characterized as follows: the short exact sequence  $0 \rightarrow \mathcal{O}_{C_P} \rightarrow \mathcal{O}_{C_P}(q) \rightarrow T_q C_P \rightarrow 0$  induces the connecting homomorphism

$$T_q C_P \rightarrow H^1(C_P, \mathcal{O}_{C_P}).$$

The image of the vector  $\frac{\partial}{\partial(x-x(q))} \in T_q C_P$  in  $H^1(C_P, \mathcal{O}_{C_P})$  corresponds to  $X_q$  under the Serre duality.

**Remark 3.5.** If  $Q$  is an infinite subset of  $C_P^0$ , the vectors  $X_q$  ( $q \in Q$ ) generate the full space of translation invariant vector fields. Indeed, this is equivalent to the triviality of the cokernel of

$$\bigoplus_{q \in Q} T_q C_P \rightarrow H^1(C_P, \mathcal{O}_{C_P}),$$

which is dual to the kernel of

$$H^0(C_P, \Omega_{C_P}^1) \rightarrow \prod_{q \in Q} T_q^* C_P;$$

but this kernel is trivial by the simple fact that any non-zero differential form has only finitely many zeros.

The main result in this subsection is the following.

**Theorem 3.6** (Cf. [6] 2.2). *Let  $a \in \mathbb{P}^1$  be a point such that  $\pi : C_P \rightarrow \mathbb{P}^1$  is unramified over  $a$ , and let  $\pi^{-1}(a) = \{q_1, \dots, q_r\}$ . Then, for each  $p \geq 1$ , the vector field  $\tilde{\Upsilon}_a^{(p)}|_{M_P/G_r}$  coincides with  $y(q_1)^p X_{q_1} + \dots + y(q_r)^p X_{q_r}$ .*

**Proof.** Let  $A(x) \in M_P$ . Then  $A(a)$  has  $r$  distinct eigenvalues  $y(q_1), \dots, y(q_r)$ . For each  $q \in \pi^{-1}(a)$ , we write  $\Pi_q \in M_r(\mathbb{C})$  for the projector to the eigenspace of  $y(q)$ , and we define a vector field  $\dot{A}_q$  on  $M_P$  by

$$\dot{A}_q(A(x)) = \frac{1}{x - a} [\Pi_q, A(x)].$$

Since  $\Upsilon_a^{(p)}|_{M_P/G_r} = y(q_1)^p \dot{A}_{q_1} + \dots + y(q_r)^p \dot{A}_{q_r}$ , the theorem is reduced to the following lemma. □

**Lemma 3.7.** *We have  $\eta_*(\Upsilon_a^{(p)})(A(x)) = X_q(\eta(A(x)))$  for any  $q \in \pi^{-1}(a)$ ,  $A(x) \in M_P$ .*

**Proof.** In this proof, we omit to indicate  $P$  and write  $C = C_P$ ,  $J = J_P$  etc. Let  $C_\epsilon$  be the scheme whose underlying topological space is  $C$  but with the structure sheaf  $\mathcal{O}_C[\epsilon]$ ,  $\epsilon^2 = 0$ . For  $L \in J$ , the tangent space  $T_L J$  is in one-to-one correspondence with the set of invertible sheaves on  $C_\epsilon$ , which reduce to  $L$  modulo  $\epsilon$ . If  $q \in C^0$  and  $L \in J'$ , the vector  $X_q(L)$  corresponding to the invertible sheaf  $L_q^\epsilon$  is given by

$$H^0(U, L_q^\epsilon) = \left\{ s + \epsilon t \mid \begin{array}{l} s \in H^0(U, L), t \in H^0(U, L(q)), \\ s/(x - a) + t \text{ is holomorphic at } q \end{array} \right\}$$

for an open set  $U$  of  $C$  (cf. [6] 2.2).

Recall that the set  $M_P$  is in one-to-one correspondence with the set of pairs  $(L, v)$  where  $L \in J'$  and  $v$  is an isomorphism  $H^0(C, L(1)) \xrightarrow{\cong} S_1(x) \oplus \mathbb{C}^{\oplus r-1}$  (cf. Remark 2.9). If  $A(x) \in M_P$  corresponds to  $(L, v)$ , the tangent space  $T_{A(x)} M_P$  is in one-to-one correspondence with the pairs of  $(L^\epsilon, v^\epsilon)$  where  $L^\epsilon$  is an invertible sheaf on  $C_\epsilon$  which reduces to  $L$  modulo  $\epsilon$ , and  $v^\epsilon$  is an isomorphism  $(S_1(x) \oplus \mathbb{C}^{\oplus r-1}) \otimes \mathbb{C}[\epsilon] \cong H^0(C_\epsilon, L^\epsilon(1))$  of  $\mathbb{C}[\epsilon]$ -modules, which reduces to  $v$  modulo  $\epsilon$ . A vector  $\dot{A}(x) \in T_{A(x)} M_P \subset T_{A(x)} M_{sm}(r, d) \cong M(r, d)$  corresponds to a pair  $(L^\epsilon, v_\epsilon)$  iff

$$\begin{array}{ccc} (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1}) & \xrightarrow{\cong_{v^\epsilon}} & H^0(C_\epsilon, L^\epsilon(1)) \\ A(x) + \epsilon \dot{A}(x) \downarrow & & \downarrow v \\ (S_{d+1}^\epsilon \oplus S_d^{\epsilon \oplus r-1}) & \xrightarrow{\cong_{v^\epsilon(d)}} & H^0(C_\epsilon, L^\epsilon(d+1)) \end{array} \tag{3.4}$$

commutes. Here we denote  $S_k^\epsilon = S_k(x) \otimes \mathbb{C}[\epsilon]$ .

Now let  $q \in C^0$ . Let  $A(x) \in M_P$  and let  $(L, v)$  be the corresponding pair. Recall that  $L_q^\epsilon$  is the invertible sheaf on  $C_\epsilon$  corresponding to  $X_q(L)$ . In order to complete the proof, we are going to construct an isomorphism



$v_q^\epsilon : S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1} \xrightarrow{\cong} H^0(C_\epsilon, L_q^\epsilon(1))$  such that  $v_q^\epsilon$  reduces to  $v$  modulo  $\epsilon$ , and that the diagram

$$\begin{CD} (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{\oplus r-1}) @>v_q^\epsilon>> H^0(C_\epsilon, L_q^\epsilon(1)) \\ @V{A(x)+\epsilon \dot{A}_q(x)}VV @VV{y}V \\ (S_{d+1}^\epsilon \oplus S_d^{\epsilon \oplus r-1}) @>v_q^{\epsilon(d)}>> H^0(C_\epsilon, L_q^\epsilon(d+1)) \end{CD} \tag{3.5}$$

commutes.

Let  $a = \pi(q)$  and write  $\pi^{-1}(a) = \{q_1 = q, q_2, \dots, q_r\}$ . There exists a section  $s_i \in H^0(C, L(1))$  which does not vanish at  $q_i$  but vanish at  $q_j$  for  $j \neq i$ . However, such an  $s_i$  is not unique. We specify a choice of  $s_i$  as follows. We write  $f_0, f_1, \dots, f_{r-1} \in H^0(C, L(1))$  for the images of  $(1, (0, \dots, 0)), (0, (1, 0, \dots, 0)), \dots, (0, (0, \dots, 1))$  under the isomorphism  $v$ . Then  $((x - a)f_0, f_0, f_1, \dots, f_{r-1})$  is a  $\mathbb{C}$ -basis of  $H^0(C, L(1))$  (and  $(x - a)f_0$  is a  $\mathbb{C}$ -base of  $H^0(C, L)$ ). On the other hand,  $((x - a)f_0, s_1, \dots, s_r)$  is also a basis of  $H^0(C, L(1))$ . Thus we can write

$$\begin{aligned} ((x - a)f_0, s_1, \dots, s_r) &= ((x - a)f_0, f_0, f_1, \dots, f_{r-1}) \cdot \tilde{\Lambda}, \\ \tilde{\Lambda} &= \begin{pmatrix} 1 & * \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_r) \in GL_r(\mathbb{C}). \end{aligned}$$

We can choose  $s_1, \dots, s_r$  so that  $\tilde{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix}$ . This condition determines  $s_i$  up to a multiplication by a non-zero scalar. By definition we have  $((x - a)f_0/s_i)(q_i) = 0$  and  $s_j/s_i(q_i) = \delta_{i,j}$ . Hence, if we set  $\mathbf{f} := ((f_j/s_i)(q_i))_{ij}$ , then  $\mathbf{f} \cdot \Lambda = \mathbb{I}_r$ .

Now we define  $v_q^\epsilon$  to be the composition of

$$\begin{aligned} \sigma : H^0(C, L(1)) \oplus H^0(C, L(1))\epsilon &\xrightarrow{\cong} H^0(C_\epsilon, L_q^\epsilon(1)) \\ (t_1, t_2\epsilon) &\mapsto t_1 + \left( t_2 - \frac{t_1}{s_1}(q) \frac{s_1}{x-a} \right) \epsilon \end{aligned} \tag{3.6}$$

with an isomorphism

$$v \otimes id_{\mathbb{C}[\epsilon]} : (S_1^\epsilon \oplus \mathbb{C}[\epsilon]^{r-1}) \xrightarrow{\cong} H^0(C, L(1)) \otimes \mathbb{C}[\epsilon] = H^0(C, L(1)) \oplus H^0(C, L(1))\epsilon.$$

The change of  $s_1$  by a scalar multiplication does not affect the definition of this map.

It is immediate that  $v_q^\epsilon \text{ mod } \epsilon$  is  $v_q$ . We check the commutativity of (3.5). We write  $\vec{f} = (f_0, \dots, f_{r-1})$  and  $\vec{f}/s_i(q) = (f_0/s_i(q), \dots, f_{r-1}/s_i(q))$ . Then the map (3.6) can be written in terms of matrices

$$\sigma(\vec{f}, \epsilon \vec{f}) = \vec{f} + \epsilon \left( \vec{f} - \frac{1}{x-a} \vec{f} \cdot \Pi \right), \quad \Pi = \tilde{\lambda}_1 \cdot \vec{f}/s_1(q) \in M_r(\mathbb{C}).$$

Therefore, the commutativity of (3.5) means

$$\vec{f} \tilde{\Lambda}(x) \left( \mathbb{I} - \frac{\epsilon}{x-a} \Pi \right) = \vec{f} \left( \mathbb{I} - \frac{\epsilon}{x-a} \Pi \right) (A(x) + \epsilon \dot{A}_q(x)),$$

which follows if we have  $\Pi = \Pi_{q_1}$ . To show the last assertion, we note that the equation  $ys_i = \vec{f}A(x)\tilde{\lambda}_i$  holds in  $H^0(C, L(d+1))$ . Thus we have  $\mathbf{f}A(a)\Lambda = \text{diag}(y(q_1), \dots, y(q_r))$ . Since  $\mathbf{f} = \Lambda^{-1}$ , this means  $\tilde{\lambda}_i$  is an eigenvector of  $A(a)$  belonging to the eigenvalue  $y(q_i)$ . In particular,  $\Pi = \tilde{\lambda}_1 \cdot \vec{f}/s_1(q_1)$  is the projector  $\Pi_{q_1}$ . This completes the proof.  $\square$

By Lemma 3.4(3) and Remark 3.5, we obtain

**Corollary 3.8.** *The space of vector fields on  $M_{ir}(r, d)/G_r$  generated by  $\tilde{Y}_j^{(p)}$  ( $1 \leq p \leq r - 1, 0 \leq j \leq pd - 2$ ) is  $g$ -dimensional.*

### 3.3. Hamiltonian structure

In this subsection, we show that the vector fields  $\tilde{Y}_a^{(p)}$  on  $M_{ir}(r, d)/G_r$  are Hamiltonian, following the method of [6] Section 5 (see also [12] Section 15, [14]).

Let  $a_1, \dots, a_{d+2}$  be distinct points in  $\mathbb{C}$ , and  $\varphi : M(r, d) \rightarrow M_r(\mathbb{C})^{d+2}$  be a map defined by

$$\varphi(A(x)) = (c_1 A(a_1), \dots, c_{d+2} A(a_{d+2})). \quad (3.7)$$

Here  $c_\alpha = P_\alpha(a_\alpha)^{-1}$  with  $P_\alpha(x) = \prod_{\rho \neq \alpha} (x - a_\rho)$ . This map is injective, and the preimage of  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{d+2}) \in \varphi(M(r, d))$  is obtained as  $\varphi^{-1}(\mathbf{Y}) = \sum_{\alpha=1}^{d+2} Y_\alpha P_\alpha(x)$  by Lagrange's interpolation formula.

We set the coordinate on  $M_r(\mathbb{C})^{d+2}$  by using  $y_{ij}^\alpha$  ( $1 \leq \alpha \leq d+2$ ,  $1 \leq i, j \leq r$ ) as  $Y_\alpha = (y_{ij}^\alpha)_{1 \leq i, j \leq r} \in M_r(\mathbb{C})$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{d+2}) \in M_r(\mathbb{C})^{d+2}$ . We define the  $G_r$ -action on  $M_r(\mathbb{C})^{d+2}$  by

$$g(x) : (Y_\alpha)_{1 \leq \alpha \leq d+2} \mapsto (g(a_\alpha)^{-1} Y_\alpha g(a_\alpha))_{1 \leq \alpha \leq d+2}, \quad (3.8)$$

which is compatible with the  $G_r$ -action on  $M(r, d)$ . We equip  $M_r(\mathbb{C})^{d+2}$  with the Poisson bracket which comes from that of  $gl_r(\mathbb{C}) \cong M_r(\mathbb{C})$ :

$$\{y_{ij}^\alpha, y_{kl}^\beta\} = \delta_{\alpha, \beta} (\delta_{j, k} y_{il}^\alpha - \delta_{l, i} y_{kj}^\alpha). \quad (3.9)$$

The associated Casimir functions are  $t_{k, \alpha} = \text{tr}(Y_\alpha^k)$  for  $1 \leq \alpha \leq d+2$ ,  $k \in \mathbb{Z}_{>0}$ .

For  $E \in \text{Lie } G_r$ , we introduce the Hamiltonian functions  $H_E$  on  $M_r(\mathbb{C})^{d+2}$ :

$$H_{E_{1j}} = \sum_{\alpha} y_{j1}^\alpha, \quad H_{E'_{1j}} = \sum_{\alpha} a_\alpha y_{j1}^\alpha, \quad H_{E_{ij}} = \sum_{\alpha} y_{ji}^\alpha, \quad \text{for } 2 \leq i, j \leq r.$$

These satisfy  $H_{[E, E']} = \{H_E, H_{E'}\}$  for any  $E, E' \in \text{Lie } G_r$ . Each  $H_E$  generates a vector field on  $M_r(\mathbb{C})^{d+2}$  compatible with  $X_E$  (3.3) on  $M(r, d)$  via the map  $\varphi$ . The associated moment map  $\mu : M_r(\mathbb{C})^{d+2} \rightarrow (\text{Lie } G_r)^*$  is the unique map which satisfies  $H_E(\mathbf{Y}) = \langle \mu(\mathbf{Y}), E \rangle$  for all  $\mathbf{Y} \in M_r(\mathbb{C})^{d+2}$  and  $E \in \text{Lie } G_r$ . Here  $\langle \cdot, \cdot \rangle$  is the pairing between  $(\text{Lie } G_r)^*$  and  $\text{Lie } G_r$ .

**Lemma 3.9.** (1) The image of  $\varphi$  is an affine subvariety of  $M_r(\mathbb{C})^{d+2}$  determined as the intersection of  $\mu^{-1}(0)$  and  $t_1^{-1}(0)$ , where  $t_1 = \sum_{\alpha} t_{1, \alpha}$ .

(2) The Poisson structure (3.9) induces the Poisson structure on  $\varphi(M_{ir}(r, d))/G_r$ , and hence on  $M_{ir}(r, d)/G_r$  via  $\varphi$ .

**Proof.** 1: The image  $\varphi(M(r, d))$  of  $\varphi$  is a subvariety of  $M_r(\mathbb{C})^{d+2}$  determined by the following conditions:

$$\begin{aligned} \sum_{\alpha=1}^{d+2} y_{11}^\alpha &= 0, \\ \sum_{\alpha=1}^{d+2} y_{j1}^\alpha &= 0, \quad \sum_{\alpha=1}^{d+2} a_\alpha y_{j1}^\alpha = 0, \quad \sum_{\alpha=1}^{d+2} y_{ji}^\alpha = 0, \quad \text{for } 2 \leq i, j \leq r. \end{aligned} \quad (3.10)$$

We see that the last three conditions are nothing but the defining equations for  $\mu^{-1}(0)$  (i.e. the zero of the Hamiltonian functions  $H_E$ ). Summing up the first one and the last one for  $2 \leq i = j \leq r$ , we obtain the defining equation for  $t_1^{-1}(0)$ .

2: Recall that the action of  $G_r$  on  $\varphi(M_{ir}(r, d)) \subset M_r(\mathbb{C})_0^{d+2}$  is free, and that  $\varphi(M_{ir}(r, d)) \subset \mu^{-1}(0) \cap t_1^{-1}(0)$ . Then the Poisson structure (3.9) on  $M_r(\mathbb{C})^{d+2}$  induces the Poisson structure on the quotient space  $\varphi(M_{ir}(r, d))/G_r$ . This is passed to the Poisson structure on  $M_{ir}(r, d)/G_r$  by  $\varphi$ .  $\square$

The following lemma is shown by a direct computation.

**Lemma 3.10.** The vector fields  $(p+1) \prod_{\alpha=1}^{d+2} (a - a_\alpha) \tilde{Y}_a^{(p)}$  on  $M_{ir}(r, d)/G_r$  is Hamiltonian. They are generated by the  $G_r$ -invariant function  $\text{tr} A(a)^{p+1}$  on  $M_{ir}(r, d)$  with respect to the Poisson bracket of Lemma 3.9(2).

Summarizing Theorems 2.8 and 3.6 and Lemma 3.10, we conclude that

**Theorem 3.11** (Cf. [1] 5.3). *The Hamiltonian system  $\psi|_{M_{ir}(r,d)/G_r} : M_{ir}(r,d)/G_r \rightarrow V(r,d)$  is completely integrable. In particular, the general level set is isomorphic to an open subvariety of a Jacobian. More precisely, we have  $M_P/G_r \cong J'_P$  if  $P \in V_{sm}(r,d)$ .*

#### 4. Generalization of even Mumford system

##### 4.1. Matrix realization of the affine Jacobian

In this section, we construct a family of subsystems of  $M_{ir}(r,d)/G_r$  whose general level set is isomorphic to the complement of the union of  $r$  translates of the theta divisor in the Jacobian.

In the following, we write  $A(x) \in M(r,d)$  as

$$A(x) = \begin{pmatrix} v(x) & {}^t\vec{w}(x) \\ \vec{u}(x) & T(x) \end{pmatrix}, \tag{4.1}$$

where  $v(x) \in S_d(x)$ ,  $\vec{u}(x) \in S_{d-1}(x)^{\oplus r-1}$ ,  $\vec{w}(x) \in S_{d+1}(x)^{\oplus r-1}$  and  $T(x) \in M_{r-1}(S_d(x))$ . The coefficients of  $x^k$  ( $k \geq 0$ ) in  $v(x)$ ,  $\vec{w}(x)$ ,  $\vec{u}(x)$  and  $T(x)$  will be denoted by  $v_k, \vec{w}_k, \vec{u}_k$  and  $T_k$ . For  $A(x) \in M(r,d)$ , we define

$$D(A(x); x) = (\vec{u}(x), T(x)\vec{u}(x), \dots, T(x)^{r-2}\vec{u}(x)) \in M_{r-1}(\mathbb{C}[x]), \tag{4.2}$$

$$D(A(x); \infty) = (\vec{u}_{d-1}, T_d\vec{u}_{d-1}, \dots, T_d^{r-2}\vec{u}_{d-1}) \in M_{r-1}(\mathbb{C}). \tag{4.3}$$

Note that  $\det D(A(x); x)$  is a polynomial in  $x$  of degree at most  $g$ , and that the coefficients of  $x^g$  is  $\det D(A(x); \infty)$ .

For each  $c \in \mathbb{P}^1$ , we define the subspaces  $\mathcal{M}_c, \mathcal{M}_c^{ir}$  and  $\mathcal{M}_{c,P}$  of  $M(r,d)$ :

$$\mathcal{M}_c = \{A(x) \in M(r,d) \mid \det D(A(x); c) \neq 0\},$$

$$\mathcal{M}_c^{ir} = \mathcal{M}_c \cap M_{ir}(r,d),$$

$$\mathcal{M}_{c,P} = \mathcal{M}_c \cap M_P.$$

**Lemma 4.1.** 1. *The subset  $\mathcal{M}_c$  is invariant under the action of  $G_r$  on  $M(r,d)$ .*

2. *The action of  $G_r$  on  $\mathcal{M}_c$  is free.*

3. *Let  $c_1, \dots, c_{g+1}$  be distinct points on  $\mathbb{P}^1$ . Then we have*

$$M_{ir}(r,d) \subset \bigcup_{i=1}^{g+1} \mathcal{M}_{c_i} = \bigcup_{c \in \mathbb{P}^1} \mathcal{M}_c \subset M(r,d).$$

**Proof.** Let  $A(x) \in \mathcal{M}_c$  and  $g(x) = \begin{pmatrix} 1 & {}^t\vec{b}(x) \\ 0 & B \end{pmatrix} \in G_r$ .

1: This follows from the relation  $\det D(g(A(x)); x) = \det B^{-1} \cdot \det D(A(x); x)$ .

2: A computation

$$g(A(x)) = \begin{pmatrix} v - {}^t\vec{b} \cdot B^{-1}\vec{u} & {}^t\vec{w} \cdot B + v {}^t\vec{b} - {}^t\vec{b} B^{-1}\vec{u} {}^t\vec{b} - {}^t\vec{b} B^{-1}TB \\ B^{-1}\vec{u} & B^{-1}\vec{u} {}^t\vec{b} + B^{-1}TB \end{pmatrix} \tag{4.4}$$

shows that the condition  $g(A(x)) = A(x)$  implies  $BD(A(x); x) = D(A(x); x)$  and  ${}^t\vec{b}D(A(x); x) = 0$ . If we further assume  $A(x) \in \mathcal{M}_c$ , then we obtain  $B = \mathbb{I}_{r-1}$  and  $\vec{b} = 0$ .

3: The equality in the middle holds since  $\deg_x D(A(x); x) \leq g$ . We show the left inclusion. Assume  $A(x) \notin \mathcal{M}_c$  for all  $c \in \mathbb{P}^1$ . Then  $D(A(x); x)$  is identically zero. Hence we have

$$\det \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, A(x)^{r-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0,$$

which implies that the column vectors span a proper subspace in  $\mathbb{C}(x)^{\oplus r}$  invariant under  $A(x)$ . Therefore the characteristic polynomial of  $A(x)$  is reducible if  $A(x) \notin \mathcal{M}_c$ .  $\square$

This lemma implies that  $\mathcal{M}_c^{ir}/G_r$  is a subsystem of the completely integrable system  $M_{ir}(r, d)/G_r$ . The general level set is described in the following:

**Proposition 4.2.** *Let  $c \in \mathbb{P}^1$  and  $P \in V_{sm}(r, d)$  such that  $\pi : C_P \rightarrow \mathbb{P}^1$  is unramified over  $c$ . Then the level set  $\mathcal{M}_{c,P}/G_r$  of  $\mathcal{M}_c^{ir}/G_r$  is isomorphic to  $J_P \setminus \left(\bigcup_{q \in \pi^{-1}(c)} \Theta_q\right)$ .*

**Proof.** Let  $A(x) \in M_P$  and let  $L \in J'_P$  be the image of  $A(x)$  under the map  $M_P \rightarrow M_P/G_r \cong J'_P$ . According to Proposition 2.10 and Theorem 2.8,  $L$  is in  $\bigcup_{q \in \pi^{-1}(c)} \Theta_q$  if and only if the first entry of any eigenvector of  ${}^tA(c)$  is nonzero. Thus the following lemma on linear algebra completes the proof.  $\square$

**Lemma 4.3.** *Let  $C \in M_r(\mathbb{C})$  be a semi-simple matrix. Writing  ${}^tC = \begin{pmatrix} * & * \\ \vec{c} & C_0 \end{pmatrix}$  with  $C_0 \in M_{r-1}(\mathbb{C})$  and  $\vec{c} \in \mathbb{C}^{r-1}$ , we set  $D = {}^t(\vec{c}, C_0\vec{c}, \dots, C_0^{r-2}\vec{c}) \in M_{r-1}(\mathbb{C})$ . We write  $W$  for the subspace of  $\mathbb{C}^r$  generated by all eigenvectors of  $C$  whose first entries are zero. Then we have  $\dim W = r - 1 - \text{rank}D$ .*

**Proof.** Define  $i : \mathbb{C}^{r-1} \rightarrow \mathbb{C}^r$  by setting the first entry to be zero, and let  $V_0 = i(\mathbb{C}^{r-1})$ . Let  $W_0 = \{i(\vec{w}) \in V_0 \mid \vec{w} \in \mathbb{C}^{r-1}, D\vec{w} = 0\}$ . Since  $\dim W_0 = r - 1 - \text{rank}D$ , it is enough to show  $W = W_0$ . The lemma below shows that  $W$  is the maximal subspace of  $V_0$  which satisfies the condition  $CW \subset W$ . Since  $CW_0 \subset W_0$ , we have  $W_0 \subset W$ . To show the converse, we take  $\vec{w} \in W$ . Since  $CW \subset W$ , we have  $C^k\vec{w} \in W (\subset V_0)$  for all  $k \geq 0$ . By writing down the condition  $C^k\vec{w} \in V_0$  for  $k = 0, 1, \dots$ , we see  $\vec{w} \in W_0$ . This shows  $W \subset W_0$  and we have finished.  $\square$

**Lemma 4.4.** *Let  $f : V \rightarrow V$  be a semi-simple endomorphism of a finite dimensional  $\mathbb{C}$ -vector space. For a subspace  $V'$  of  $V$ , we write  $Ev(V')$  for the set of eigenvectors of  $f$  in  $V'$ . Let  $W$  be a subspace of  $V$ . Let  $W_{st}$  be the maximal subspace in  $W$  which satisfies  $f(W_{st}) \subset W_{st}$ , and let  $W_{\text{eig}}$  be the subspace of  $V$  generated by  $Ev(W)$ . Then we have  $W_{st} = W_{\text{eig}}$ .*

**Proof.** We have  $W_{\text{eig}} \subset W_{st}$  because  $f(W_{\text{eig}}) \subset W_{\text{eig}}$ . It holds that

$$W_{st} \stackrel{(1)}{=} \langle Ev(W_{st}) \rangle \stackrel{(2)}{\subset} \langle Ev(W) \rangle \stackrel{(3)}{=} W_{\text{eig}}.$$

Here (1), (2) and (3) follows by the semi-simplicity of  $f$ , by  $Ev(W_{st}) \subset Ev(W)$  and by definition, respectively.  $\square$

We summarize our main result.

**Theorem 4.5.** *The Hamiltonian system  $\psi|_{\mathcal{M}_c^{ir}/G_r} : \mathcal{M}_c^{ir}/G_r \rightarrow V(r, d)$  is algebraically completely integrable. In particular the general level set is isomorphic to an affine subvariety of a Jacobian. More precisely, if  $P \in V_{sm}(r, d)$  and if  $\pi : C_P \rightarrow \mathbb{P}^1$  is unramified over  $c$ , we have  $\mathcal{M}_{c,P}/G_r \cong J_P^g \setminus \left(\bigcup_{q \in \pi^{-1}(c)} \Theta_q\right)$ .*

**Remark 4.6.** The Hamiltonian vector fields  $\tilde{Y}_a^{(p)}$  are defined on  $\mathcal{M}_c/G_r$  (not only on  $\mathcal{M}_c^{ir}/G_r$ ) because of Lemma 4.1-2.

#### 4.2. Space of representatives

We introduce a space of representatives of  $\mathcal{M}_c/G_r$ . For Beauville’s system, Donagi and Markman [8] constructed such a space of representatives.

We define subspaces  $\mathcal{S}_c$  of  $M(r, d)$  for  $c \in \mathbb{P}^1$  as follows:

$$\begin{aligned} \mathcal{S}_c &= \left\{ A(x) \in M(r, d) \mid A(x) = \begin{pmatrix} v^{(0)} & {}^t\vec{w}^{(0)} \\ \vec{v} & \tau \end{pmatrix} + (x - c) \begin{pmatrix} v^{(1)} & {}^t\vec{w}^{(1)} \\ \vec{u}^{(1)} & T^{(1)} \end{pmatrix} \right. \\ &\quad \left. + \text{higher terms in } (x - c), T^{(1)} \in \mathcal{T} \right\}, \quad \text{for } c \in \mathbb{C}, \\ \mathcal{S}_\infty &= \left\{ A(x) \in M(r, d) \mid A(x) = \begin{pmatrix} 0 & {}^t\vec{w}_{d+1} \\ \vec{0} & O \end{pmatrix} x^{d+1} + \begin{pmatrix} v_d & {}^t\vec{w}_d \\ \vec{0} & \tau \end{pmatrix} x^d + \begin{pmatrix} v_{d-1} & {}^t\vec{w}_{d-1} \\ \vec{v} & T_{d-1} \end{pmatrix} x^{d-1} \right. \\ &\quad \left. + \text{lower terms in } x, T_{d-1} \in \mathcal{T} \right\}, \quad \text{for } c = \infty. \end{aligned}$$

Here  $\tau, \vec{v}$  and the set  $\mathcal{T}$  is as follows:

$$\tau = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in M_{r-1}(\mathbb{C}), \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{r-1}, \tag{4.5}$$

$$\mathcal{T} = \{\rho \in M_{r-1}(\mathbb{C}) \mid \rho_{1j} = 0 \text{ for } j = 1, \dots, r-1\}.$$

By definition,  $\mathcal{S}_c \subset \mathcal{M}_c$  since  $\det D(A(x); x) = 1$  for all  $A(x) \in \mathcal{S}_c$ .

**Proposition 4.7.** For  $c \in \mathbb{P}^1$ , the map given by  $\mathcal{S}_c \times G_r \rightarrow \mathcal{M}_c; (S(x), g(x)) \mapsto g(S(x))$  is an isomorphism. Thus the space  $\mathcal{S}_c$  is a set of representatives of  $\mathcal{M}_c/G_r$ .

This is a consequence of the following lemma:

**Lemma 4.8.** Let  $c \in \mathbb{P}^1$ .

- (1) If  $A(x) \in \mathcal{M}_c$ , then there exists  $g(x) \in G_r$  such that  $g(A(x)) \in \mathcal{S}_c$ .
- (2) If  $g(S(x)) = \tilde{S}(x)$  with  $S(x), \tilde{S}(x) \in \mathcal{S}_c$  and  $g(x) \in G_r$ , then we have  $g(x) = \mathbb{I}_r$ .

**Proof.** 1: We give a proof for  $c \neq \infty$ . (The case of  $c = \infty$  can be shown in a similar way.) Define  $B \in M_{r-1}(\mathbb{C})$  by

$$B = (\vec{u}(c), \zeta_1 \vec{u}(c), \dots, \zeta_{r-2} \vec{u}(c)).$$

Here  $\zeta_i$  ( $1 \leq i \leq r-2$ )  $\in M_{r-1}(\mathbb{C})$  are defined by

$$\zeta_i = T(c)^i + \beta_1 T(c)^{i-1} + \beta_2 T(c)^{i-2} + \cdots + \beta_i \mathbb{I}_{r-1},$$

where  $\beta_i$  ( $1 \leq i \leq r-1$ ) are the coefficients of  $y^i$  in the characteristic polynomial of  $T(c)$ :  $\det(y\mathbb{I}_{r-1} - T(c)) = y^{r-1} + \beta_1 y^{r-2} + \cdots + \beta_{r-1}$ . Since we have assumed  $A(x) \in \mathcal{M}_c$ ,  $B$  is invertible. Then we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix} A(x) \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} * & * \\ \vec{v} & \tau' \end{pmatrix} + (x-c) \begin{pmatrix} * & * \\ * & T \end{pmatrix} + \text{higher terms in } (x-c),$$

where

$$\tau' = \begin{pmatrix} -\beta_1 & -\beta_2 & \cdots & -\beta_{r-1} \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad T \in M_{r-1}(\mathbb{C}).$$

We define  $\vec{b}_1$  and  $\vec{b}_0$  by

$$\vec{b}_1 c + \vec{b}_0 = {}^t(\beta_1, \dots, \beta_{r-1}), \quad \vec{b}_1 = -{}^t(T_{11}, T_{12}, \dots, T_{1r-1}).$$

Consequently we obtain the matrix

$$g(x) = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & {}^t\vec{b}_1 x + {}^t\vec{b}_0 \\ 0 & 1 \end{pmatrix},$$

which satisfies  $g(A(x)) \in \mathcal{S}_c$ .

2: By expanding the relation  $g(S(x)) = \tilde{S}(x)$  in  $(x-c)$  and comparing the coefficient matrices of  $(x-c)^0$  and  $(x-c)^1$ , we see  $g(x) = \mathbb{I}_r$ .  $\square$

### 4.3. Integrable structure of $\mathcal{S}_\infty$

Now we set  $c = \infty$ . We study an explicit relation between  $\mathcal{S}_{\infty, P}$  and  $\text{Div}_{\text{eff}}^g(C_P)$ , then give a description of the vector field on  $\mathcal{S}_\infty$ . These two results may be regarded as the counterparts of the studies on Beauville’s system by Smirnov and Zeitlin [19] Section 4.1-2, and by Fu [10] respectively.

Let  $P \in V_{sm}(r, d)$  be such that  $\infty \in \mathbb{P}^1$  is not a ramification point of  $\pi$ , and set  $\mathcal{S}_{\infty, P} = \mathcal{S}_{\infty} \cap M_P$ . We study the relation between  $\mathcal{S}_{\infty, P}$  and  $\text{Div}_{\text{eff}}^g(C_P)$  by applying the method of Sklyanin [17] (the separation of variables). Let  $\tau : \text{Div}_{\text{eff}}^g(C_P) \rightarrow J_P^g$  be the Abel–Jacobi map. Its restriction  $\tau|_{\tau^{-1}(J'_P)}$  is injective, because the complete linear system of  $L \in J'_P$  is of dimension zero (cf. Lemma 2.5). By abuse of notation, we write  $\tau^{-1}$  for the composition of

$$J'_P \xrightarrow{\cong} \tau^{-1}(J'_P) \hookrightarrow \text{Div}_{\text{eff}}^g(C_P).$$

Our aim is to give an explicit description of the composition  $\kappa$  of

$$\mathcal{S}_{\infty, P} \xrightarrow{\cong} \mathcal{M}_{\infty, P}/G_r \xrightarrow{\cong} J_P^g \setminus \left( \bigcup_{q \in \pi^{-1}(\infty)} \Theta_q \right) \subset J'_P \xrightarrow{\tau^{-1}} \text{Div}_{\text{eff}}^g(C_P).$$

Unfortunately, our result is limited to a subset of  $\mathcal{S}_{\infty, P}$  due to technical difficulties. Define

$$\mathcal{S}'_{\infty, P} = \{A(x) \in \mathcal{S}_{\infty, P} \mid \text{all roots of } \det D(A(x); x) \text{ are simple and belong to } \pi(C_P^0)\}.$$

Note that  $\det D(A(x); x)$  of  $A(x) \in \mathcal{S}_{\infty, P}$  is of degree  $g$  by the definition of  $\mathcal{S}_{\infty, P}$ .

**Proposition 4.9.** *Let  $A(x) \in \mathcal{S}'_{\infty, P}$ . Denote by  $x_1, \dots, x_g$  the simple roots of  $\det D(A(x); x) = 0$ . Let  $\vec{v} \in \mathbb{C}^{r-1}$  be any vector satisfying*

$$\det(\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x)) \neq 0. \tag{4.6}$$

With this  $\vec{v}$ , define

$$y_i := \frac{\det(T(x)\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x))}{\det(\vec{v}, \vec{u}(x), \dots, T(x)^{r-3}\vec{u}(x))} \Big|_{x=x_i}. \tag{4.7}$$

(This is independent of the choice of  $\vec{v}$ .) Then we have  $\kappa(A(x)) = \sum_{i=1}^g (x_i, y_i)$ .

**Proof.** The assumption that  $x_i$  ( $1 \leq i \leq g$ ) is a simple root of  $\det D(A(x); x)$  implies that the rank of  $D(A(x); x_i)$  is  $r - 2$ . By Lemma 4.3, there exists a unique eigenvector of  ${}^t A(x_i)$  whose first component is zero. Denote the eigenvalue by  $\alpha_i$ . Then by Proposition 2.10, the invertible sheaf  $L$  corresponding to  $A(x)$  satisfies  $L \in \bigcap_{i=1}^g \Theta_{(x_i, \alpha_i)}$ . Because of the injectivity of  $\tau|_{\tau^{-1}(J'_P)}$  mentioned above, we see  $\kappa(A(x)) = \tau^{-1}(L) = \sum_{i=1}^g (x_i, \alpha_i)$ . Thus what we have to show is that  $y_i = \alpha_i$ .

For simplicity, we show the case of  $i = 1$ . Since the eigenvalue  $\alpha_1$  of  $A(x_1)$  is also an eigenvalue of  $T(x_1)$ , there exists an eigenvector  $\vec{\mu}'$  of  $T(x_1)$  of the eigenvalue  $\alpha_1$ . It is easy to show  $\det(\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1)) \neq 0$ , and we obtain

$$\alpha_1 = \frac{\det(T(x_1)\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1))}{\det(\vec{\mu}', \vec{u}(x_1), \dots, T(x_1)^{r-3}\vec{u}(x_1))}.$$

Let  $\vec{v} \in \mathbb{C}^{r-1}$  be a vector satisfying (4.6). Then there exist rational functions  $\beta(x), \beta_0(x), \dots, \beta_{r-3}(x) \in \mathbb{C}(x)$  such that

$$\vec{v} = \beta(x)\vec{\mu}' + \sum_{k=0}^{r-3} \beta_k(x)T(x)^k\vec{u}(x).$$

Here  $\beta(x) \neq 0$  by the assumption on  $\vec{v}$ . Now it is immediate to check that  $y_1 = \alpha_1$ . □

Next we describe the vector field on  $\mathcal{S}_{\infty}$  induced from (3.1), using the following lemma:

**Lemma 4.10.** *Let  $X$  be a vector field on  $\mathcal{M}_{\infty} \simeq \mathcal{S}_{\infty} \times G_r$ . The isomorphism  $\Phi : \mathcal{M}_{\infty} \xrightarrow{\sim} \mathcal{S}_{\infty} \times G_r; A(x) \mapsto (S(x), g(x))$  induces the decomposition of  $X$  as  $\Phi_*X = F + G$ , where  $F \in H^0(\mathcal{S}_{\infty} \times G_r, T\mathcal{S}_{\infty})$  and  $G \in H^0(\mathcal{S}_{\infty} \times G_r, TG_r)$ . Then*

$$X(A(x)) = g(F(S(x), g(x))) - [g(x)^{-1}G(S(x), g(x)), A(x)]. \tag{4.8}$$

Here we identify  $T_{g(x)}G_r$  with  $\text{Lie}G_r$ , and  $T_{S(x)}\mathcal{S}_{\infty}$  with the subspace of  $M(r, d)$  via the inclusion  $\mathcal{S}_{\infty} \hookrightarrow M(r, d)$ .

The proof is left to the reader. The Hamiltonian vector field on  $\mathcal{S}_\infty$  becomes as follows:

**Proposition 4.11.** *The projection of the vector field (3.1) onto  $\mathcal{S}_\infty$  is*

$$F_a^{(p)}(A(x)) = \frac{1}{x-a} [A(a)^p, A(x)] + \left[ \begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ \vec{0} & C_p \end{pmatrix}, A(x) \right] \quad \text{at } A(x) \in \mathcal{S}_\infty. \tag{4.9}$$

Here  $(\vec{\gamma}_p, \vec{\beta}_p, C_p) \in \mathbb{C}^{r-1} \oplus \mathbb{C}^{r-1} \oplus M_{r-1}(\mathbb{C})$  is a unique solution of

$$\begin{aligned} C_p \cdot \vec{v} &= (\tau - v_d \mathbb{I}_{r-1}) \cdot \vec{h}_p, \\ \vec{v} \cdot {}^t\vec{\gamma}_p - [C_p, \tau] &= \vec{h}_p \cdot {}^t\vec{w}_{d+1}, \\ (\vec{v} \cdot {}^t\vec{\beta}_p + \vec{u}_{d-2} \cdot {}^t\vec{\gamma}_p - C_p T_{d-1})_{1,i} &= (\vec{h}_p \cdot ({}^t\vec{w}_d + a {}^t\vec{w}_{d+1}) + J_p \tau)_{1,i}, \quad \text{for } 1 \leq i \leq r-1, \end{aligned} \tag{4.10}$$

where  $\tau$  and  $\vec{v}$  are defined in (4.5), and  $\vec{h}_p$  and  $J_p$  are

$$A(a)^p = \begin{pmatrix} * & * \\ \vec{h}_p & J_p \end{pmatrix}. \tag{4.11}$$

**Proof.** The equations (4.10) are obtained by solving (4.8) for  $F(S(x), g(x))$  and  $G(S(x), g(x))$  at  $X = \mathcal{Y}_a^{(p)}$  and  $g = \mathbb{I}_r$ . Eq. (4.8) becomes

$$T_a^{(p)}(A(x)) = F_a^{(p)}(A(x)) - [G_a^{(p)}(A(x)), A(x)], \tag{4.12}$$

where  $F_a^{(p)}(A(x))$  is of the form

$$F_a^{(p)}(A(x)) = x^{d+1} \begin{pmatrix} 0 & * \\ \vec{0} & O \end{pmatrix} + x^d \begin{pmatrix} * & * \\ \vec{0} & O \end{pmatrix} + x^{d-1} \begin{pmatrix} * & * \\ \vec{0} & \rho \end{pmatrix} + \text{lower terms in } x.$$

Here  $\rho \in \mathcal{T}$  (4.5), and  $G_a^{(p)}(A(x)) \in \text{Lie } G_r$  is of the form

$$G_a^{(p)}(A(x)) = \begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ \vec{0} & C_p \end{pmatrix} \quad (\vec{\gamma}_p, \vec{\beta}_p \in \mathbb{C}^{r-1}, C_p \in M_{r-1}(\mathbb{C})).$$

The matrix  $G_a^{(p)}(A(x))$  is determined as follows. In the LHS of (4.12), the  $(i, 1)$ -entries ( $2 \leq i \leq r$ ) and  $(i, j)$ -entries ( $2 \leq i, j \leq r$ ) are

$$\begin{aligned} (v_d \mathbb{I}_{r-1} - \tau) \cdot \vec{h}_p x^{d-1} &+ \text{lower order in } x, \\ \vec{h}_p \cdot {}^t\vec{w}_{d+1} x^d &+ (\vec{h}_p \cdot ({}^t\vec{w}_d + a {}^t\vec{w}_{d+1}) + [J_p, \tau]) x^{d-1} + \text{lower order in } x. \end{aligned}$$

In the RHS of (4.12), the  $(i, 1)$ -entries ( $2 \leq i \leq r$ ) and  $(i, j)$ -entries ( $2 \leq i, j \leq r$ ) are

$$\begin{aligned} -C_p \cdot \vec{v} x^{d-1} &+ \dots, \\ (\vec{v} \cdot {}^t\vec{\gamma}_p - [C_p, \tau]) x^d &+ (\vec{v} \cdot {}^t\vec{\beta}_p + \vec{u}_{d-2} \cdot {}^t\vec{\gamma}_p - [C_p, T_{d-1}] + \rho) x^{d-1} + \dots. \end{aligned}$$

We obtain the Eqs. (4.10) for  $(\vec{\gamma}_p, \vec{\beta}_p, C_p)$  by comparing the LHS and the RHS.

The solution to Eqs. (4.10) is unique since the first and second equations completely determine  $C_p$  and  $\vec{\gamma}_p$  and then the third equation completely determines the value of  $\vec{\beta}_p$ .  $\square$

#### 4.4. Examples

The case of  $r = 2$ : we have the space of representatives as

$$\mathcal{S}_\infty = \left\{ A(x) = \begin{pmatrix} v(x) & w(x) \\ u(x) & t(x) \end{pmatrix} \right.$$

$$= \left( \begin{matrix} 0 & w_{d+1} \\ 0 & 0 \end{matrix} \right) x^{d+1} + \left( \begin{matrix} v_d & w_d \\ 0 & 0 \end{matrix} \right) x^d + \left( \begin{matrix} v_{d-1} & w_{d-1} \\ 1 & 0 \end{matrix} \right) x^{d-1} + \text{lower terms in } x \Bigg\}.$$

For  $P \in V_{sm}(r, d)$ , the genus of the curve  $C_P$  is  $d - 1$ . The isomorphism given in Proposition 4.9 becomes very simple:  $x_k$  ( $k = 1, \dots, d - 1$ ) are the zeros of  $u(x)$  and  $y_k = t(x_k)$ . The vector field on  $\mathcal{S}_\infty$  (4.9) becomes

$$F_a^{(1)}(A(x)) = \left[ \frac{1}{x-a} A(a) + u(a) \begin{pmatrix} 0 & (x+a-u_{d-2})w_{d+1} + w_d \\ 0 & -v_d \end{pmatrix}, A(x) \right].$$

Let  $V = \{P(x, y) \in V(2, d) \mid s_1(x) \equiv 0\}$ . The restriction  $\psi^{-1}(V) \cap \mathcal{S}_\infty \rightarrow V$  of our system  $\psi|_{\mathcal{S}_\infty} : \mathcal{S}_\infty \rightarrow V(2, d)$  coincides with the even Mumford system introduced by Vanhaeche [21].

The case of  $r = 3$ : this is a new system.  $\mathcal{S}_\infty$  is written as

$$\mathcal{S}_\infty = \left\{ A(x) = \begin{pmatrix} v(x) & w^{(1)}(x) & w^{(2)}(x) \\ u^{(1)}(x) & T^{(1,1)}(x) & T^{(1,2)}(x) \\ u^{(2)}(x) & T^{(2,1)}(x) & T^{(2,2)}(x) \end{pmatrix} \right. \\ = \left( \begin{matrix} 0 & w_{d+1}^{(1)} & w_{d+1}^{(2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right) x^{d+1} + \left( \begin{matrix} v_d & w_d^{(1)} & w_d^{(2)} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right) x^d + \left( \begin{matrix} v_{d-1} & w_{d-1}^{(1)} & w_{d-1}^{(2)} \\ 1 & 0 & 0 \\ 0 & T_{d-1}^{(2,1)} & T_{d-1}^{(2,2)} \end{matrix} \right) x^{d-1} \\ \left. + \text{lower terms in } x \right\}.$$

For  $P \in V_{sm}(r, d)$ , the genus  $g$  of  $C_P$  is  $3d - 2$ . The isomorphism given in Proposition 4.9 becomes as follows:  $x_k$  are the zeros of  $D(A(x))$  (4.2) and  $y_k$  (4.7) has two equivalent descriptions:

$$y_k = \frac{u^{(2)}(x)T^{(1,1)}(x) - u^{(1)}(x)T^{(2,1)}(x)}{u^{(2)}(x)} \Bigg|_{x=x_k} \quad \text{or} \quad \frac{u^{(1)}(x)T^{(2,2)}(x) - u^{(2)}(x)T^{(1,2)}(x)}{u^{(1)}(x)} \Bigg|_{x=x_k}.$$

The vector field on  $\mathcal{S}_\infty$  is written as

$$F_a^{(p)}(A(x)) = \left[ \frac{1}{x-a} A(a) + \begin{pmatrix} 0 & {}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p \\ & C_p \end{pmatrix}, A(x) \right] \quad \text{for } p = 1, 2,$$

where

$${}^t\vec{\gamma}_p x + {}^t\vec{\beta}_p = h_p^{(1)} \left( (x+a-u_{d-2}^{(1)}) {}^t\vec{w}_{d+1} + {}^t\vec{w}_d \right) \\ + {}^t(h_p^{(2)} w_{d+1}^{(2)} (x + T_{d-1}^{(2,1)} - u_{d-2}^{(1)}) + (J_p)_{1,2}, h_p^{(2)} w_{d+1}^{(2)} T_{d-1}^{(2,2)}) \\ C_p = h_p^{(1)} \begin{pmatrix} -v_d & 0 \\ 1 & -v_d \end{pmatrix} + h_p^{(2)} \begin{pmatrix} 1 & w_{d+1}^{(2)} \\ -v_d & -w_{d+1}^{(1)} \end{pmatrix}.$$

Here  $J_p$  and  $\vec{h}_p = \begin{pmatrix} h_p^{(1)} \\ h_p^{(2)} \end{pmatrix}$  are given at (4.11).

**Acknowledgement**

The subsection on related works was included at the suggestion of the referee. The authors thank the referee for this advice.

**References**

[1] S. Abenda, Y. Fodorov, On the weak Kowalevski–Painlevé property for hyperelliptically separable systems, Acta. Appl. Math. 60 (2000) 137–178.



- [2] M.R. Adams, J. Harnad, E. Previato, Isospectral Hamiltonian flows in finite and infinite dimensions, I. Generalized Moser systems and moment maps into loop algebras, *Comm. Math. Phys.* 117 (1988) 451–500.
- [3] M.R. Adams, J. Harnad, J. Hurtubise, Isospectral Hamiltonian flows in finite and infinite dimensions, II. Integration of flows, *Comm. Math. Phys.* 134 (1990) 555–585.
- [4] M. Adler, P. van Moerbeke, The complex geometry of the Kowalewski–Painlevé analysis, *Invent. Math.* 97 (1989) 3–51.
- [5] M. Adler, P. van Moerbeke, P. Vanhaecke, Algebraic integrability, Painlevé geometry and Lie algebras, in: *Ergeb. Math. Grenzgeb.* 3. Folge, vol. 47, Springer-Verlag, Berlin, 2004.
- [6] A. Beauville, Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables, *Acta. Math.* 164 (1990) 211–235.
- [7] A. Beauville, M.S. Narasimhan, S. Ramanan, Spectral curves and the generalized theta divisor, *J. Reine Angew. Math.* 398 (1989) 169–179.
- [8] R. Donagi, E. Markman, Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles, in: *Lecture Notes in Mathematics*, vol. 1620, 1996, pp. 1–119.
- [9] R.L. Fernandes, P. Vanhaecke, Hyperelliptic prym varieties and integrable systems, *Comm. Math. Phys.* 221 (2001) 169–196.
- [10] B. Fu, Champs de vecteurs invariants par translation sur les jacobiennes affines des courbes spectrales, *C. R. Math. Acad. Sci. Paris* 337 (2) (2003) 105–110.
- [11] R. Inoue, T. Yamazaki, Cohomological study on variants of the Mumford system, and integrability of the Noumi–Yamada system, *Comm. Math. Phys.* 265 (2006) 699–719.
- [12] A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, Berlin, 1976.
- [13] D. Mumford, *Tata Lectures on Theta II*, Birkhäuser, 1984.
- [14] M.A. Olshanetsky, A.M. Perelomov, Integrable systems and finite-dimensional Lie algebras, in: *Encyclopedia of Mathematical Sciences*, vol. 16, Springer-Verlag, Berlin, 1994, pp. 87–116.
- [15] M. Pendroni, P. Vanhaecke, A Lie algebraic generalization of the Mumford system, its symmetries and its multi-Hamiltonian structure, *Regul. Chaotic Dyn.* 3 (1998) 132–160.
- [16] A.G. Reyman, M.A. Semenov-Tian-Shansky, Group-theoretical methods in the theory of finite-dimensional integrable systems, in: *Encyclopedia of Mathematical Sciences*, vol. 16, Springer-Verlag, Berlin, 1994, pp. 116–225.
- [17] E.K. Sklyanin, Separation of variables — New trends, *Prog. Theor. Phys. Suppl.* 118 (1995) 35–60.
- [18] A. Nakayashiki, F.A. Smirnov, Cohomologies of affine Jacobi varieties and integrable systems, *Comm. Math. Phys.* 217 (2001) 623–652.
- [19] F.A. Smirnov, V. Zeitlin, Affine Jacobi varieties of spectral curves and integrable models, [math-ph/0203037](https://arxiv.org/abs/math-ph/0203037).
- [20] P. Vanhaecke, Linearising two-dimensional integrable systems and the construction of action-angle variables, *Math. Z.* 211 (1992) 265–313.
- [21] P. Vanhaecke, Integrable systems in the realm of algebraic geometry, in: *Lecture Notes in Mathematics*, vol. 1638, 2001.