Symbolic Computation of Quantities Associated With Time-Periodic Dynamical Systems[1]

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*Many dynamical systems can be modeled by a set of linear/nonlinear ordinary differential equations with periodic time-varying coefficients. The state transition matrix (STM) $\boldsymbol{\Phi}(t,\alpha)$, associated with the linear part of the equation, can be expressed in terms of the periodic Lyapunov–Floquét (L-F) transformation matrix $\boldsymbol{Q}(t,\alpha)$ and a time-invariant matrix $\boldsymbol{R}(\alpha)$ containing a set of symbolic system parameters α. Computation of $\boldsymbol{Q}(t,\alpha)$ and $\boldsymbol{R}(\alpha)$ in symbolic form as a function of α is of paramount importance in stability, bifurcation analysis, and control system design. In earlier studies, since $\boldsymbol{Q}(t,\alpha)$ and $\boldsymbol{R}(\alpha)$ were available only in numerical forms, general results for parameter unfolding and control system design could not be obtained in the entire parameter space. In 2009, an attempt was made by Butcher et al. (2009, "Magnus' Expansion for Time-Periodic Systems: Parameter Dependent Approximations," Commun. Nonlinear Sci. Numer. Simul., **14**(12), pp. 4226–4245) to compute the $\boldsymbol{Q}(t,\alpha)$ matrix in a symbolic form using the Magnus expansions with some success. In this work, an efficient technique for symbolic computation of $\boldsymbol{Q}(t,\alpha)$ and $\boldsymbol{R}(\alpha)$ matrices is presented. First, $\boldsymbol{\Phi}(t,\alpha)$ is computed symbolically using the shifted Chebyshev polynomials and Picard iteration method as suggested in the literature. Then, $\boldsymbol{R}(\alpha)$ is computed using a Gaussian quadrature integral formula. Finally, $\boldsymbol{Q}(t,\alpha)$ is computed using the matrix exponential summation method. Using* MATHEMATICA*, this approach has successfully been applied to the well-known Mathieu equation and a four-dimensional time-periodic system in order to demonstrate the applications of the proposed method to linear as well as nonlinear problems.* [DOI: 10.1115/1.4033382]

1 Introduction

Dynamic systems such as asymmetric rotors, parametrically excited pendulums, columns, helicopter blades, automotive components, and several others can be modeled by a set of ordinary differential equations with periodically time-varying coefficients.

In 1991, Sinha and Wu [1] proposed an innovative and efficient numerical scheme for the analysis of linear systems with periodically varying parameters. The approach is based on the idea that the state vector and the periodic matrix of the system can be expanded in terms of Chebyshev polynomials over the principal period. Such an expansion converts the original problem into a set of linear algebraic equations from which the solution in the interval of one period can be obtained. Further, the technique is combined with Floquét theory to provide the stability conditions via the eigenanalysis procedure. Sinha et al. [2] computed the L-F transformation matrix $\mathbf{Q}(t)$ and $\mathbf{R}$ by factoring the $\boldsymbol{\Phi}(t)$ matrix. Sinha and Joseph [3] introduced optimal control theory in conjunction with Floquét analysis to design full state and observer-based controllers for periodic systems. Sinha and Pandiyan [4] used these transformed equations to construct solutions of nonlinear time-periodic systems via time-dependent normal form and center manifold theories. For the first time, Sinha and Butcher [5] presented a symbolic computation of the STM $\boldsymbol{\Phi}(t,\alpha)$ for linear time-periodic dynamical systems. Dávid and Sinha [6] presented a local semi-analytical method of quantitative bifurcation analysis for parameter unfolding in time-periodic nonlinear systems by using the L-F transformation in the neighborhood of a bifurcation point.

The critical limitation of these previous methods is the inability to determine the L-F transformation matrix in a symbolic form. Since the L-F transformation was available only in numerical form, general results for parameter unfolding and control system design [7] could not be obtained in the entire parameter space. Motivated by these issues, in 2009 Butcher et al. [8] presented a technique to compute the L-F transformation $\mathbf{Q}(t,\alpha)$ in a symbolic form using the Magnus expansion. Although the approach appears to be quite general, the algorithm can get complicated and the convergence could be problematic for large values of system parameters. The stability boundaries obtained in Ref. [8] for the Mathieu equation do not match the exact boundaries even for relatively smaller values of system parameters. The authors do not attempt to improve the accuracy due to the complexity involved in construction of the additional "trees." Nevertheless, the Magnus expansion method could be advantageous since one does not have to compute the monodromy matrix in a symbolic form before computing the L-F transformation matrix.

In this work, a technique for symbolic computation of L-F transformation $\mathbf{Q}(t,\alpha)$ and $\mathbf{R}(\alpha)$ matrices is presented. First, the STM $\boldsymbol{\Phi}(t,\alpha)$ is computed in a symbolic form based on Chebyshev polynomial expansion technique as described in Ref. [5]. Once $\Phi(T,\alpha)$ is known, $\mathbf{R}(\alpha)$ can be computed using a quadrature integral formula. Using MATHEMATICA, this approach has successfully been applied to the well-known Mathieu equation and a time-periodic double inverted pendulum system in order to demonstrate the applicability of the proposed work.

2 Floquét Theory

Floquét theory focuses on predicting the stability and response of linear differential equations with periodic coefficients. The stability conditions are based on the characteristic multipliers (Floquét multipliers) of the STM $\boldsymbol{\Phi}(t)$ evaluated at the end of one principle period, known as the monodromy or the Floquét

[1]Presented at the IDETC/CIE 2015 Conference, Aug. 2–5, 2015, Boston, MA.
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Contributed by the Design Engineering Division of ASME for publication in the JOURNAL OF COMPUTATIONAL AND NONLINEAR DYNAMICS. Manuscript received July 1, 2015; final manuscript received March 30, 2016; published online May 13, 2016. Assoc. Editor: Bogdan I. Epureanu.

transition matrix (FTM) $\mathbf{\Phi}(T)$. Floquét theory is extremely useful in stability analysis, as once the problem is solved for one full period, solutions are known for all time t.

Consider a time-periodic system in state-space form given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{1}$$

Let $\mathbf{\Phi}(t)$ be the STM of Eq. (1) with $\mathbf{\Phi}(0) = \mathbf{I}$, the identity matrix. Then, by Yakubovich and Starzhinski [9]

$$\dot{\mathbf{\Phi}}(t) = \mathbf{A}(t)\mathbf{\Phi}(t); \quad \mathbf{\Phi}(0) = \mathbf{I} \tag{2}$$

and

$$\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)\mathbf{F} \tag{3}$$

which implies

$$\mathbf{\Phi}(T) = \mathbf{F} \tag{4}$$

where $\mathbf{F}$ is a constant matrix and

$$\dot{\mathbf{\Phi}}(t+T, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t+T, t_0) \tag{5}$$

For time greater than one principle period, the STM is given by

$$\mathbf{\Phi}(\delta) = \mathbf{\Phi}(t)\mathbf{\Phi}^n(T) \tag{6}$$

where $\delta = t + nT$, $t \in [0, T]$, $n = 1, 2, 3\ldots$ Equation (6) is significant in the study of periodic differential equations since it infers that if a solution is known for the time variations in the principle period, then the solution is known for all time t.

3 Computation of the STM in a Symbolic Form

As indicated above, Sinha and Butcher [5] have outlined a technique for computing the fundamental solution matrix for a linear time-periodic dynamical system explicitly as a function of system parameters via Picard iteration and expansion in shifted Chebyshev polynomials. Hence, it is possible to express the local stability conditions as a function of system parameters α in a closed form. A brief description of the methodology is given below for completeness.

Consider the linear time-periodic system

$$\dot{\mathbf{x}}(t, \alpha) = \mathbf{A}(t, \alpha)\mathbf{x}(t, \alpha), \quad \mathbf{x}(0, \alpha) = \mathbf{x}^0 \tag{7}$$

The $n \times n$ matrix $\mathbf{A}(t, \alpha) = \mathbf{A}(t + T, \alpha)$, where T is the principle period of the system. The fundamental solution matrix $\mathbf{\Phi}(t, \alpha)$ of Eq. (7) satisfies $\dot{\mathbf{\Phi}}(t, \alpha) = \mathbf{A}(t, \alpha)\mathbf{\Phi}(t, \alpha)$, where $\mathbf{\Phi}(0, \alpha) = \mathbf{I}$ and the solution for the given initial conditions may be expressed as $\mathbf{x}(t, \alpha) = \mathbf{\Phi}(t, \alpha)\mathbf{x}^0$.

An equivalent integral form of Eq. (7) is

$$\mathbf{x}(t, \alpha) = \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau, \alpha)\mathbf{x}(\tau, \alpha)d\tau \tag{8}$$

As the zeroth approximation, let $\mathbf{x}^0(t, \alpha) = \mathbf{x}(0, \alpha) = \mathbf{x}^0$. Use of Eq. (8) to determine the $(k+1)$th term is then

$$\begin{aligned}\mathbf{x}^{(k+1)}(t, \alpha) &= \mathbf{x}^0 + \int_0^t \mathbf{A}(\tau_k, \alpha)\mathbf{x}^{(k)}(\tau_k, \alpha)d\tau_k \\ &= \left[\mathbf{I} + \int_0^t \mathbf{A}(\tau_k, \alpha)d\tau_k \right. \\ &\quad + \int_0^t \mathbf{A}(\tau_k, \alpha)d\tau_0 \int_0^{\tau_k} \mathbf{A}(\tau_{k-1}, \alpha)d\tau_{k-1}d\tau_k + \ldots \\ &\quad \left. + \int_0^t \mathbf{A}(\tau_k, \alpha)\ldots \int_0^{\tau_1} \mathbf{A}(\tau_0, \alpha)d\tau_0 \cdots d\tau_k \right]\mathbf{x}^0 \end{aligned} \tag{9}$$

where $\tau_0, \tau_1, \ldots, \tau_k$ are dummy variables. This series of integrals is an approximation to the fundamental matrix $\mathbf{\Phi}(t, \alpha)$ because it is truncated at a finite number of terms, while the true solution is an infinite series.

If $\mathbf{A}(t, \alpha) = \mathbf{A}(\alpha)$, a constant matrix, then this series results in the power series definition of the exponential solution of Eq. (9), such that

$$\mathbf{x}(t, \alpha) = e^{\mathbf{X}(\alpha)t}\mathbf{x}^0 = \left[\mathbf{I} + \mathbf{X}(\alpha)t + \frac{[\mathbf{X}(\alpha)t]^2}{2!} + \ldots + \frac{[\mathbf{X}(\alpha)t]^n}{n!}\right]\mathbf{x}^0 \tag{10}$$

Unfortunately, the symbolic evaluation of the fundamental matrix via Eq. (9), in general, leads to complicated expressions for $\mathbf{\Phi}(t, \alpha)$ and is not efficient due to the necessary repeated integration by parts. Instead, the following approach is taken, which results in a more efficient approximation of $\mathbf{\Phi}(t, \alpha)$.

The transformation $t = T\tau$ is applied to Eq. (7), which normalizes the system matrix's principal period to one. The equation then becomes

$$\frac{d\mathbf{x}(\tau, \alpha)}{d\tau} = \overline{\mathbf{A}}(\tau, \alpha)\mathbf{x}(\tau, \alpha) \tag{11}$$

where $\overline{\mathbf{A}}(\tau + 1, \alpha) = \overline{\mathbf{A}}(\tau, \alpha), \mathbf{x}(0, \alpha) = \mathbf{x}^0$, and $\overline{\mathbf{A}}(\,)$ is expressed as $\overline{\mathbf{A}}(\tau, \alpha) = \overline{\mathbf{A}}_1(\alpha)f_1(\tau) + \cdots + \overline{\mathbf{A}}_r(\alpha)f_r(\tau)$, $f_i(\tau) = f_i(\tau + 1)$, and $\overline{\mathbf{A}}_i(\alpha) = \hat{\mathbf{T}}\mathbf{A}_i(\alpha), i = 1, \ldots, r$. The *Chebyshev polynomial matrix* $\hat{\mathbf{T}}^{\mathbf{T}}(\tau)$ is used to expand the normalized system matrix in m shifted Chebyshev polynomials of the first kind as

$$\overline{\mathbf{A}}(\tau, \alpha) = \hat{\mathbf{T}}^{\mathbf{T}}(\tau)\mathbf{D}(\alpha) \tag{12}$$

where the $nm \times n$ Chebyshev coefficient matrix $\mathbf{D}(\alpha)$ is defined as

$$\mathbf{D}(\alpha) = \sum_{i=1}^{r} \overline{\mathbf{A}}_i(\alpha) \otimes \mathbf{d}_i \tag{13}$$

The $m \times 1$ column vectors $\mathbf{d}_i$ contain the known coefficients of the Chebyshev expansion of the one-periodic functions, and $\otimes$ is the Kronecker product.

Then, using the integration operational and product operational matrices (see Ref. [5] for details)

$$\mathbf{\Phi}^{(p,m)}(\tau, \alpha) = \hat{\mathbf{T}}^T(\tau)\left[\hat{\mathbf{I}} + \left(\sum_{k=1}^{p} [\mathbf{L}(\alpha)]^{k-1}\right)\mathbf{P}(\alpha)\right] = \hat{\mathbf{T}}^T(\tau)\mathbf{B}(\alpha) \tag{14}$$

where $\mathbf{B}(\alpha)$ contains the Chebyshev coefficients of the elements of $\mathbf{\Phi}(\tau, \alpha)$. By selecting a value for p, the number of Picard iterations, this truncated expression gives an approximate solution to any desired degree of accuracy. While this is valid only in the interval $t \in [0, T]$ or $\tau \in [0, 1]$, the solution can be easily extended for $t > T(\tau > 1)$ by utilizing the formula

$$\mathbf{\Phi}^{(p,m)}(\tau, \alpha) = \mathbf{\Phi}^{(p,m)}(\eta, \alpha)[\mathbf{\Phi}^{(p,m)}(1, \alpha)]^k \tag{15}$$

where $\tau = k + \eta, \eta \in [0, 1], k = 1, 2, \ldots$ The matrix $\mathbf{\Phi}^{(p,m)}(1, \alpha)$ is the FTM whose eigenvalues (Floquét multipliers) determine the stability characteristics of the system. While these expressions are in terms of normalized time, the substitution $\tau = t/T$ yields the result in real time.

4 Computation of the L-F Transformation Matrix in a Symbolic Form

As a corollary to the Floquét theory, the L-F theorem states that the STM $\mathbf{\Phi}(t, \alpha)$ of Eq. (7) can be written as the product of two $n \times n$ matrices

$$\begin{aligned}\mathbf{\Phi}(t,\alpha) &= \mathbf{L}(t,\alpha)e^{\mathbf{C}(\alpha)t}\\ \mathbf{\Phi}(0,\alpha) &= \mathbf{L}(0,\alpha) = \mathbf{L}(T,\alpha) = \mathbf{I}\end{aligned} \tag{16}$$

where $\mathbf{L}(t,\alpha)$ is a T -periodic $n \times n$ matrix and $\mathbf{C}(\alpha)$ is a constant $n \times n$ matrix. In general, $\mathbf{L}$ and $\mathbf{C}$ are complex. $\mathbf{\Phi}(t,\alpha)$ can also be factored as

$$\begin{aligned}\mathbf{\Phi}(t,\alpha) &= \mathbf{Q}(t,\alpha)e^{\mathbf{R}(\alpha)t}\\ \mathbf{\Phi}(0,\alpha) &= \mathbf{Q}(0,\alpha) = \mathbf{Q}(2T,\alpha) = \mathbf{I}\end{aligned} \tag{17}$$

where $\mathbf{Q}(t,\alpha)$ is real and $2T$-periodic and $\mathbf{R}(\alpha)$ is a real matrix.

A note of interest: If all of the Floquét multipliers lie in the left-half of the complex plane, then $\mathbf{Q}(t)$ is $2T$-periodic and has symmetry of $\mathbf{Q}(t+T) = -\mathbf{Q}(t)$. However, if all the Floquét multipliers lie in the right-half of the complex plane, the real and complex L-F transformations coincide, both being T-periodic and real.

Obviously, one must first compute the STM $\mathbf{\Phi}(t,\alpha)$ as outlined in Sec. 3. In this paper, only the real transformation will be considered. The computation of $\mathbf{Q}(t,\alpha)$ and $\mathbf{R}(\alpha)$ is not a simple task, except in the special class of commutative systems [10].

Applying the $2T$-periodic L-F transformation

$$\mathbf{x}(t,\alpha) = \mathbf{Q}(t,\alpha)\mathbf{z}(t,\alpha) \tag{18}$$

to Eq. (7) produces a real representation as

$$\dot{\mathbf{z}}(t,\alpha) = \mathbf{R}(\alpha)\mathbf{z}(t,\alpha) \tag{19}$$

where

$$\mathbf{R}(\alpha) = \mathbf{Q}^{-1}(t,\alpha)\left[\mathbf{A}(t,\alpha)\mathbf{Q}(t,\alpha) - \dot{\mathbf{Q}}(t,\alpha)\right] \tag{20}$$

Evaluating Eq. (17) at $t = 2T$

$$\mathbf{\Phi}(2T,\alpha) = \mathbf{\Phi}^2(T,\alpha) = \mathbf{I}e^{2R(\alpha)T} \tag{21}$$

then it follows that

$$\mathbf{R}(\alpha) = \frac{1}{2T}\ln\left(\mathbf{\Phi}^2(T,\alpha)\right) \tag{22}$$

Assuming $\mathbf{R}(\alpha)$ is known, an approximation of $\mathbf{Q}(t,\alpha)$ can be computed from

$$\mathbf{Q}(t,\alpha) = \mathbf{\Phi}(t,\alpha)e^{-\mathbf{R}(\alpha)t} \tag{23}$$

For symbolic computation, Eq. (23) is written as

$$\hat{\mathbf{Q}}(t,\alpha) = \mathbf{\Phi}(t,\alpha)\left[\sum_{k=0}^{k=N}\frac{(-\mathbf{R})^k t^k}{k!}\right] \tag{24}$$

Several authors have presented feasible methods of computing the natural logarithm of symbolic matrices. In particular, this paper presents two approaches based on series expansion and one based on Gaussian quadrature [11–13] as compiled by Dieci et al. [11].

4.1 Series Method 1. Let

$$\mathbf{A} = \mathbf{I} - \mathbf{M} \tag{25}$$

where $\mathbf{A}$ is a dummy matrix and $\mathbf{M}$ is the matrix of which the real natural logarithm is desired. Assuming $\rho(\mathbf{A}) < 1$, where $\rho(\mathbf{A})$ is the spectral radius of $\mathbf{A}$, we have

$$\ln(\mathbf{M}) = \ln(\mathbf{I} - \mathbf{A}) = -\sum_{k=1}^{\infty}\frac{\mathbf{A}^k}{k} \tag{26}$$

This method is severely limited by the constraint placed of the spectral radius of the matrix, as the series will only converge if the magnitudes of all of the eigenvalues of the matrix are less than one (i.e., the matrix $\mathbf{A}$ associated with a discrete-time system is stable or matrix $\mathbf{M}$ associated with a continuous time system is unstable). In a symbolic form, the parameters are not known in advance; thus, it is impossible to predict if this series will converge.

4.2 Series Method 2. Since $\ln(\mathbf{I}+\mathbf{Y}) - \ln(\mathbf{I}-\mathbf{Y}) = \ln((\mathbf{I}+\mathbf{Y})(\mathbf{I}-\mathbf{Y})^{-1})$, using conformal transformation $\mathbf{M} = (\mathbf{Y}-\mathbf{I})(\mathbf{Y}+\mathbf{I})^{-1}$ and results of *series method 1*, it can be shown that

$$\ln(\mathbf{M}) = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left[(\mathbf{M}-\mathbf{I})(\mathbf{M}+\mathbf{I})^{-1}\right]^{2k+1} \tag{27}$$

The restriction of *series method 1* becomes $\mathbb{R}(\Lambda(\mathbf{M})) > 0$, where $\Lambda(\mathbf{M}) = \{\lambda_i(\mathbf{M});\ i = 1, 2, ..., n\}$ indicates the spectrum of $\mathbf{M}$. Hence, for this series to converge, the real parts of all eigenvalues must be greater than zero (i.e., the matrix $\mathbf{M}$ associated with a continuous time system is unstable). Similar to *series method 1*, the eigenvalues are not known beforehand; thus, it is not possible to determine whether this series will converge.

As previously mentioned, these methods are not ideal for symbolic computations as the determination of spectral radius and spectrum requires eigenanalysis. While possible in a symbolic form, this typically requires a great deal of computing power. Also, as bifurcation analysis is performed near the stability boundaries, these series expansion will likely converge very slowly.

4.3 Integral Quadrature (IQ) Method. The IQ method is an alternative technique for determining the real natural logarithm of a matrix based on a continuous model. This method determines the desired natural logarithm of a given matrix based on the solution of an ordinary differential equation as shown below.

Let the s dependent matrix $\mathbf{X}(s)$ be defined as

$$\mathbf{X}(s) : e^{\mathbf{X}(s)} = (\mathbf{M}-\mathbf{I})s + \mathbf{I}, \quad 0 \le s \le 1 \tag{28}$$

Thus, $\mathbf{X}(s)$ is well defined, real, $s \in [0,1]$, and $\mathbf{X}(1)$ defines $\log(\mathbf{M})$ based on the following theorem [11]:

Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be nonsingular. Then, there exists a real $\mathbf{X} = \ln(\mathbf{M})$ if and only if $\mathbf{M}$ has an even number of Jordan blocks of each size for every negative eigenvalue. If $\mathbf{M}$ has any eigenvalue on the negative real axis, then no real logarithm of $\mathbf{M}$ can be a primary matrix function of $\mathbf{M}$.

$\mathbf{X}(s)$ must satisfy the ODE

$$\begin{aligned}\dot{\mathbf{X}} &= (\mathbf{M}-\mathbf{I})e^{-\mathbf{X}(s)}, \quad 0 \le s \le 1\\ \mathbf{X}(0) &= 0\end{aligned} \tag{29}$$

and $\mathbf{M}$ and $e^{\mathbf{X}(s)}$ commute such that $\mathbf{M}e^{\mathbf{X}(s)} = e^{\mathbf{X}(s)}\mathbf{M}$.

By Eq. (28), the explicit solution can be written as

$$\mathbf{X}(t) = \int_0^s (\mathbf{M}-\mathbf{I})((\mathbf{M}-\mathbf{I})\beta + \mathbf{I})^{-1}d\beta, \quad 0 \le s \le 1 \tag{30}$$

where β is a dummy variable. Hence

$$\ln(\mathbf{M}) = \mathbf{X}(1) = \int_0^1 (\mathbf{M}-\mathbf{I})((\mathbf{M}-\mathbf{I})s + \mathbf{I})^{-1}ds \tag{31}$$

Computation of the natural logarithm of the matrix $\mathbf{M}$ can then be achieved by any type of numerical integration technique,

such as Simpson's method, quadrature rules, etc., whereas $((\mathbf{M}-\mathbf{I})s+\mathbf{I})^{-1}$ may be computed via Cayley–Hamilton theorem since it does not require computation of eigenvalues. In this work, Gaussian quadrature will be implemented to approximate $\mathbf{R}(\alpha)$, represented by $\hat{\mathbf{R}}(\alpha)$. Unlike the previously stated series methods, the IQ method is not constrained by eigenvalue limitations, and thus, this method converges for all parameter values. Applying this result to Eq. (22), we can determine $\hat{\mathbf{R}}(\alpha)$ as

$$\hat{\mathbf{R}}(\alpha)=\frac{1}{2T}\left(\mathbf{\Phi}^2(T,\alpha)-\mathbf{I}\right)\int_0^1\left(\left(\mathbf{\Phi}^2(T,\alpha)-\mathbf{I}\right)s+\mathbf{I}\right)^{-1}ds \tag{32}$$

5 Application to the Damped Mathieu Equation (DME)

Consider the well-known Mathieu equation

$$\ddot{x}+d\dot{x}+(a+b\cos\omega t)x=0 \tag{33}$$

which has a period of $T=2\pi/\omega$, where $\omega=2\pi$ for simplicity and a, b, and d are the system parameters denoted by α. The time t is normalized as $\tau=t/T$ and Eq. (33) is written in the state-space form as

$$\dot{\mathbf{x}}=\left[\overline{\mathbf{A}}_c(a,d)+\overline{\mathbf{A}}_p(b)\cos(2\pi\tau)\right]\mathbf{x} \tag{34}$$

where

$$\overline{\mathbf{A}}_c(a,d)=\begin{bmatrix}0 & 1\\ -a & -d\end{bmatrix},\quad \overline{\mathbf{A}}_p(b)=\begin{bmatrix}0 & 0\\ -b & 0\end{bmatrix} \tag{35}$$

$\mathbf{x}^T=(x_1x_2)$ and the derivatives are with respect to τ. Then, the STM of the original second-order system is given by Eq. (14) as (see Sec. 3)

$$\mathbf{\Phi}^{(24,15)}(\tau,\alpha)=\hat{\mathbf{T}}^T(\tau)\overline{\mathbf{B}}(\alpha)=\hat{\mathbf{T}}^{\mathbf{T}}(\tau)\begin{bmatrix}\frac{\omega}{2\pi}I_m & 0\\ 0 & I_m\end{bmatrix}\mathbf{B}(\alpha)\begin{bmatrix}\frac{2\pi}{\omega} & 0\\ 0 & 1\end{bmatrix} \tag{36}$$

The "product operational matrices" for $f_c(\tau)=1$ and $f_p(\tau)=\cos 2\pi\tau$ are utilized in the approximation. Computation was performed with $m=15$ Chebyshev expansion terms and $p=24$ Picard iterations, which was shown to be sufficiently accurate by Sinha and Butcher [5]. All computations were performed on a 64-bit Windows 7 Home Premium PC with a 4.0 GHz AMD FX™ 8350 8-core processor and 8 GB of RAM. The STM $\mathbf{\Phi}(\tau,\alpha)$ in terms of the parameters a, b, and d was first computed in a symbolic form. This process required 48.22 s of CPU time.

The IQ method was implemented to determine $\hat{\mathbf{R}}(\alpha)$, the approximate value of $\mathbf{R}(\alpha)$ matrix, by Eq. (32).

A computation-time study was performed to determine the amount of CPU time required for a given number of nodes used and is shown in Fig. 1.

Clearly, the CPU time required varies linearly with the number of Gaussian nodes implemented in the numerical integration approximation for $\hat{\mathbf{R}}(\alpha)$. Figure 2 shows the CPU time required to compute $\hat{\mathbf{Q}}(\tau,\alpha)$ by Eq. (24) for a given number of summation terms.

5.1 Stable Case. The parameter set $a=0.5, b=4.0$, and $d=0.3$ produces characteristic multipliers of $\{0.578814\pm 0.637019i\}$ with absolute values of 0.860708, indicating that this parameter set is stable. Figure 3 displays a plot of the "relative error" between $\hat{\mathbf{R}}$ computed using Eq. (32) and $\mathbf{R}$ obtained from Eq. (22) wherein the $\Phi(T)$ matrix was computed numerically using a

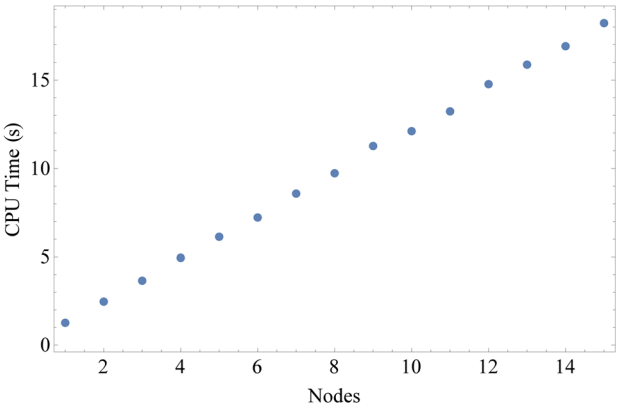

Fig. 1 CPU time to compute $\hat{\mathbf{R}}(\alpha)$ versus number of Gaussian nodes for the DME

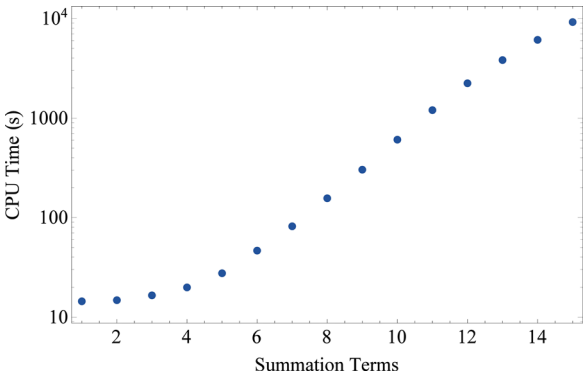

Fig. 2 CPU time to compute $\hat{\mathbf{Q}}(t,\alpha)$ versus number of summation terms for the DME

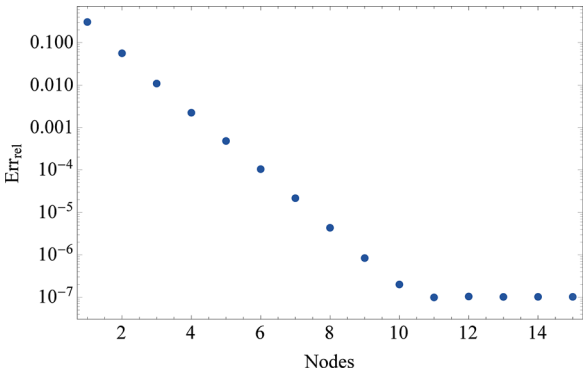

Fig. 3 Relative error versus number of Gaussian nodes of the $\hat{\mathbf{R}}$ matrix (stable DME)

Runge–Kutta type algorithm available in MATHEMATICA. Relative error is defined as

$$\text{Err}_{\text{rel}}=\frac{\|\hat{\mathbf{R}}-\mathbf{R}\|_f}{\|\mathbf{R}\|_f} \tag{37}$$

where $\hat{\mathbf{R}}$ is defined as the approximate solution, $\mathbf{R}$ is the numerical solution, and $\|\bullet\|_f$ is the Frobenius norm. It is easily noticed that in a simple 2×2 system, the relative error decreases rapidly with the increase of Gaussian nodes. It was noted that 11 nodes are required to achieve a relative error on the order of 1×10^{-7}, which corresponds to a CPU time of only 2.93 s. Beyond 11 nodes, numerical error due to machine-precision values occurs. Implementing 11 nodes, $\hat{\mathbf{R}}$ evaluates to

$$\hat{\mathbf{R}} = \begin{pmatrix} 0.03798 & 1.2532 \\ -0.5822 & -0.3379 \end{pmatrix} \tag{38}$$

where $\hat{\mathbf{R}}$ is truncated at four decimal places.

Figure 4 illustrates the time-invariant nature of the $\hat{\mathbf{R}}$ matrix over two periods.

$\hat{\mathbf{Q}}(t)$ is then computed by Eq. (24), and the relative error (using an expression similar to Eq. (37)) is shown in Fig. 5.

Computing $\hat{\mathbf{Q}}(2T)$ with ten summation terms, we obtain

$$\mathbf{Q}(2T) = \begin{bmatrix} 1 & 9.08387 \times 10^{-9} \\ -4.2201 \times 10^{-9} & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{39}$$

As both characteristic multipliers are on the right-half of the complex plane, $\hat{\mathbf{Q}}(\tau)$ is T-periodic, as shown in Fig. 6.

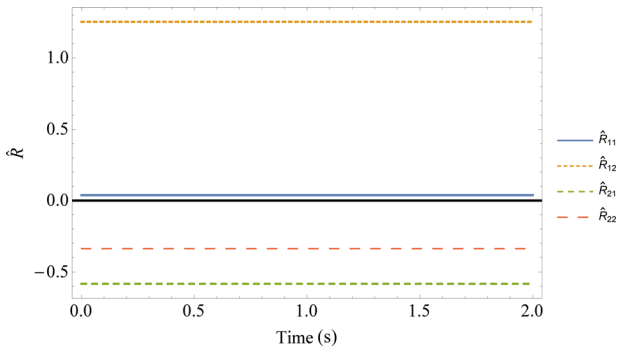

Fig. 4 $\hat{\mathbf{R}}$ matrix over two periods of the DME (stable)

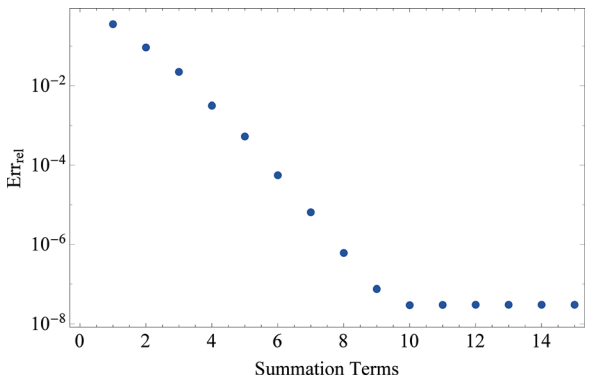

Fig. 5 Relative error versus number of summation terms of the $\hat{\mathbf{Q}}$ matrix (stable DME)

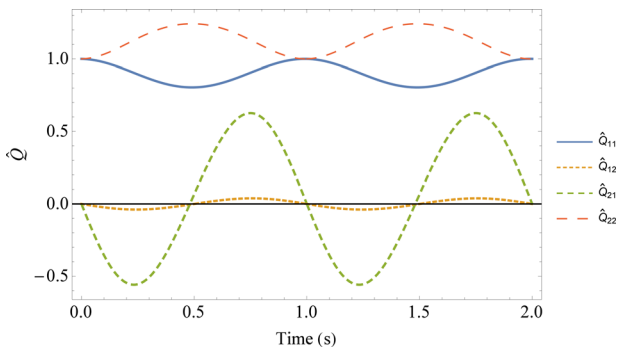

Fig. 6 T-periodic $\hat{\mathbf{Q}}(t)$ matrix of the DME (stable)

5.2 Unstable Case. The parameter set $a = 12.0, b = 7.0$, and $d = 0.3$ produces characteristic multipliers of $\{-1.28285, -0.577091\}$ with absolute values of $\{1.28285, 0.577091\}$, which makes this parameter set unstable. It was observed that seven nodes are required to achieve a relative error on the order of 1×10^{-6}. In this case, beyond seven nodes, the numerical computation error begins to occur due to the nature of machine-precision values. Implementing seven nodes, $\hat{\mathbf{R}}$ evaluates to

$$\hat{\mathbf{R}} = \begin{pmatrix} -0.1577 & -0.0514 \\ -3.0961 & -0.1422 \end{pmatrix} \tag{40}$$

where $\hat{\mathbf{R}}$ is truncated at four decimals.

$\hat{\mathbf{Q}}(\tau)$ is then computed by Eq. (24). Computing $\hat{\mathbf{Q}}(2T)$ with 11 expansion terms, we obtain

$$\hat{\mathbf{Q}}(2T) = \begin{bmatrix} 1 & -1.172 \times 10^{-9} \\ -7.055 \times 10^{-8} & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

As both characteristic multipliers are on the left-half of the complex plane, $\hat{\mathbf{Q}}(\tau)$ is $2T$-periodic, as shown in Fig. 7.

Thus, this method of analysis is feasible for application to stable as well as unstable systems in a symbolic form. Due to the nature of the polynomial expansions, matrix elements of $\mathbf{Q}$ as well as $\mathbf{R}$ contain very large order polynomials involving parameters a, b, and d. An abbreviated version of the symbolic elements of $\mathbf{R}$ matrix is shown in the Appendix.

5.3 Critical Case. For nonlinear systems, it is desirable to ascertain the type of bifurcation a system goes through when the control parameter is given a small perturbation η from its critical value (say α_c). This change in control parameter ($\alpha = \alpha_c + \eta$) changes the linear part of the equation, and thus, the eigenvalues of the matrix can be directly related to η.

However, such a procedure cannot be directly applied to a time-periodic system due to the fact that the linear part is time-periodic. Nevertheless, one can use the L-F transformation $\mathbf{Q}(t, \alpha)$ and obtain Eq. (19) where the $\mathbf{R}(\alpha)$ matrix is time-invariant. Then, one can relate the changes in the control parameter α to the eigenvalues of $\mathbf{R}(\alpha)$. But, in order to do so, $\mathbf{Q}(t, \alpha)$ must be symbolically computed such that $\mathbf{R}(\alpha)$ is also in a symbolic form. Since $\mathbf{Q}(t, \alpha)$ and $\mathbf{R}(\alpha)$ are available in symbolic form, the "parameter unfolding" in bifurcation of time-periodic systems can be easily studied. Recently, Dávid and Sinha [6] used a numerical form of $\mathbf{Q}(t)$ (evaluated at $\alpha = \alpha_c$) and attempted to approximate $\mathbf{R}(\alpha_c + \eta)$ using Taylor series expansion and "curve fitting" methods. Using the results of the present work, $\hat{\mathbf{R}}(\alpha_c + \eta)$ can be computed almost exactly for all values of η. The approximate values of $\mathbf{R}$ as computed in Ref. [6] are compared with the results obtained here for the Mathieu equation

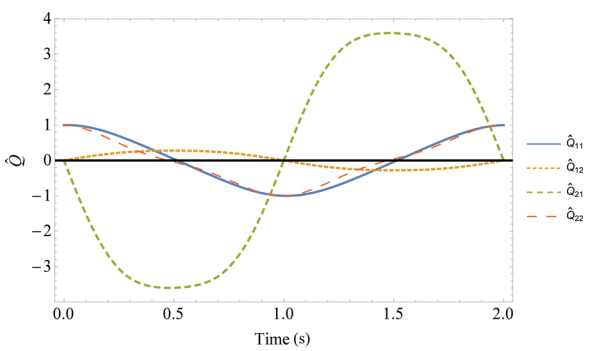

Fig. 7 2T-periodic $\hat{\mathbf{Q}}(t)$ matrix of the DME (unstable)

Table 1 Critical eigenvalue μ_c as a function of η

Bifurcation parameter $\eta = a - a_c$	Critical eigenvalue μ_c				
	Numerical value	Symbolic method	Linear relation sensitivity analysis [6]	Quadratic relation sensitivity analysis [6]	Quadratic relation curve fitting [6]
0.0001	0.000086777	0.000086777	0.000086800	0.000086781	0.000086484
0.001	0.00086615	0.00086615	0.00086800	0.00086615	0.00085553
0.005	0.0042941	0.0042941	0.0043400	0.0042963	0.0042484
0.01	0.0084981	0.0084981	0.0086800	0.0084946	0.0084234
0.02	0.016645	0.016645	0.017360	0.016618	0.016554
0.03	0.024460	0.024460	0.026040	0.024371	0.024391
0.04	0.031958	0.031958	0.034720	0.031754	0.031934
0.05	0.039155	0.039155	0.043400	0.038765	0.039185
0.06	0.046065	0.046065	0.052080	0.045405	0.046142
0.08	0.059069	0.059069	0.694400	0.057574	0.059178
0.1	0.071057	0.071057	0.086800	0.068260	0.071040
0.15	0.097048	0.097048	0.130200	0.088484	Not valid

$$\ddot{x} + d\dot{x} + (a_c + b\cos\omega t)x = 0 \tag{42}$$

Selecting parameters of $a_c = 3.91778734, b = 4.0, d = 0.31623$, and $\omega = 2.0$, we obtain characteristic multipliers $\{0.9999999957, 0.3702910945\}$ indicating that the system undergoes a symmetry breaking bifurcation. For increased accuracy, $\mathbf{\Phi}(\tau, \alpha)$ was computed with $m = 32$ Chebyshev expansion terms and $p = 30$ Picard iterations. With $\hat{\mathbf{R}}(\alpha)$ and $\hat{\mathbf{Q}}(\tau, \alpha)$ in symbolic forms, we may fix the values of b and d and investigate the dynamics of the bifurcation by introducing small changes in a_c. Twelve Gaussian nodes were implemented in the following analysis. $\hat{\mathbf{R}}$ at a_c evaluated to

$$\hat{\mathbf{R}} = \begin{pmatrix} -0.48907 & 0.01542 \\ 15.99016 & -0.50439 \end{pmatrix} \tag{43}$$

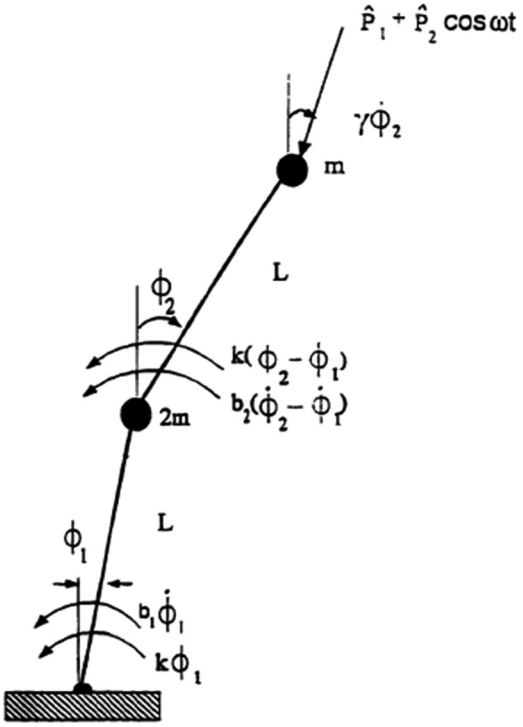

Fig. 8 Inverted double pendulum (IDP) [8]

With eigenvalues of $\hat{\mathbf{R}}$ being $\{-0.993466, -4.28304 \times 10^{-9} \approx 0\}$ where the latter eigenvalue is known as the critical eigenvalue μ_c. A small bifurcation parameter η was introduced into the symbolic computation of $\hat{\mathbf{R}}(a)$ such that $a = a_c + \eta$ and the change in the critical eigenvalue was investigated and shown in Table 1.

This data is compared to results obtained by Dávid and Sinha [6] in which linear $(\mu_c = 0.868\eta)$, quadratic $(\mu_c = 0.868\eta - 1.854\eta^2)$, and curve fitting $(\mu_c = 0.865\eta - 1.635\eta^2)$ methods were implemented to approximate the critical eigenvalues. The "numerical values" in the table were computed using a Runge–Kutta type algorithm. It is noticed that the values computed via the symbolic method presented in this paper match exactly (truncated to at max six decimals) with the numerical values for all bifurcation parameters tested. The curve fitting method is the second most accurate approximation. However, the versal deformation of the normal form must remain on the center manifold in order to provide an accurate approximation of the system dynamics. Thus, the bifurcation parameter can only be increased until the critical eigenvalue is at least one order of magnitude less than the stable eigenvalue. The symbolic method is not limited by working on the center manifold, which means that η may be of any magnitude as long as the approximation of the STM and the time-invariant matrix is of high enough accuracy to provide correct results.

6 Application to an Inverted Double Pendulum (IDP)

To prove the viability of this method's application to higher order systems, consider the inverted double pendulum subjected to a constant and periodically varying follower force as shown in Fig. 8.

The time-periodic equations of motion for this system as given in Refs. [4,8] are

$$\begin{aligned} &3\ddot{\phi}_1 + \cos(\phi_2 - \phi_1)\ddot{\phi}_2 - \sin(\phi_2 - \phi_1)\dot{\phi}_2^2 + (B_1 + B_2)\dot{\phi}_1 - B_2\dot{\phi}_2 \\ &\quad + 2\overline{k}\phi_1 - \overline{k}\phi_2 - \overline{p}(t)\sin(\phi_1 - \gamma\phi_2) = 0 \\ &\cos(\phi_2 - \phi_1)\ddot{\phi}_1 + \ddot{\phi}_2 + \sin(\phi_2 - \phi_1)\dot{\phi}_1^2 - B_2\dot{\phi}_1 + B_2\dot{\phi}_2 \\ &\quad - \overline{k}\phi_1 + \overline{k}\phi_2 - \overline{p}(t)\sin((1 - \gamma)\phi_2) = 0 \end{aligned} \tag{44}$$

where $\overline{k} = k/ml^2$ is the normalized stiffness, $B_1 = b_1/ml^2$ and $B_2 = b_2/ml^2$ are the normalized damping constants, $\overline{p}(t) = (\hat{P}_1 + \hat{P}_2\cos\omega t)/ml = P_1 + P_2\cos\omega t$ is the normalized applied load, γ is load the direction parameter, and ω is the excitation frequency of the applied load. Denoting $\mathbf{x}^{\mathrm{T}} = (x_1\ x_2\ x_3\ x_4) = (\phi_1\ \phi_2\ \phi_3\ \phi_4)$ as the state vector and retaining only linear terms, Eq. (44) in state-space form becomes

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\overline{p}(t)-3\overline{k}}{2} & \overline{k}-\dfrac{\overline{p}(t)}{2} & \dfrac{-B_1}{2}-B_2 & B_2 \\ \dfrac{5\overline{k}-\overline{p}(t)}{2} & -2\overline{k}+\left(\dfrac{3}{2}-\gamma\right)\overline{p}(t) & \dfrac{B_1}{2}+2B_2 & -2B_2 \end{bmatrix} \times \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} \tag{45}$$

The STM $\boldsymbol{\Phi}(t,\overline{p})$ is computed using the symbolic method as previously detailed via Eq. (14). Once again, the approximation was performed with $m = 15$ Chebyshev expansion terms and $p = 24$ Picard iterations

$$\boldsymbol{\Phi}^{(24,15)}(\tau,\alpha) = \hat{\mathbf{T}}^{\mathrm{T}}(\tau)\begin{bmatrix}\dfrac{\omega}{2\pi}\mathbf{I}_{2m} & 0\\ 0 & \mathbf{I}_{2m}\end{bmatrix}\mathbf{B}(\alpha)\begin{bmatrix}\dfrac{2\pi}{\omega}\mathbf{I}_2 & 0\\ 0 & \mathbf{I}_2\end{bmatrix} \tag{46}$$

Values of $\overline{k} = 1$, $B_1 = B_2 = 1.0$, $\omega = 2\pi$, $\gamma = 1$, and $P_1 = 1.0$ were chosen while P_2 remained symbolic. A total CPU time of 175.95 s was required to perform the computation of $\boldsymbol{\Phi}(t,\overline{p})$. As in the Mathieu equation example, the CPU time required varies linearly with the number of Gaussian nodes implemented in the integral approximation for $\hat{\mathbf{R}}(\alpha)$.

Figure 9 shows the CPU time required to compute $\hat{\mathbf{Q}}(t,\alpha)$ for a given number of summation terms. Due to the increased size in the system matrix, the time required is greatly increased compared to the Mathieu system.

Figure 10 displays a plot of the relative error of the symbolically approximated $\hat{\mathbf{R}}$ matrix compared to $\mathbf{R}$ obtained from a Runge–Kutta type numerical solution. $P_2 = 0.7$ was selected for a numerical comparison.

Again, for the larger 4×4 system, the relative error decreases linearly with an increase in Gaussian nodes. It was noted that 48 nodes are required to achieve a relative error on the order of 1×10^{-7}; this corresponds to a CPU time of 2.11 s in symbolic form. Beyond 48 nodes, numerical error due to the use of machine-precision values occurs. Implementing 48 nodes, $\hat{\mathbf{R}}$ evaluates to

$$\hat{\mathbf{R}} = \begin{pmatrix} -0.0190 & 0.0188 & 0.9844 & 0.0147 \\ 0.0412 & -0.0387 & 0.0205 & 0.9824 \\ -0.9692 & 0.4631 & -1.4889 & 0.9838 \\ 1.9302 & -1.4244 & 2.4565 & -1.9531 \end{pmatrix} \tag{47}$$

Figure 11 shows the values of $\hat{\mathbf{R}}_{31}, \hat{\mathbf{R}}_{32}, \hat{\mathbf{R}}_{33}$, and $\hat{\mathbf{R}}_{34}$ with respect to time. Other elements show similar behavior.

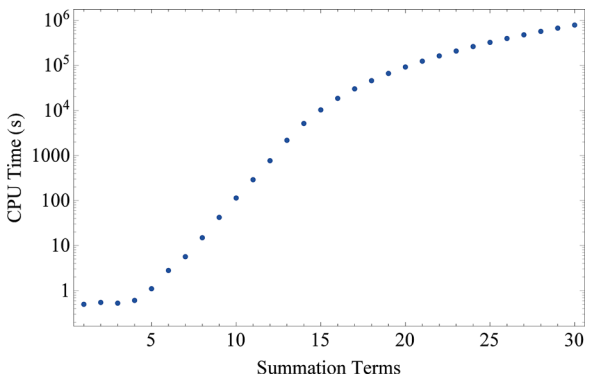

Fig. 9 CPU time to compute $\hat{\mathbf{Q}}(t,\alpha)$ versus number of summation terms for the IDP

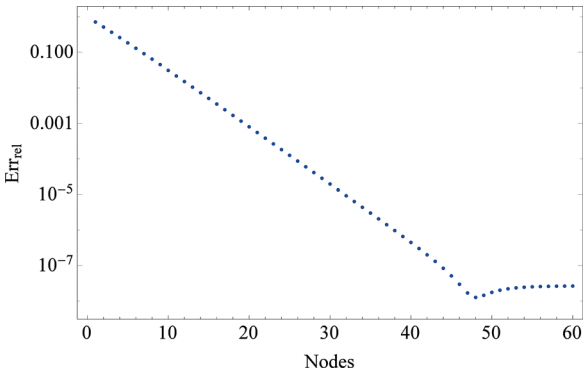

Fig. 10 Relative error versus number of Gaussian nodes of the $\hat{\mathbf{R}}$ matrix for the IDP

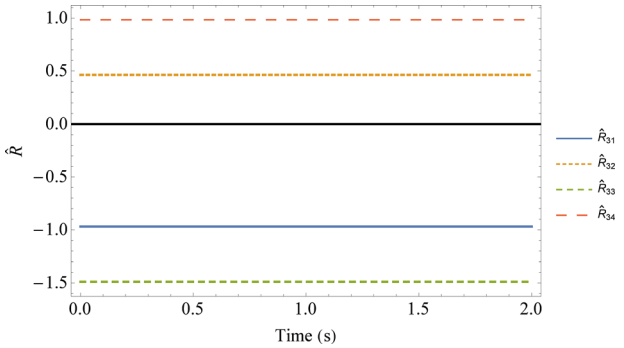

Fig. 11 $\hat{\mathbf{R}}$ matrix over two periods of the IDP

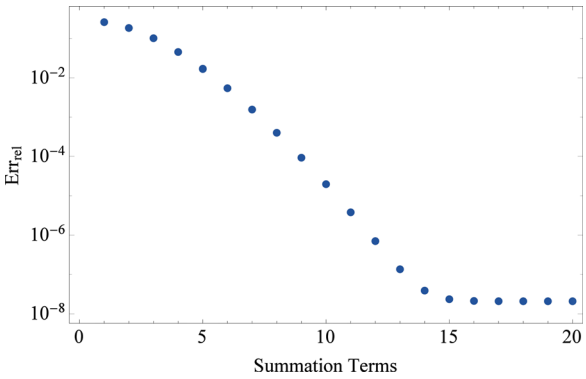

Fig. 12 Relative error versus number of expansion terms of the $\hat{\mathbf{Q}}$ matrix for the IDP

$\hat{\mathbf{Q}}(t)$ is then computed by Eq. (24), and the relative error is shown in Fig. 12.

Computing $\hat{\mathbf{Q}}(T)$ with 15 expansion terms, we obtain

$$\hat{\mathbf{Q}}(T) = 1 \times 10^{-8}\begin{bmatrix} 1\times10^8 & 9.6 & 1.4 & -1.1 \\ -2.5 & 1\times10^8 & -2.3 & 1.8 \\ -3.6 & 2.3 & 1\times10^8 & 2.6 \\ 5.9 & 3.8 & 5.6 & 1\times10^8 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

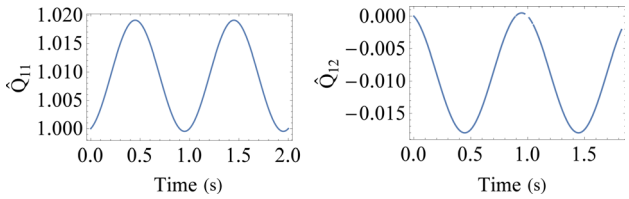

Fig. 13 *T*-periodic $\hat{\mathbf{Q}}_{11}$ and $\hat{\mathbf{Q}}_{12}$ of the IDP

The Floquét multipliers are $\{0.8453 \pm 0.4125i, 0.3664, 0.0931\}$; thus, we can expect a T-periodic $\hat{\mathbf{Q}}(t)$. Figure 13 shows elements $\hat{\mathbf{Q}}_{11}$ and $\hat{\mathbf{Q}}_{12}$. Other elements show similar behavior.

7 Conclusions

For a linear system with time-periodic coefficients, the STM, $\mathbf{\Phi}(t, \alpha)$, can be written as $\mathbf{\Phi}(t, \alpha) = \mathbf{Q}(t, \alpha)e^{\mathbf{R}(\alpha)t}$, where α is a set of system parameters. In this work, a technique for the computation of the real L-F transformation matrix $\mathbf{Q}(t, \alpha)$ and the time-invariant matrix $\mathbf{R}(\alpha)$ in a symbolic form as a function of system parameters is presented. This allows symbolic computation of general results for parameter unfolding and control system design in the entire parameter space. First, implementing the Chebyshev expansion and Picard iteration methods, the STM and FTMs are computed in a symbolic form. Then, the time-invariant matrix $\mathbf{R}(\alpha)$ is computed from the symbolic form of $\mathbf{\Phi}(t, \alpha)$ using an integral representation and a quadrature method. Finally, $\mathbf{Q}(t, \alpha)$ is computed from $\mathbf{\Phi}(t, \alpha)$ using the series representation of the exponential of $\mathbf{R}(\alpha)$. Two alternate methods for computing $\mathbf{R}(\alpha)$, based on natural logarithmic series expansions, are also discussed. However, these methods are not suitable for symbolic computations since the stability properties of a system cannot be ascertained beforehand. The symbolic computation of $\mathbf{Q}(t, \alpha)$ and $\mathbf{R}(t, \alpha)$, associated with the DME, is presented for stable, unstable, and critical cases. Bifurcation and parameter unfolding are investigated for the critical case and compared to the results available in the literature. The comparison clearly demonstrates the superiority of the method suggested here. The stable case of a linearized inverted double pendulum is presented to illustrate the application to a moderately large system. The technique presented here is computationally efficient and practical for the cases considered, and is expected to be convergent over a wide parameter space. It is anticipated that these results will be used to obtain general results for parameter unfolding and control system design for a large class of problems in the near future.

Appendix

Elements of $\hat{\mathbf{R}}$ matrix for the DME (33) in a symbolic form

$$
\begin{aligned}
\hat{R}_{11} = {} & 1.872573307371286 \times 10^{-10}(-2.670122435430305 \times 10^{9} + 4.450204059050507 \\
& \times 10^{8}a - 2.225102029525254 \times 10^{7}a^{2} + 529786.1975060159a^{3} \\
& - 7358.141632028073a^{4} + 66.89219665480027a^{5} \\
& - 0.4287961324026424a^{6} - 1.352699826777344 \times 10^{8}b \\
& + 8839282.420491958ab - 231643.30434715538a^{2}b \\
& + 3368.9080533924807a^{3}b - 31.426045754925273a^{4}b \\
& + 924908.5956588635b^{2} - 64405.79420981872ab^{2} \\
& + 1628.7244935426115a^{2}b^{2} - 22.169294178669865a^{3}b^{2} \\
& + 8281.832112179707b^{3} - 337.22773449070974ab^{3} \\
& + 5.761230159219807a^{2}b^{3} - 40.037819766278 69b^{4} \\
& + 1.3046913456297837ab^{4} + 1.335061217715152 \times 10^{9}d \\
& - 2.2251020295253 \times 10^{8}ad + 1.112551014762627 \times 10^{7}a^{2}d \\
& - 264893.0987530069a^{3}d + 3679.070816013979a^{4}d \\
& - 33.4460983274013a^{5}d + 6.763499133979191 \times 10^{7}bd \\
& - 4419641.250426704abd + 115821.64330868478a^{2}bd \\
& - 1684.454522849935a^{3}bd + 15.713031193858347a^{4}bd \\
& - 462454.5284305774b^{2}d + 32203.062496390437ab^{2}d - \cdots)
\end{aligned}
$$

$$\begin{aligned}\hat{R}_{12} = {} & 0.5 + 9.042058724009175 \times 10^{-11} b - 1.507009787334862 \times 10^{-11} ab \\ & + 7.535048936674315 \times 10^{-13} a^2 b - 1.794059270636751 \times 10^{-14}\, a^3 b \\ & + 2.491748986995513 \times 10^{-16} a^4 b - 2.265226351814089 \times 10^{-18}\, a^5 b \\ & + 1.45206817424006 \times 10^{-20} a^6 b - 0.006332585367603382 b^2 \\ & + 0.0010554308950646916\, ab^2 - 0.0000527715447603568 6 a^2 b^2 \\ & + 0.00000125646535150976 a^3 b^2 - 1.745090766037826 \times 10^{-8} a^4 b^2 \\ & + 1.586446151040215 \times 10^{-10}\, a^5 b^2 - 1.016952660923396 \times 10^{-12}\, a^6 b^2 \\ & + 4.118016558611256 \times 10^{-9}\, b^3 - 0.000032505790683846 81 ab^3 \\ & + 0.00000212409607191077 6 a^2 b^3 - 5.566423778958834 \times 10^{-8} a^3 b^3 \\ & + 8.095531994779215 \times 10^{-10}\, a^4 b^3 - 8.037133248104041 \times 10^{-12} a^5 b^3 \\ & + 5.152008492375305 \times 10^{-14}\, a^6 b^3 - 0.000020385703198466296 b^4 \\ & - 0.00000153802334988729 6 ab^4 + 4.785637935945735 \times 10^{-8} a^2 b^4 \\ & - 8.421527369426231 \times 10^{-10}\, a^3 b^4 + \cdots)\end{aligned}$$

$$\begin{aligned}\hat{R}_{21} = {} & 0.5(1. - 1.1666666666666667 a + 0.3416666666666667 a^2 \\ & - 0.04166666666666664 a^3 \\ & + 0.0017361111111111111 a^4 + 0.050660591770180376 b \\ & + 3.858063913067739 \times 10^{-10} ab + 0.000086753620807157 6 a^2 b \\ & + 1.607526630444891 \times 10^{-11}\, a^3 b - 0.00034639182959781 b^2 \\ & + 0.00082760621168266 79 ab^2 - 0.00043610152598726404 a^2 b^2 \\ & + 0.00004532781162524261 5 a^3 b^2 - 9.036849559831769 \times 10^{-7} a^4 b^2 \\ & - 0.00000310166754987961 6 b^3 - 2.617062408976375 \times 10^{-10} ab^3 \\ & - 2.026652394423692 \times 10^{-9} a^2 b^3 - 1.090861047324918 \times 10^{-11} a^3 b^3 \\ & + 1.499475051593519 \times 10^{-8} b^4 - 2.390332229458579 \times 10^{-8} ab^4 \\ & + 1.831846715511766 \times 10^{-7}\, a^2 b^4 - 9.970715157984674 \times 10^{-9} a^3 b^4 \\ & + 1.175970959525255 \times 10^{-10}\, a^4 b^4 - 1.0829946632876 \times 10^{-13}\, a^2 b^5 \\ & + 2.837999913433251 \times 10^{-15} a^3 b^5 - 9.891268988303464 \times 10^{-12} a^2 b^6 \\ & + 2.592128730676135 \times 10^{-13} a^3 b^6 + 3.127824311340369 \times 10^{-18} a^2 b^7 + \cdots)\end{aligned}$$

$$\begin{aligned}\hat{R}_{22} = {} & 1.204428287761545 \times 10^{-10} (4.15134700000261 \times 10^{9} - 2.07567360000013 \times 10^{9} a \\ & + 1.729728000000108 \times 10^{8} a^2 - 5765760.000000372\, a^3 \\ & + 102960.00000000637\, a^4 - 1144.000000000076\, a^5 \\ & + 8.666666666667421\, a^6 - 2.628871372952065 \times 10^{7} b^2 \\ & + 1717847.653801569\, ab^2 + 654.7181605948956\, a^3 b^2 \\ & - 6.106677256168282\, a^4 b^2 + 0.4353529998731057\, b^3 \\ & + 1390.43267100043 b^4 - 49.62117930219122\, ab^4 \\ & + 2471040.000000155\, a^3 d - 45760.000000003456\, a^4 d \\ & + 520.0000000000373\, a^5 d - 4.000000000000536 5\, a^6 d \\ & + 1.051548530201969 \times 10^{8} bd - 6871394.555945888\, abd \\ & - 2618.891143524505\, a^3 bd + 24.429442749897632\, a^4 bd \\ & + 1.242535847489643 \times 10^{7}\, b^2 d - 808857.5193181505\, ab^2 d \\ & - 310.11582980409406\, a^3 b^2\, d + 2.9039299571796535\, a^4 bd \\ & - 6438.307548258003 b^3 d + 262.41232611737684\, ab^3 d \\ & - 664.1194363173282 b^4 d + 23.795444946916003 ab^4 d \\ & - 617760.0000000391 a^3 d^2 + 11440.000000000833 a^4 d^2 \\ & - 130.00000000001162\, a^5\, d^2 + 1. a^6 d^2 - \cdots)\end{aligned}$$

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