# Rigorous derivation of the formula for the buckling load in axially compressed circular cylindrical shells

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**Abstract**

The goal of this paper is to apply the recently developed theory of buckling of arbitrary slender bodies to a tractable yet non-trivial example of buckling in axially compressed circular cylindrical shells, regarded as three-dimensional hyperelastic bodies. The theory is based on a mathematically rigorous asymptotic analysis of the second variation of 3D, fully nonlinear elastic energy, as the shell's thickness goes to zero. Our main results are a rigorous proof of the classical formula for buckling load and the explicit expressions for the relative amplitudes of displacement components in single Fourier harmonics buckling modes, whose wave numbers are described by Koiter's circle. This work is also a part of an effort to understand the root causes of high sensitivity of the buckling load of axially compressed cylindrical shells to imperfections of load and shape.

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# 1 Introduction

The buckling of rods, shells and plates is traditionally described in mechanics textbooks as an instability in the framework of nonlinear shell theory obtained by semi-rigorous dimension reduction of three-dimensional nonlinear elasticity. While these theories are effective in describing large deformations of rods and shells (including buckling), their heuristic nature obscures the source of the discrepancy between theoretical and experimental results, as is the case for axially compressed circular cylindrical shells [18]. At the same time, a rigorously derived theory of bending of shells [3] captures deformations in the vicinity of relatively smooth isometries of the middle surface. While a plane and a line has a large space of smooth isometries, the cylindrical surface has none. This fact explains higher axial rigidity of cylindrical shells than plates and rods of the same thickness. Our approach, originating in [6], combines the study of critical loads of Holden [7] with rigorous asymptotic analysis of the second variation functional for slender structures.

It is universally accepted (see [1, 14, 17]) that sensitivity to imperfections is the cause of the discrepancy between the experiment and the classical formula for buckling load [12, 15]. In any attempt to understand the mechanisms of high imperfection sensitivity one needs to start by determining whether the tacit assumptions of the classical derivation contribute to the discrepancy of theory with experiment. The result of our analysis is twofold. On the one hand we produce a mathematically rigorous proof of the classical formula. On the other we develop the machinery necessary for rigorous asymptotic analysis of imperfections of load and shape.

Imperfection sensitivity is commonly explained by the instability of equilibrium states in the vicinity of the buckling point on the bifurcation diagram [10, 17, 2]. However, the exact mechanisms of imperfection sensitivity are not fully understood, and there is no reliable theory capable of predicting experimentally observed buckling loads [11, 18, 8]. While a full bifurcation analysis is necessary to understand the stability of equilibria near the critical point, our method's singular focus on the stability of the trivial branch gives access to the scaling behavior of key measures of structural stability in the thin shell limit. We have argued in [**?**] that axially compressed circular cylindrical shells are susceptible to scaling instability of the critical load, whereby the scaling exponent, and not just its coefficient, can be affected by imperfections. The new analytical tools developed in [4] give hope for a path towards quantification of imperfection sensitivity.

Our approach is based on the observation that the prebuckled state is well approximated by linear elasticity [6]. At the critical load, the linear elastic stress reaches a level at which the trivial branch becomes unstable in the framework of fully non-linear 3D hyperelasticity. The origin of this instability is completely geometric: the frame-indifference of the energy density function implies[1] non-convexity in the compressive strain region. Since buckling occurs at relatively small compressive loads, the material's stress-strain response is locally linear. This explains why all classical asymptotic formulas for buckling loads, such as Euler's formula for columns, involve only linear elastic moduli and hold regardless of the constitutive model.

[1]The assumption that the reference configuration is stress-free is also essential.

The significance of our approach is two-fold. First, it provides a common platform to study buckling of arbitrary slender bodies. Second, its conclusions are mathematically rigorous and its underlying assumptions are explicitly specified. Our analysis is powered by asymptotically sharp Korn-like inequalities [9, 13], where instead of bounding the $L^2$ norm of the displacement gradient by the $L^2$ norm of the strain tensor, we bound the $L^2$ norm of individual components of the gradient by the $L^2$ norm of the strain tensor. These inequalities have been derived in our companion paper [4]. The method of buckling equivalence [6, **?**] applied in this paper provides flexibility by furnishing a systematic way of discarding asymptotically insignificant terms, while simplifying the variational functionals that characterize buckling.

The paper is organized as follows. In Section 2, we describe the loading and corresponding trivial branch of an axially compressed cylindrical shell treated as 3-dimensional hyperelastic body[2]. We then formulate the problem of the asymptotic analysis of the second variation of energy. In Section 3, we give the rigorous derivation of the classical buckling load and identify the explicit form of buckling modes. Our two most delicate results are a rigorous proof of the existence of a buckling mode that is a single Fourier harmonic and the linearization of the dependence of this buckling mode on the radial variable—the two assumptions that are commonly made in the classical derivation of the critical load formula. This paper is self-contained in the sense that all the results that we need from our prior work are restated. For rigorous proofs of these statements one does need to consult our companion papers [4, 5].

# 2 Axially compressed cylindrical shell

In this section we will give a mathematical formulation of the problem of buckling of axially compressed cylindrical shell.

## 2.1 Boundary conditions and trivial branch

Consider the circular cylindrical shell given in cylindrical coordinates $(r, \theta, z)$ as follows:

$$\mathcal{C}_h = I_h \times \mathbb{T} \times [0, L], \qquad I_h = [1 - h/2, 1 + h/2],$$

where $\mathbb{T}$ is a 1-dimensional torus (circle) describing $2\pi$-periodicity in $\theta$. Here $h$ is the slenderness parameter, equal to the ratio of the shell thickness to the radius. In this paper we consider the axial compression of the shell where the Lipschitz deformation $\boldsymbol{y} : \mathcal{C}_h \to \mathbb{R}^3$ satisfies the boundary conditions, given in cylindrical coordinates by

$$y_\theta(r, \theta, 0) = y_z(r, \theta, 0) = y_\theta(r, \theta, L) = 0, \qquad y_z(r, \theta, L) = (1 - \lambda)L. \tag{2.1}$$

The loading is parametrized by the compressive strain $\lambda > 0$ in the axial direction. The trivial deformation $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}$ satisfies the boundary conditions for $\lambda = 0$. By a *stable*

[2] While hyperelasticity is hardly the "ultimate" theory of elasticity, it is sufficiently general to permit rigorous study of the asymptotics of the buckling load of a slender structure.

*deformation* we mean a Lipschitz function $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$, satisfying boundary conditions (2.1) and being a weak local minimizer[3] of the energy functional

$$\mathcal{E}(\boldsymbol{y}) = \int_{\mathcal{C}_h} W(\nabla \boldsymbol{y}) d\boldsymbol{x}$$

among all Lipschitz functions satisfying (2.1). The energy density function $W(\boldsymbol{F})$ is assumed to be three times continuously differentiable in a neighborhood of $\boldsymbol{F} = \boldsymbol{I}$. The key (and universal) properties of $W(\boldsymbol{F})$ are

(P1) Absence of prestress: $W_{\boldsymbol{F}}(\boldsymbol{I}) = \boldsymbol{0}$;

(P2) Frame indifference: $W(\boldsymbol{R}\boldsymbol{F}) = W(\boldsymbol{F})$ for every $\boldsymbol{R} \in SO(3)$;

(P3) Local stability of the reference configuration $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}$:

$$\langle \mathsf{L}_0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle > \alpha_{\mathsf{L}_0} |\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi} \in \mathrm{Sym}(\mathbb{R}^3), \tag{2.2}$$

where $\mathrm{Sym}(\mathbb{R}^3)$ is the space of symmetric $3 \times 3$ matrices, and $\mathsf{L}_0 = W_{\boldsymbol{F}\boldsymbol{F}}(\boldsymbol{I})$ is the linearly elastic tensor of material properties.

Here, and elsewhere we use the notation $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \mathrm{Tr}\,(\boldsymbol{A}\boldsymbol{B}^T)$ for the Frobenius inner product on the space of $3 \times 3$ matrices. While this is not needed for general theory, in this paper we will also assume that $W(\boldsymbol{F})$ is isotropic:

$$W(\boldsymbol{F}\boldsymbol{R}) = W(\boldsymbol{F}) \text{ for every } \boldsymbol{R} \in SO(3). \tag{2.3}$$

Our goal is to examine stability of the homogeneous trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ given in cylindrical coordinates by

$$y_r = (a(\lambda) + 1)r, \qquad y_\theta = 0, \qquad y_z = (1 - \lambda)z, \tag{2.4}$$

where the function $a(\lambda)$ is determined by the natural boundary conditions at the lateral boundary of the shell

$$\boldsymbol{P}(\nabla \boldsymbol{y})\boldsymbol{e}_r = \boldsymbol{0}, \quad r = 1 \pm \frac{h}{2}, \tag{2.5}$$

where $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ is the moving orthonormal frame associated with cylindrical coordinates, and $\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F})$, the gradient of $W$ with respect to $\boldsymbol{F}$, is the Piola-Kirchhoff stress tensor.

Lemma **2.1.** *Assume that $W(\boldsymbol{F})$ is three times continuously differentiable in a neighborhood of $\boldsymbol{F} = \boldsymbol{I}$, satisfies properties (P1)–(P3) and is isotropic (i.e. satisfies (2.3)). Then there exists a unique function $a(\lambda)$, of class $C^2$ on a neighborhood of 0, such that $a(0) = 0$ and the natural boundary conditions (2.5) are satisfied*

[3]A deformation $\boldsymbol{y}$ is called a weak local minimizer, if it delivers the smallest value of the energy $\mathcal{E}(\boldsymbol{y})$ among all Lipschitz function satisfying boundary conditions (2.1) that are sufficiently close to $\boldsymbol{y}$ in the $W^{1,\infty}$ norm.

*Proof.* By (P2) $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{F}^T\boldsymbol{F})$. The function $\hat{W}(\boldsymbol{C})$ is three times continuously differentiable in a neighborhood of $\boldsymbol{C} = \boldsymbol{I}$. Thus,

$$\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F}) = 2\boldsymbol{F}\hat{W}_{\boldsymbol{C}}(\boldsymbol{F}^T\boldsymbol{F}).$$

The isotropy (2.3) implies that $\hat{W}(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T) = \hat{W}(\boldsymbol{C})$ for all $\boldsymbol{R} \in SO(3)$. Differentiating this relation in $\boldsymbol{R}$ at $\boldsymbol{R} = \boldsymbol{I}$ we conclude that $\hat{W}_{\boldsymbol{C}}(\boldsymbol{C})$ must commute with $\boldsymbol{C}$. In particular, this implies that the matrix $\hat{W}_{\boldsymbol{C}}(\boldsymbol{C})$ must be diagonal, whenever $\boldsymbol{C}$ is diagonal. We compute that in cylindrical coordinates

$$\boldsymbol{F} = \nabla\boldsymbol{y} = \begin{bmatrix} 1+a & 0 & 0 \\ 0 & 1+a & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}, \quad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \begin{bmatrix} (1+a)^2 & 0 & 0 \\ 0 & (1+a)^2 & 0 \\ 0 & 0 & (1-\lambda)^2 \end{bmatrix}$$

Hence, $\boldsymbol{P}(\boldsymbol{F})$ is diagonal, and boundary conditions (2.5) reduce to a single scalar equation

$$\hat{W}_{\boldsymbol{C}}((1+a)^2(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta) + (1-\lambda)^2\boldsymbol{e}_z \otimes \boldsymbol{e}_z)\boldsymbol{e}_r \cdot \boldsymbol{e}_r = 0, \tag{2.6}$$

where the left-hand side of (2.6) is a twice continuously differentiable function of $(\lambda, a)$. Condition (P1) implies that $(\lambda, a) = (0, 0)$ is a solution. The conclusion of the lemma is guaranteed by the implicit function theorem, provided the derivative of the left-hand side in (2.6) with respect to $a$ is non-zero when $(\lambda, a) = (0, 0)$:

$$\frac{1}{2}\mathsf{L}_0(\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta)\boldsymbol{e}_r \cdot \boldsymbol{e}_r \neq 0. \tag{2.7}$$

When $\mathsf{L}_0$ is isotropic, the non-degeneracy condition (2.7) becomes $\kappa + \mu/3 \neq 0$, which is satisfied due to (P3). Here $\kappa$ and $\mu$ are the bulk and shear moduli, respectively. □

It is important, that as $h \to 0$, the trivial branch does not blow up. In fact, in our case the trivial branch is independent of $h$.

The general theory of buckling [6] is designed to detect the first instability of a trivial branch in a slender body $\Omega_h$ that is well-described by linear elasticity. Here is the formal definition from [6, ?].

**Definition 2.2.** *We call the family of Lipschitz equilibria*[4] $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ *of* $\mathcal{E}(\boldsymbol{y})$ *a* ***regular trivial branch*** *if there exist* $h_0 > 0$ *and* $\lambda_0 > 0$, *so that for every* $h \in [0, h_0]$ *and* $\lambda \in [0, \lambda_0]$

*(i)* $\boldsymbol{y}(\boldsymbol{x}; h, 0) = \boldsymbol{x}$

*(ii)*

$$\left\|\frac{\partial(\nabla\boldsymbol{y})}{\partial\lambda}(\boldsymbol{x}; h, \lambda) - \nabla\boldsymbol{u}^h(\boldsymbol{x})\right\|_{L^\infty(\Omega_h)} \leq C\lambda, \tag{2.8}$$

[4]We restrict our attention to Lipschitz equilibria for technical simplicity of the theory. On the one hand, in most cases of interest, and for a cylindrical shells in particular, the strains and stresses in the trivial branch are uniformly bounded. On the other, the presence or absence of higher spatial derivatives of strains and stresses are immaterial for the theory, and hence no assumptions about them are made.

*where the constant $C$ is independent of $h$ and $\lambda$.*

We remark, that the leading order asymptotics $\boldsymbol{u}^h(\boldsymbol{x})$ of the nonlinear trivial branch is nothing else but a linear elastic displacement, that can be found by solving the equations of linear elasticity $\nabla \cdot (\mathsf{L}_0 e(\boldsymbol{u}^h)) = \boldsymbol{0}$, augmented by the appropriate boundary conditions. Here $e(\boldsymbol{u}^h) = \dfrac{1}{2}(\nabla \boldsymbol{u}^h + (\nabla \boldsymbol{u}^h)^T)$ is the linear elastic strain. The linear elastic trivial branch $\lambda \boldsymbol{u}^h(\boldsymbol{x})$ depends only on the linear elastic moduli $\mathsf{L}_0$, unlike the model-dependent nonlinear trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$.

The fact that our trivial branch (2.4) satisfies all conditions in Definition 2.2 is easy to verify. Here

$$\boldsymbol{u}^h(\boldsymbol{x}) = \left.\frac{\partial \boldsymbol{y}(\boldsymbol{x}; h, \lambda)}{\partial \lambda}\right|_{\lambda=0} = a'(0) r \boldsymbol{e}_r - z \boldsymbol{e}_z = \nu r \boldsymbol{e}_r - z \boldsymbol{e}_z$$

is independent of $h$. Here we computed that $a'(0) = \nu$ (Poisson's ratio) by differentiating (2.6) in $\lambda$ at $\lambda = 0$.

## 2.2 Stability of the trivial branch

We describe critical strain $\lambda_{\text{crit}}$ in terms of the second variation of energy

$$\delta^2 \mathcal{E}(\boldsymbol{\phi}; h, \lambda) = \int_{\mathcal{C}_h} \langle W_{\boldsymbol{FF}}(\nabla \boldsymbol{y}(\boldsymbol{x}; h, \lambda)) \nabla \boldsymbol{\phi}, \nabla \boldsymbol{\phi} \rangle d\boldsymbol{x}, \tag{2.9}$$

defined on the space of admissible variations

$$V_h^\circ = \{\boldsymbol{\phi} \in W^{1,\infty}(\mathcal{C}_h; \mathbb{R}^3) : \phi_\theta(r, \theta, 0) = \phi_z(r, \theta, 0) = \phi_\theta(r, \theta, L) = \phi_z(r, \theta, L) = 0\}.$$

We remark that this approach is capable of capturing fractional powers of $h$ in the asymptotics of the critical load, while the bifurcation analysis with the asymptotic expansion in $h$ as in [16] can produce only integral exponents.

By density of $W^{1,\infty}(\mathcal{C}_h; \mathbb{R}^3)$ in $W^{1,2}(\mathcal{C}_h; \mathbb{R}^3)$ we extend the space of admissible variations from $V_h^\circ$ to its closure $V_h$ in $W^{1,2}$.

$$V_h = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h; \mathbb{R}^3) : \phi_\theta(r, \theta, 0) = \phi_z(r, \theta, 0) = \phi_\theta(r, \theta, L) = \phi_z(r, \theta, L) = 0\}. \tag{2.10}$$

The critical strain $\lambda_{\text{crit}}$ can be defined as follows.

$$\lambda_{\text{crit}}(h) = \inf\{\lambda > 0 : \delta^2 \mathcal{E}(\boldsymbol{\phi}; h, \lambda) < 0 \text{ for some } \boldsymbol{\phi} \in V_h\}. \tag{2.11}$$

While this definition is unambiguous, it is inconvenient, since the critical strain depends on the choice of the nonlinear energy density function. Instead, we will focus only on the leading order asymptotics of the critical strain, as $h \to 0$. The corresponding buckling mode, to be defined below, will also be understood in an asymptotic sense. Following [**?**] we define

**Definition 2.3.** *We say that a function $\lambda(h) \to 0$, as $h \to 0$ is a buckling load if*

$$\lim_{h \to 0} \frac{\lambda(h)}{\lambda_{\text{crit}}(h)} = 1. \tag{2.12}$$

*A **buckling mode** is a family of variations $\boldsymbol{\phi}_h \in V_h \setminus \{0\}$, such that*

$$\lim_{h\to 0} \frac{\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda_{\rm crit}(h))}{\lambda_{\rm crit}(h)\dfrac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h; h, \lambda_{\rm crit}(h))} = 0. \tag{2.13}$$

The definition of the buckling mode (2.13) requires an explanation. The natural definition would be the variation $\boldsymbol{\phi}_h^* \in V_h \setminus \{0\}$, such that $\delta^2\mathcal{E}(\boldsymbol{\phi}_h^*; h, \lambda_{\rm crit}(h)) = 0$. Such a variation might not exist in the first place, and we are only interested in the *asymptotics* of the critical load, as $h \to 0$ in the sense of (2.12). That means that we can only distinguish between $\lambda_{\rm crit}(h)$ and $\lambda_\epsilon(h) = \lambda_{\rm crit}(h)(1+\epsilon)$, provided $\epsilon$ is a fixed non-zero number. Assuming that $\delta^2\mathcal{E}(\boldsymbol{\phi}_h^*; h, \lambda_{\rm crit}(h)) = 0$ we estimate

$$\delta^2\mathcal{E}(\boldsymbol{\phi}_h^*; h, \lambda_{\rm crit}(h)(1+\epsilon)) \approx \lambda_{\rm crit}(h)\epsilon\frac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h^*; h, \lambda_{\rm crit}(h)).$$

This means that for the purposes of asymptotics we should not distinguish differences in values of second variation that are infinitesimal, compared to

$$\lambda_{\rm crit}(h)\frac{\partial(\delta^2\mathcal{E})}{\partial\lambda}(\boldsymbol{\phi}_h^*; h, \lambda_{\rm crit}(h)).$$

Thus, in Definition 2.3 we declare that any variation $\boldsymbol{\phi}_h$ satisfying (2.13) makes the second variation $\delta^2\mathcal{E}(\boldsymbol{\phi}_h; h, \lambda_{\rm crit}(h))$ indistinguishable from 0 for our purposes.

Targeting only the leading order asymptotics allows us to determine critical strain and buckling modes from a *constitutively linearized* second variation, which is obtained by first approximating $W_{\boldsymbol{FF}}(\nabla\boldsymbol{y}(\boldsymbol{x}; h, \lambda))$ with $W_{\boldsymbol{FF}}(\boldsymbol{I} + \lambda\nabla\boldsymbol{u}^h)$ and then computing its leading order asymptotics as $(h, \lambda) \to (0, 0)$. This was done rigorously in [6, **?**], where it was shown that the *constitutively linearized* second variation $\delta^2\mathcal{E}_{\rm cl}(\boldsymbol{\phi}; h, \lambda)$ has the form

$$\delta^2\mathcal{E}_{\rm cl}(\boldsymbol{\phi}; h, \lambda) = \int_{\mathcal{C}_h} \{\langle \mathsf{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle + \lambda\langle\boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle\}d\boldsymbol{x}, \qquad \boldsymbol{\phi} \in V_h, \tag{2.14}$$

where $\boldsymbol{\sigma}_h$ is the linear elastic stress

$$\boldsymbol{\sigma}_h(\boldsymbol{x}) = \mathsf{L}_0 e(\boldsymbol{u}^h(\boldsymbol{x})). \tag{2.15}$$

Since the first term in (2.14) is always non-negative we define the set

$$\mathcal{A}_h = \left\{\boldsymbol{\phi} \in V_h : \langle\boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle < 0\right\} \tag{2.16}$$

of potentially destabilizing variations[5]. Replacing the second variation of energy by its constitutively linearized version in (2.11) we obtain the constitutively linearized version

[5]The set $\mathcal{A}_h$ could be empty, in which case there are no destabilizing variations, so that the trivial branch remains stable in a neighborhood of $(0, 0)$ in the $(h, \lambda)$-plane.

$\lambda_{\rm cl}(h)$ of the critical load. It was shown in [6] that $\lambda_{\rm cl}(h)$ can also be regarded as a Korn constant and defined in terms of the Rayleigh quotient

$$\mathfrak{R}(h,\boldsymbol{\phi})=-\frac{\int_{\Omega_h}\langle \mathsf{L}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\int_{\Omega_h}\langle \boldsymbol{\sigma}_h,\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle d\boldsymbol{x}}. \tag{2.17}$$

If we use constitutively linearized second variation in Definition 2.3 we also obtain the constitutively linearized version of the critical mode. This motivates the following definition.

**Definition 2.4.** *The* ***constitutively linearized buckling load*** $\lambda_{\rm cl}(h)$ *is defined by*

$$\lambda_{\rm cl}(h)=\inf_{\boldsymbol{\phi}\in\mathcal{A}_h}\mathfrak{R}(h,\boldsymbol{\phi}). \tag{2.18}$$

*We say that the family of variations* $\{\boldsymbol{\phi}_h\in\mathcal{A}_h : h\in(0,h_0)\}$ *is a* ***constitutively linearized buckling mode*** *if*

$$\lim_{h\to 0}\frac{\mathfrak{R}(h,\boldsymbol{\phi}_h)}{\lambda_{\rm cl}(h)}=1. \tag{2.19}$$

The functional $\mathfrak{R}(h,\boldsymbol{\phi})$ expresses the relative strength of the destabilizing compressive stress, measured by the functional

$$\mathfrak{C}_h(\boldsymbol{\phi})=\int_{\Omega_h}\langle \boldsymbol{\sigma}_h,\nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}\rangle d\boldsymbol{x} \tag{2.20}$$

and the reserve of structural stability measured by the functional

$$\mathfrak{S}_h(\boldsymbol{\phi})=\int_{\Omega_h}\langle \mathsf{L}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}. \tag{2.21}$$

In [6] we have defined a measure of "slenderness" of the body in terms of the Korn constant

$$K(V_h)=\inf_{\boldsymbol{\phi}\in V_h}\frac{\|e(\boldsymbol{\phi})\|^2_{L^2(\Omega_h)}}{\|\nabla\boldsymbol{\phi}\|^2_{L^2(\Omega_h)}}. \tag{2.22}$$

It is obvious (see [6]), that if $K(V_h)$ stays uniformly positive, then so does the constitutively linearized second variation $\delta^2\mathcal{E}_{\rm cl}(\boldsymbol{\phi};h,\lambda(h))$ as a quadratic form on $V_h$, for any $\lambda(h)\to 0$, as $h\to 0$. We therefore regard the structure as a slender one if its Korn constant is small.

**Definition 2.5.** *We say that the body* $\Omega_h$ *is* ***slender*** *if* $K(V_h)>0$ *and*

$$\lim_{h\to 0}K(V_h)=0. \tag{2.23}$$

This notion of slenderness requires not only geometric slenderness of the domain but also traction-dominated boundary conditions conveniently encoded in the closed subspace $V_h$, satisfying $W_0^{1,2}(\Omega_h;\mathbb{R}^3)\subset V_h\subset W^{1,2}(\Omega_h;\mathbb{R}^3)$. (See [6] for the detailed discussion of this definition of slenderness.) We can now state the theorem proved in [**?**] that gives sufficient conditions under which the constitutively linearized buckling load and buckling mode, defined in (2.18)–(2.19), correctly describe the asymptotic behavior of fully nonlinear buckling load and buckling mode, i.e. verify Definition 2.3.

THEOREM **2.6.** *Suppose that the body is slender in the sense of Definition 2.5. Assume that the constitutively linearized critical load $\lambda_{\rm cl}(h)$, defined in (2.18) satisfies $\lambda_{\rm cl}(h) > 0$ for all sufficiently small $h$ and*

$$\lim_{h\to 0}\frac{\lambda_{\rm cl}(h)^2}{K(V_h)} = 0. \tag{2.24}$$

*Then $\lambda_{\rm cl}(h)$ is the buckling load and any constitutively linearized buckling mode $\boldsymbol{\phi}_h$ is a buckling mode in the sense of Definition 2.3.*

Now we will show that Theorem 2.6 applies to the axially compressed circular cylindrical shells. The asymptotics of the Korn constant $K(V_h)$, as $h \to 0$, was established in [4]. $\tilde{\mathcal{A}}_h$

THEOREM **2.7.** *Let $V_h$ be given by (2.10). Then, there exist positive constants $c < C$, depending only on $L$*[6]*, such that*

$$ch^{3/2} \le K(V_h) \le Ch^{3/2}. \tag{2.25}$$

In order to establish (2.24) we need to estimate $\lambda_{\rm cl}(h)$. For the trivial branch (2.4) we compute

$$\boldsymbol{\sigma}_h = -E\boldsymbol{e}_z \otimes \boldsymbol{e}_z, \tag{2.26}$$

where $E$ is the Young's modulus. Hence,

$$\mathfrak{C}_h(\boldsymbol{\phi}) = -E(\|\phi_{r,z}\|^2 + \|\phi_{z,z}\|^2 + \|\phi_{\theta,z}\|^2), \tag{2.27}$$

where from now on $\|\cdot\|$ will always denote the $L^2$-norm on $\mathcal{C}_h$. In order to estimate $\lambda_{\rm cl}(h)$ we need to prove Korn-like inequalities for the gradient components, $\phi_{r,z}$, $\phi_{z,z}$, and $\phi_{\theta,z}$. This was done in [4].

THEOREM **2.8.** *There exist a constant $C > 0$ depending only on $L$ such that for any $\boldsymbol{\phi} \in V_h$ one has,*

$$\|\phi_{\theta,z}\|^2 \le \frac{C}{\sqrt{h}}\|e(\boldsymbol{\phi})\|^2, \tag{2.28}$$

$$\|\phi_{r,z}\|^2 \le \frac{C}{h}\|e(\boldsymbol{\phi})\|^2. \tag{2.29}$$

*Moreover, the powers of $h$ in the inequalities (2.25)–(2.29) are optimal, achieved* simultaneously *by the ansatz*

$$\begin{cases} \phi_r^h(r,\theta,z) = & -W_{,\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) \\ \phi_\theta^h(r,\theta,z) = & r\sqrt[4]{h}W_{,\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) + \frac{r-1}{\sqrt[4]{h}}W_{,\eta\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \\ \phi_z^h(r,\theta,z) = & (r-1)W_{,\eta\eta z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) - \sqrt{h}W_{,z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \end{cases} \tag{2.30}$$

*where $W(\eta, z)$ can be any smooth compactly supported function on $\mathbb{R} \times (0, L)$, with the understanding that the above formulas hold on a single period $\theta \in [0, 2\pi]$, while the function $\boldsymbol{\phi}^h(r,\theta,z)$ is $2\pi$-periodic in $\theta$.*

[6] Here we consider cylindrical shells of radius 1 and lengths that are uniformly bounded away from 0 and infinity. We therefore do not examine explicit dependence of our constants on $L$, since it does not affect our results. The cases $L \to \infty$ or $L \to 0$ are beyond the scope of this paper.

**Corollary 2.9.**

$$ch \leq \lambda_{\mathrm{cl}}(h) \leq Ch. \tag{2.31}$$

*Proof.* This is an immediate consequence of Theorem 2.8. The lower bound follows from inequalities (2.2), (2.28) and (2.29) (and also an obvious inequality $\|\phi_{z,z}\| \leq \|e(\boldsymbol{\phi})\|$). The upper bound follows from using a test function (2.30) in the constitutively linearized second variation. $\square$

Inequalities (2.25) and (2.31) imply that the condition (2.24) in Theorem 2.6 is satisfied for the axially compressed circular cylindrical shell.

## 2.3 Buckling equivalence

The problem of finding the asymptotic behavior of the critical strain $\lambda_{\mathrm{crit}}$ and the corresponding buckling mode, as $h \to 0$ now reduces to minimization of the Rayleigh quotient (2.17), which is expressed entirely in terms of linear elastic data. Even though this already represents a significant simplification of our problem, its explicit solution is still technically difficult. However, the asymptotic flexibility of the notion of buckling load and buckling mode permits us to replace $\mathfrak{R}(h, \boldsymbol{\phi}_h)$ with an equivalent, but simpler functional. The notion of buckling equivalence was introduced in [6] and developed further in [**?**]. Here we give the relevant definition and theorems for the sake of completeness.

**Definition 2.10.** *Assume that $J(h, \boldsymbol{\phi})$ is a variational functional defined on $\mathcal{B}_h \subset \mathcal{A}_h$. We say that the pair $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$* ***characterizes buckling*** *if the following three conditions are satisfied*

*(a) Characterization of the buckling load: If*

$$\lambda(h) = \inf_{\boldsymbol{\phi} \in \mathcal{B}_h} J(h, \boldsymbol{\phi}),$$

*then $\lambda(h)$ is a buckling load in the sense of Definition 2.3.*

*(b) Minimizing property of the buckling mode: If $\boldsymbol{\phi}_h \in \mathcal{B}_h$ is a buckling mode in the sense of Definition 2.3, then*

$$\lim_{h \to 0} \frac{J(h, \boldsymbol{\phi}_h)}{\lambda(h)} = 1. \tag{2.32}$$

*(c) Characterization of the buckling mode: If $\boldsymbol{\phi}_h \in \mathcal{B}_h$ satisfies (2.32) then it is a buckling mode.*

**Definition 2.11.** *Two pairs $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ and $(\mathcal{B}'_h, J'(h, \boldsymbol{\phi}))$ are called* ***buckling equivalent*** *if the pair $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ characterizes buckling if and only if $(\mathcal{B}'_h, J'(h, \boldsymbol{\phi}))$ does.*

Of course this definition becomes meaningful only if the pairs $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ and $(\mathcal{B}'_h, J'(h, \boldsymbol{\phi}))$ are related.

The following lemma has been proved in [**?**].

LEMMA **2.12.** *Suppose the pair* $(\mathcal{B}_h, J(h, \boldsymbol{\phi}))$ *characterizes buckling. Let* $\mathcal{B}'_h \subset \mathcal{B}_h$ *be such that it contains a buckling mode. Then the pair* $(\mathcal{B}'_h, J(h, \boldsymbol{\phi}))$ *characterizes buckling*[7]*.*

The key tool for simplification of functionals characterizing buckling is the following theorem, [**?**].

THEOREM **2.13** (Buckling equivalence)**.** *Suppose that* $\lambda(h)$ *is a buckling load in the sense of Definition 2.3. If either*

$$\lim_{h\to 0} \lambda(h) \sup_{\boldsymbol{\phi}\in\mathcal{B}_h} \left| \frac{1}{J_1(h, \boldsymbol{\phi})} - \frac{1}{J_2(h, \boldsymbol{\phi})} \right| = 0, \tag{2.33}$$

*or*

$$\lim_{h\to 0} \frac{1}{\lambda(h)} \sup_{\boldsymbol{\phi}\in\mathcal{B}_h} |J_1(h, \boldsymbol{\phi}) - J_2(h, \boldsymbol{\phi})| = 0, \tag{2.34}$$

*then the pairs* $(\mathcal{B}_h, J_1(h, \boldsymbol{\phi}))$ *and* $(\mathcal{B}_h, J_2(h, \boldsymbol{\phi}))$ *are buckling equivalent in the sense of Definition 2.11.*

As an application we will simplify the denominator in the functional $\mathfrak{R}(h, \boldsymbol{\phi})$, given by (2.17). Theorem 2.8 suggests that $\|\phi_{r,z}\|^2$ can be much larger than $\|\phi_{z,z}\|^2$ and $\|\phi_{\theta,z}\|^2$. Hence, we will prove that we can replace $\mathfrak{C}_h(\boldsymbol{\phi})$, given by (2.27), with $-E\|\phi_{r,z}\|^2$. Hence, we define a simplified functional

$$\mathfrak{R}_1(h, \boldsymbol{\phi}) = \frac{\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\int_{\mathcal{C}_h} |\phi_{r,z}|^2 d\boldsymbol{x}}, \qquad \widehat{\mathsf{L}}_0 = \frac{\mathsf{L}_0}{E}.$$

LEMMA **2.14.** *The pair* $(\mathcal{A}_h, \mathfrak{R}_1(h, \boldsymbol{\phi}))$ *characterizes buckling.*

*Proof.* By Theorem 2.8 we have

$$\left| \frac{1}{\mathfrak{R}(h, \boldsymbol{\phi})} - \frac{1}{\mathfrak{R}_1(h, \boldsymbol{\phi})} \right| = \frac{\|\phi_{\theta,z}\|^2 + \|\phi_{z,z}\|^2}{\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}} \le \frac{C}{\sqrt{h}}.$$

for every $\boldsymbol{\phi} \in V_h$. Condition (2.33) now follows from (2.31). Thus, by Theorem 2.13, the pair $(\mathcal{A}_h, \mathfrak{R}_1(h, \boldsymbol{\phi}))$ characterizes buckling. □

# 3 Rigorous derivation of the classical formula for the buckling load

In this section we prove the classical asymptotic formula for the critical strain [12, 15]

$$\lambda_{\text{crit}}(h) \sim \frac{h}{\sqrt{3(1-\nu^2)}}. \tag{3.1}$$

[7]This lemma highlights the fact that Part (b) in Definition 2.10 is designed to capture a buckling mode. We make no attempt to characterize an infinite set of geometrically distinct, yet energetically equivalent buckling modes that exist in our example.

### 3.1 Restriction to a single Fourier mode

The goal of this section is to show that even if we shrink the space of admissible variations to the set of single Fourier modes in $(\theta, z)$, we still retain the ability to characterize buckling. The first step is to define Fourier modes by constructing an appropriate $2L$-periodic extension of $\boldsymbol{\phi}$ in $z$ variable. Since, no continuous $2L$-periodic extension $\widetilde{\boldsymbol{\phi}}$ has the property that $e(\widetilde{\boldsymbol{\phi}})(r,\theta,-z) = \pm e(\boldsymbol{\phi})(r,\theta,z)$, we will have to navigate around various sign changes in components of $e(\boldsymbol{\phi})$. We can handle this difficulty if $\mathsf{L}_0$ is isotropic, which we have already assumed. It is easy to check that there are only two possibilities that work[8]: odd extension for $\phi_r$, $\phi_\theta$, even for $\phi_z$, and even for $\phi_r$, $\phi_\theta$, odd for $\phi_z$. Since, $\phi_r$ is unconstrained at the boundary $z = 0, L$, only the latter possibility is available to us. Hence, we expand $\phi_r$ and $\phi_\theta$ is the cosine series in $z$, while $\phi_z$ is represented by the sine series:

$$\begin{cases} \phi_r(r,\theta,z) = \displaystyle\sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty} \widehat{\phi}_r(r,m,n)e^{in\theta}\cos\left(\frac{\pi m z}{L}\right), \\ \phi_\theta(r,\theta,z) = \displaystyle\sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty} \widehat{\phi}_\theta(r,m,n)e^{in\theta}\cos\left(\frac{\pi m z}{L}\right), \\ \phi_z(r,\theta,z) = \displaystyle\sum_{n\in\mathbb{Z}}\sum_{m=0}^{\infty} \widehat{\phi}_z(r,m,n)e^{in\theta}\sin\left(\frac{\pi m z}{L}\right). \end{cases} \tag{3.2}$$

While functions in $V_h$ can be represented by the expansion (3.2), single Fourier modes do not belong to $V_h$. Yet, the convenience of working with such simple test functions outweighs this unfortunate circumstance, and hence, we switch (for the duration of technical calculations) to the space

$$\tilde{V}_h = \left\{ \boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : \phi_z(r,\theta,0) = \phi_z(r,\theta,L) = \int_0^L \phi_\theta(r,\theta,z)dz = 0 \ \forall (r,\theta) \in I_h \times \mathbb{T} \right\}. \tag{3.3}$$

We will come back at the very end to the space $V_h$ to get the desired result for our original boundary conditions.

The space $\tilde{V}_h$ appears in our companion paper [4] as $V_h^3$, where the inequalities (2.25), (2.28) and (2.29) have been proved for it. As a consequence, the estimates (2.31) hold for

$$\widetilde{\lambda}(h) = \inf_{\boldsymbol{\phi}\in\tilde{\mathcal{A}}_h} \mathfrak{R}(h,\boldsymbol{\phi}), \tag{3.4}$$

where $\tilde{\mathcal{A}}_h$ is given by (2.16) with $V_h$ replaced by $\tilde{V}_h$. We conclude that the pair $(\tilde{\mathcal{A}}_h, \mathfrak{R}(h,\boldsymbol{\phi}))$ characterizes buckling (for the new boundary conditions associated with the space $\tilde{V}_h$). In that case the proof of Lemma 2.14 carries with no change for the space $\tilde{V}_h$. Hence, the pair $(\tilde{\mathcal{A}}_h, \mathfrak{R}_1(h,\boldsymbol{\phi}))$ characterizes buckling as well.

We now define the single Fourier mode spaces $\mathcal{F}(m,n)$. For any complex-valued function $\boldsymbol{f}(r) = (f_r(r), f_\theta(r), f_\theta(r))$ and any $m \geq 1$, $n \geq 0$ we define

$$\boldsymbol{\Phi}_{m,n}(\boldsymbol{f}) = \left(f_r(r)\cos\left(\frac{\pi m z}{L}\right), f_\theta(r)\cos\left(\frac{\pi m z}{L}\right), f_z(r)\sin\left(\frac{\pi m z}{L}\right)\right)e^{in\theta},$$

[8] Meaning that each component of $e(\boldsymbol{\phi})$ and its trace either changes sign or remains unchanged.

and

$$\mathcal{F}(m,n)=\{\mathfrak{Re}(\boldsymbol{\Phi}_{m,n}(\boldsymbol{f})):\boldsymbol{f}\in C^1(I_h;\mathbb{C}^3)\},\quad m\ge 1,\ n\ge 0. \tag{3.5}$$

We define

$$\widetilde{\lambda}_1(h)=\inf_{\boldsymbol{\phi}\in\tilde{\mathcal{A}}_h}\mathfrak{R}_1(h,\boldsymbol{\phi}),\qquad \tilde{\lambda}_{m,n}(h)=\inf_{\boldsymbol{\phi}\in\mathcal{F}(m,n)}\mathfrak{R}_1(h,\boldsymbol{\phi}). \tag{3.6}$$

THEOREM **3.1.**

*(i)*

$$\widetilde{\lambda}_1(h)=\inf_{\substack{m\ge 1\\ n\ge 0}}\tilde{\lambda}_{m,n}(h). \tag{3.7}$$

*(ii) The infimum in (3.7) is attained at $m=m(h)$ and $n=n(h)$ satisfying*

$$m(h)\le\frac{C}{\sqrt{h}},\qquad \frac{n(h)^2}{m(h)}\le\frac{C}{\sqrt{h}} \tag{3.8}$$

*for some constant $C$ depending only on $L$.*

*(iii) Let $(m(h),n(h))$ be a minimizer in (3.7). Then the pair $(\mathcal{F}(m(h),n(h)),\mathfrak{R}_1(h,\boldsymbol{\phi}))$ characterizes buckling in the sense of Definition 2.10.*

*Proof.* Part (i). Let

$$\alpha(h)=\inf_{\substack{m\ge 1\\ n\ge 0}}\tilde{\lambda}_{m,n}(h).$$

It is clear that $\tilde{\lambda}_{m,n}(h)\ge\widetilde{\lambda}_1(h)$ for any $m\ge 1$ and $n\ge 0$, since $\mathcal{F}(m,n)\subset\tilde{\mathcal{A}}_h$. Therefore, $\alpha(h)\ge\widetilde{\lambda}_1(h)$.

Let us prove the reverse inequality. By definition of $\alpha(h)$ we have

$$\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}\ge\alpha(h)\|\phi_{r,z}\|^2 \tag{3.9}$$

for any $\boldsymbol{\phi}\in\mathcal{F}(m,n)$, and any $m\ge 1$ and $n\ge 0$. Any $\boldsymbol{\phi}\in\tilde{\mathcal{A}}_h$ can be expanded in the Fourier series in $\theta$ and $z$

$$\boldsymbol{\phi}(r,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\boldsymbol{\phi}^{(m,n)}(r,\theta,z),$$

where $\boldsymbol{\phi}^{(m,n)}(r,\theta,z)\in\mathcal{F}(m,n)$ for all $m\ge 1$, $n\ge 0$. If $\mathsf{L}_0$ is isotropic, then the sine and cosine series in $z$ do not couple and the Plancherel identity implies that the quadratic form $\langle\widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle$ diagonalizes in Fourier space:

$$\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}),e(\boldsymbol{\phi})\rangle d\boldsymbol{x}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}_{m,n}),e(\boldsymbol{\phi}_{m,n})\rangle d\boldsymbol{x}. \tag{3.10}$$

We also have

$$\|\phi_{r,z}\|^2=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\|\phi_{r,z}^{(m,n)}\|^2=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\|\phi_{r,z}^{(m,n)}\|^2.$$

Inequality (3.9) implies that

$$\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}_{m,n}), e(\boldsymbol{\phi}_{m,n})\rangle d\boldsymbol{x} \geq \alpha(h)\|\phi_{r,z}^{(m,n)}\|^2, \qquad m \geq 1,\ n \geq 0. \tag{3.11}$$

Summing up, we obtain that

$$\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x} \geq \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}_{m,n}), e(\boldsymbol{\phi}_{m,n})\rangle d\boldsymbol{x} \geq \alpha(h)\|\phi_{r,z}\|^2$$

for every $\boldsymbol{\phi} \in \tilde{\mathcal{A}}_h$. It follows that $\widetilde{\lambda}_1(h) \geq \alpha(h)$, and Part (i) is proved.

To establish Part(ii) we require a new delicate Korn-type inequality, proved in [4]. It is a weighted Korn inequality in Nazarov's terminology [13].

Theorem **3.2.** *There exists a constant $C$ depending only on $L$ such that*

$$\|\nabla\boldsymbol{\phi}\|^2 \leq C\left(\frac{\|\phi_r\|}{h} + \|e(\boldsymbol{\phi})\|\right)\|e(\boldsymbol{\phi})\|. \tag{3.12}$$

*for any $\boldsymbol{\phi} \in \tilde{V}_h$.*

Observe that, according to the estimate

$$ch \leq \widetilde{\lambda}_1(h) \leq Ch.$$

and Part (i) we have

$$\inf_{\substack{m\geq 1\\ n\geq 0}} \tilde{\lambda}_{m,n}(h) = \inf_{(m,n)\in S_h} \tilde{\lambda}_{m,n}(h),$$

where

$$S_h = \{(m,n) : \tilde{\lambda}_{m,n}(h) \leq 2Ch\}.$$

Let us show that the bounds (3.8) hold for all $(m,n) \in S_h$. In particular, the sets $S_h$ are finite for all $h > 0$, and hence, the infimum in (3.7) is attained. Let $h > 0$ and $(m,n) \in S_h$ be fixed. Then, by definition of the infimum there exists $\boldsymbol{\phi}^h \in \mathcal{F}(m,n)$ such that $\mathfrak{R}_1(h,\boldsymbol{\phi}^h) \leq 3Ch$. Hence, there exists a possibly different constant $C$ (not relabeled, but independent of $m$, $n$ and $h$), such that

$$\|e(\boldsymbol{\phi}^h)\|^2 \leq Ch\|\phi_{r,z}^h\|^2 = Cm^2h\|\phi_r^h\|^2. \tag{3.13}$$

To prove the first estimate in (3.8) we apply inequality (3.12) to $\boldsymbol{\phi}^h$ and then estimate $\|e(\boldsymbol{\phi}^h)\|$ via (3.13):

$$\frac{m^2\pi^2}{L^2}\|\phi_r^h\|^2 = \|\phi_{r,z}^h\|^2 \leq \|\nabla\boldsymbol{\phi}^h\|^2 \leq C\left(m^2h + \frac{m}{\sqrt{h}}\right)\|\phi_r^h\|^2.$$

Hence

$$h + \frac{1}{m\sqrt{h}} \geq c$$

for some constant $c > 0$, independent of $h$. Therefore, we obtain a uniform in $h \in (0, 1)$ upper bound on $m\sqrt{h}$. To estimate $n$ we write

$$n^2\|\phi_r^h\|^2 = \|\phi_{r,\theta}^h\|^2 \leq C_0(\|(\nabla\boldsymbol{\phi}^h)_{r\theta}\|^2 + \|\phi_\theta^h\|^2).$$

By the Poincaré inequality

$$\|\phi_\theta^h\|^2 \leq \frac{L^2}{\pi^2}\|\phi_{\theta,z}^h\|^2 \leq \frac{L^2}{\pi^2}\|(\nabla\boldsymbol{\phi}^h)_{\theta z}\|^2,$$

and hence $n^2\|\phi_r^h\|^2 \leq C\|(\nabla\boldsymbol{\phi}^h)\|^2$. Applying (3.12) again and estimating $\|e(\boldsymbol{\phi}^h)\|$ via (3.13) we obtain

$$n^2 \leq C\left(hm^2 + \frac{m}{\sqrt{h}}\right),$$

from which $(3.8)_2$ follows via $(3.8)_1$. Part (ii) is proved now.

Part (iii). Now, let $m(h)$, $n(h)$ be the minimizers in (3.7). It is sufficient to show, due to Lemma 2.12, that $\mathcal{F}(m(h), n(h))$ contains a buckling mode. By definition of the infimum in (3.6), for each $h \in (0, h_0)$ there exists $\boldsymbol{\psi}_h \in \mathcal{F}(m(h), n(h)) \subset \tilde{\mathcal{A}}_h$ such that

$$\widetilde{\lambda}_1(h) = \lambda_{m(h),n(h)}(h) \leq \mathfrak{R}_1(h, \boldsymbol{\psi}_h) \leq \widetilde{\lambda}_1(h) + (\widetilde{\lambda}_1(h))^2.$$

Therefore,

$$\lim_{h\to 0}\frac{\mathfrak{R}_1(h, \boldsymbol{\psi}_h)}{\widetilde{\lambda}_1(h)} = 1.$$

Hence, $\boldsymbol{\psi}_h \in \mathcal{F}(m(h), n(h))$ is a buckling mode, since the pair $(\tilde{\mathcal{A}}_h, \mathfrak{R}_1(h, \boldsymbol{\phi}))$ characterizes buckling. $\square$

## 3.2 Linearization in $r$

In this section we prove that the buckling load and a buckling mode can be captured by single Fourier harmonics whose $\theta$ and $z$ components are linear in $r$. In fact, we specify an explicit structure for buckling mode candidates. We define the linearization operator as follows:

$$\mathcal{L}(\boldsymbol{\phi}) = (\phi_r(r, \theta, z), r\phi_\theta(1, \theta, z) - (r - 1)\phi_{r,\theta}(1, \theta, z), \phi_z(1, \theta, z) - (r - 1)\phi_{r,z}(1, \theta, z)).$$

We show now that the buckling mode can be found among the linearized single Fourier modes

$$\mathcal{F}_{\text{lin}}(m, n) = \{\mathcal{L}(\boldsymbol{\phi}) : \boldsymbol{\phi} \in \mathcal{F}(m, n)\}, \qquad m \geq 1,\ n \geq 0. \tag{3.14}$$

Lemma **3.3.** *There exists $C > 0$ depending only on $L$, so that for every $h \in (0, 1)$, every $m \geq 1$ and $n \geq 0$, satisfying (3.8), and every $\boldsymbol{\phi} \in \mathcal{F}(m, n)$ we have the estimate*

$$\mathfrak{R}_1(h, \mathcal{L}(\boldsymbol{\phi})) \leq (1 + Ch)\mathfrak{R}_1(h, \boldsymbol{\phi}). \tag{3.15}$$

*Proof.* We will perform linearization in $r$ sequentially, first in $\phi_\theta$ and then in $\phi_z$.

**Step 1** (Linearization of $\phi_\theta$.) We introduce the operator of linearization of $\phi_\theta$ component.

$$\mathcal{L}_\theta(\boldsymbol{\phi}) = (\phi_r(r,\theta,z), r\phi_\theta(1,\theta,z) - (r-1)\phi_{r,\theta}(1,\theta,z), \phi_z(r,\theta,z)),$$

For $\boldsymbol{\phi} \in \mathcal{F}_{\text{lin}}(m,n)$ we define $\boldsymbol{\phi}^{(1)} = \mathcal{L}_\theta(\boldsymbol{\phi})$. Then, it is easy to see that $\boldsymbol{\phi}^{(1)} \in \mathcal{F}_{\text{lin}}(m,n)$. It is clear that

$$e(\boldsymbol{\phi}^{(1)})_{rr} = e(\boldsymbol{\phi})_{rr}, \quad e(\boldsymbol{\phi}^{(1)})_{zr} = e(\boldsymbol{\phi})_{zr}, \quad e(\boldsymbol{\phi}^{(1)})_{zz} = e(\boldsymbol{\phi})_{zz},$$

Thus we can estimate

$$\|e(\boldsymbol{\phi}^{(1)})\|^2 \le \|e(\boldsymbol{\phi})\|^2 + \|e(\boldsymbol{\phi}^{(1)})_{r\theta}\|^2 + 2\|e(\boldsymbol{\phi}^{(1)})_{\theta\theta} - e(\boldsymbol{\phi})_{\theta\theta}\|^2 + 2\|e(\boldsymbol{\phi}^{(1)})_{\theta z} - e(\boldsymbol{\phi})_{\theta z}\|^2,$$

$$\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)})) - \mathrm{Tr}\,(e(\boldsymbol{\phi}))\| = \|e(\boldsymbol{\phi}^{(1)})_{\theta\theta} - e(\boldsymbol{\phi})_{\theta\theta}\|^2.$$

We also have

$$\|e(\boldsymbol{\phi}^{(1)})_{\theta\theta} - e(\boldsymbol{\phi})_{\theta\theta}\| \le 2\|\phi^{(1)}_{\theta,\theta} - \phi_{\theta,\theta}\|, \qquad \|e(\boldsymbol{\phi}^{(1)})_{\theta z} - e(\boldsymbol{\phi})_{\theta z}\| \le \|\phi^{(1)}_{\theta,z} - \phi_{\theta,z}\|.$$

Therefore,

$$\|e(\boldsymbol{\phi}^{(1)})\|^2 \le \|e(\boldsymbol{\phi})\|^2 + \|e(\boldsymbol{\phi}^{(1)})_{r\theta}\|^2 + 2(\|\phi^{(1)}_{\theta,\theta} - \phi_{\theta,\theta}\|^2 + \|\phi^{(1)}_{\theta,z} - \phi_{\theta,z}\|^2), \tag{3.16}$$

and

$$\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)})) - \mathrm{Tr}\,(e(\boldsymbol{\phi}))\| \le 2\|\phi^{(1)}_{\theta,\theta} - \phi_{\theta,\theta}\|^2. \tag{3.17}$$

Recalling that $\{\boldsymbol{\phi}, \boldsymbol{\phi}^{(1)}\} \subset \mathcal{F}(m,n)$, and that the inequalities (3.8) imply that $n^2 \le C/h$, we obtain

$$\|\phi^{(1)}_{\theta,\theta} - \phi_{\theta,\theta}\|^2 = n^2\|\phi^{(1)}_\theta - \phi_\theta\|^2 \le \frac{C}{h}\|\phi^{(1)}_\theta - \phi_\theta\|^2, \tag{3.18}$$

due to (3.8). Similarly,

$$\|\phi^{(1)}_{\theta,z} - \phi_{\theta,z}\|^2 = \frac{\pi^2 m^2}{L^2}\|\phi^{(1)}_\theta - \phi_\theta\|^2 \le \frac{C}{h}\|\phi^{(1)}_\theta - \phi_\theta\|^2, \tag{3.19}$$

Observe that

$$\|e(\boldsymbol{\phi}^{(1)})_{r\theta}\|^2 = \|\frac{\phi_{r,\theta} - \phi_{r,\theta}(1,\theta,z)}{r}\|^2 = n^2\left\|\frac{1}{r}\int_1^r \phi_{r,r}(t,\theta,z)dt\right\|^2.$$

Using the inequality

$$\int_{I_h}\left(\int_1^r f(t)dt\right)^2 dr \le \frac{h^2}{4}\int_{I_h} f(r)^2 dr, \tag{3.20}$$

and the bounds on wave numbers (3.8) we obtain

$$\|e(\boldsymbol{\phi}^{(1)})_{r\theta}\|^2 \le 2n^2Ch^2\|\phi_{r,r}\|^2 \le Ch\|e(\boldsymbol{\phi})\|^2.$$

We now proceed to estimate $\|\phi_\theta^{(1)} - \phi_\theta\|$. Let

$$w(r,\theta,z) = \phi_{\theta,r} + \phi_{r,\theta} - \phi_\theta = 2e(\boldsymbol{\phi})_{r\theta} - (1-r)(\nabla\boldsymbol{\phi})_{r\theta}.$$

Therefore,

$$\|w\|^2 \leq 8\|e(\boldsymbol{\phi})\|^2 + h^2\|\nabla\boldsymbol{\phi}\|^2 \leq 8\|e(\boldsymbol{\phi})\|^2 + C\sqrt{h}\|e(\boldsymbol{\phi})\|^2$$

due to Korn's inequality (2.25). Thus, $\|w\| \leq C\|e(\boldsymbol{\phi})\|$. We can express $\phi_\theta^{(1)} - \phi_\theta$ in terms of $w$ as follows

$$\phi_\theta - \phi_\theta^{(1)} = \int_1^r w(t,\theta,z)dt + \int_1^r (\phi_\theta(t,\theta,z) - \phi_\theta(1,\theta,z))dt - \int_1^r (\phi_{r,\theta}(t,\theta,z) - \phi_{r,\theta}(1,\theta,z))dt.$$

Hence, by (3.20), we have

$$\|\phi_\theta - \phi_\theta^{(1)}\|^2 \leq \frac{3h^2}{4}(\|w\|^2 + \|\phi_\theta - \phi_\theta(1,\theta,z)\|^2 + \|\phi_{r,\theta} - \phi_{r,\theta}(1,\theta,z)\|^2).$$

By the Poincaré inequality followed by the application of Korn's inequality (2.25) we obtain,

$$\|\phi_\theta - \phi_\theta(1,\theta,z)\|^2 \leq h^2\|\phi_{\theta,r}\|^2 \leq C\sqrt{h}\|e(\boldsymbol{\phi})\|^2.$$

Similarly, by the Poincaré inequality and (3.8) we estimate

$$\|\phi_{r,\theta} - \phi_{r,\theta}(1,\theta,z)\|^2 = n^2\|\phi_r - \phi_r(1,\theta,z)\|^2 \leq Cn^2h^2\|\phi_{r,r}\|^2 \leq Ch\|e(\boldsymbol{\phi})\|^2.$$

We conclude that

$$\|\phi_\theta - \phi_\theta^{(1)}\|^2 \leq Ch^2\|e(\boldsymbol{\phi})\|^2.$$

Hence, (3.16) and (3.17) become respectively,

$$\|e(\boldsymbol{\phi}^{(1)})\|^2 \leq \|e(\boldsymbol{\phi})\|^2(1+Ch), \tag{3.21}$$

and

$$\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)}))\|^2 \leq \|\mathrm{Tr}\,(e(\boldsymbol{\phi}))\|^2 + Ch\|e(\boldsymbol{\phi})\|^2. \tag{3.22}$$

Hence, by (3.21), (3.22) and the coercivity of $\widehat{\mathsf{L}}_0$, we have

$$\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}^{(1)}), e(\boldsymbol{\phi}^{(1)})\rangle d\boldsymbol{x} = \frac{1}{1+\nu}\left(\frac{\nu}{1-2\nu}\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)}))\|^2 + \|e(\boldsymbol{\phi}^{(1)})\|^2\right) \leq$$
$$(1+Ch)\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}. \tag{3.23}$$

**Step 2** (Linearization of $\phi_z$.) In this step we define $\boldsymbol{\phi}^{(2)} = \mathcal{L}(\boldsymbol{\phi}) = \mathcal{L}(\boldsymbol{\phi}^{(1)})$, and proceed exactly as in Step 1. We observe that

$$e(\boldsymbol{\phi}^{(2)})_{rr} = e(\boldsymbol{\phi}^{(1)})_{rr}, \qquad e(\boldsymbol{\phi}^{(2)})_{r\theta} = e(\boldsymbol{\phi}^{(1)})_{r\theta}, \qquad e(\boldsymbol{\phi}^{(2)})_{\theta\theta} = e(\boldsymbol{\phi}^{(1)})_{\theta\theta},$$

and hence,

$$\|e(\boldsymbol{\phi}^{(2)})\|^2 \le \|e(\boldsymbol{\phi}^{(1)})\|^2 + \|e(\boldsymbol{\phi}^{(2)})_{rz}\|^2 + 2\|e(\boldsymbol{\phi}^{(2)})_{\theta z} - e(\boldsymbol{\phi}^{(1)})_{\theta z}\|^2 + 2\|e(\boldsymbol{\phi}^{(2)})_{zz} - e(\boldsymbol{\phi}^{(1)})_{zz}\|^2,$$

and

$$\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)})) - \mathrm{Tr}\,(e(\boldsymbol{\phi}^{(2)}))\| \le 2\|\phi_{z,z}^{(1)} - \phi_{z,z}^{(2)}\|^2. \tag{3.24}$$

We also have

$$\|e(\boldsymbol{\phi}^{(2)})_{\theta z} - e(\boldsymbol{\phi}^{(1)})_{\theta z}\| \le 2\|\phi_{z,\theta}^{(1)} - \phi_{z,\theta}^{(2)}\|, \qquad \|e(\boldsymbol{\phi}^{(2)})_{zz} - e(\boldsymbol{\phi}^{(1)})_{zz}\| \le \|\phi_{z,z}^{(1)} - \phi_{z,z}^{(2)}\|. \tag{3.25}$$

For functions $\{\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}\} \subset \mathcal{F}(m,n)$ we obtain

$$\|\phi_{z,\theta}^{(1)} - \phi_{z,\theta}^{(2)}\| = n\|\phi_z^{(1)} - \phi_z^{(2)}\| \le \frac{C}{h}\|\phi_z^{(1)} - \phi_z^{(2)}\|,$$

and

$$\|\phi_{z,z}^{(1)} - \phi_{z,z}^{(2)}\| = \frac{\pi m}{L}\|\phi_z^{(1)} - \phi_z^{(2)}\| \le \frac{C}{h}\|\phi_z^{(1)} - \phi_z^{(2)}\|,$$

where the bounds (3.8) on wave numbers have been used. Hence,

$$\|e(\boldsymbol{\phi}^{(2)})_{\theta z} - e(\boldsymbol{\phi}^{(1)})_{\theta z}\|^2 \le \frac{C}{h}\|\phi_z^{(1)} - \phi_z^{(2)}\|^2, \quad \|e(\boldsymbol{\phi}^{(2)})_{zz} - e(\boldsymbol{\phi}^{(1)})_{zz}\|^2 \le \frac{C}{h}\|\phi_z^{(1)} - \phi_z^{(2)}\|^2. \tag{3.26}$$

For $\|e(\boldsymbol{\phi}^{(2)})_{rz}\|$ we obtain

$$\|e(\boldsymbol{\phi}^{(2)})_{rz}\|^2 = \|\phi_{r,z} - \phi_{r,z}(1,\theta,z)\|^2 = \frac{\pi^2 m^2}{L^2}\|\phi_r - \phi_r(1,\theta,z)\|^2 = \frac{\pi^2 m^2}{L^2}\left\|\int_1^r \phi_{r,r}(t,\theta,z)dt\right\|^2.$$

Applying inequalities (3.20) and (3.8) we obtain

$$\|e(\boldsymbol{\phi}^{(2)})_{rz}\|^2 \le Cm^2h^2\|\phi_{r,r}\|^2 \le Ch\|e(\boldsymbol{\phi}^{(1)})\|^2.$$

Finally, we estimate the norm $\|\phi_z^{(1)} - \phi_z^{(2)}\|$. Integrating the identity $\phi_{z,r}^{(1)} = 2e(\boldsymbol{\phi}^{(1)})_{rz} - \phi_{r,z}^{(1)}$ we get

$$\phi_z^{(1)}(r,\theta,z) - \phi_z^{(1)}(1,\theta,z) = 2\int_1^r e(\boldsymbol{\phi}^{(1)})_{rz}(t,\theta,z)dt - \int_1^r \phi_{r,z}^{(1)}(t,\theta,z)dt.$$

Therefore,

$$\phi_z^{(1)} - \phi_z^{(2)} = 2\int_1^r e(\boldsymbol{\phi}^{(1)})_{rz}(t,\theta,z)dt - \int_1^r (\phi_{r,z}^{(1)}(t,\theta,z) - \phi_{r,z}^{(1)}(1,\theta,z))dt.$$

Applying inequalities (3.20) and (3.8) we get

$$\|\phi_z^{(1)} - \phi_z^{(2)}\|^2 \le h^2(\|e(\boldsymbol{\phi}^{(1)})\|^2 + \|\phi_{r,z}^{(1)}(r,\theta,z) - \phi_{r,z}^{(1)}(1,\theta,z)\|^2) =$$
$$h^2(\|e(\boldsymbol{\phi}^{(1)})\|^2 + \frac{\pi^2 m^2}{L^2}\left\|\int_1^r \phi_{r,r}^{(1)}(t,\theta,z)dt\right\|^2) \le h^2(\|e(\boldsymbol{\phi}^{(1)})\|^2 + \frac{\pi^2 m^2 h^2}{L^2}\|\phi_{r,r}^{(1)}\|^2) \le$$
$$h^2(1 + \frac{\pi^2 m^2 h^2}{L^2})\|e(\boldsymbol{\phi}^{(1)})\|^2 \le Ch^2\|e(\boldsymbol{\phi}^{(1)})\|^2.$$

Applying this estimate to (3.26) and (3.24) we obtain

$$\|e(\boldsymbol{\phi}^{(2)})_{\theta z}-e(\boldsymbol{\phi}^{(1)})_{\theta z}\|^2\le Ch\|e(\boldsymbol{\phi}^{(1)})\|^2,\qquad \|e(\boldsymbol{\phi}^{(2)})_{zz}-e(\boldsymbol{\phi}^{(1)})_{zz}\|^2\le Ch\|e(\boldsymbol{\phi}^{(1)})\|^2,$$

and

$$\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(2)}))\|^2\le\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)}))\|^2+Ch\|e(\boldsymbol{\phi}^{(1)})\|^2.$$

We conclude that

$$\|e(\boldsymbol{\phi}^{(2)})\|^2\le\|e(\boldsymbol{\phi}^{(1)})\|^2(1+Ch),\qquad \|\mathrm{Tr}\,(\boldsymbol{\phi}^{(2)})\|^2\le\|\mathrm{Tr}\,(e(\boldsymbol{\phi}^{(1)}))\|^2+Ch\|e(\boldsymbol{\phi}^{(1)})\|^2,$$

and hence, by coercivity of $\widehat{\mathsf{L}}_0$ we have

$$\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0e(\boldsymbol{\phi}^{(2)}),e(\boldsymbol{\phi}^{(2)})\rangle d\boldsymbol{x}\le(1+Ch)\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0e(\boldsymbol{\phi}^{(1)}),e(\boldsymbol{\phi}^{(1)})\rangle d\boldsymbol{x}. \tag{3.27}$$

Combining (3.23) and (3.27) we obtain (3.15). □

Lemma 3.3 permits us to look for a buckling mode among those single Fourier modes, whose $\theta$ and $z$ components are linear in $r$. Let $C$ be a constant, whose existence is guaranteed by Lemma 3.3. Let

$$\mathcal{M}_h=\{(m,n):n\ge 0,\ m\ge 1\text{ and inequalities (3.8) hold}\}.$$

Let

$$\mathcal{F}_{\mathrm{lin}}^h=\bigcup_{(m,n)\in\mathcal{M}_h}\mathcal{F}_{\mathrm{lin}}(m,n).$$

**Corollary 3.4.** *The pair $(\mathcal{F}_{\mathrm{lin}}^h,\mathfrak{R}_1)$ characterizes buckling.*

*Proof.* By Lemma 2.12 it is sufficient to show that $\mathcal{F}_{\mathrm{lin}}^h$ contains a buckling mode. Let $(m(h),n(h))$ be minimizers in (3.7). Then, according to Theorem 3.1, $(m(h),n(h))\in\mathcal{M}_h$ and $\mathcal{F}(m(h),n(h))$ contains a buckling mode. Let $\boldsymbol{\psi}_h\in\mathcal{F}(m(h),n(h))$ be a buckling mode. Let us show that $\mathcal{L}(\boldsymbol{\psi}_h)\in\mathcal{F}_{\mathrm{lin}}^h$ is also a buckling mode. Indeed, by Lemma 3.3

$$1\le\frac{\mathfrak{R}_1(h,\mathcal{L}(\boldsymbol{\psi}_h))}{\widetilde{\lambda}_1(h)}\le(1+Ch)\frac{\mathfrak{R}_1(h,\boldsymbol{\psi}_h)}{\widetilde{\lambda}_1(h)}.$$

Taking a limit as $h\to 0$ and using the fact that $\boldsymbol{\psi}_h$ is a buckling mode, we obtain

$$\lim_{h\to 0}\frac{\mathfrak{R}_1(h,\mathcal{L}(\boldsymbol{\psi}_h))}{\widetilde{\lambda}_1(h)}=1.$$

Hence, $\mathcal{L}(\boldsymbol{\psi}_h)$ is also a buckling mode, since, by Theorem 3.1, the pair $(\mathcal{F}(m(h),n(h)),\mathfrak{R}_1(h,\boldsymbol{\phi}))$ characterizes buckling. □

## 3.3 Simplification via buckling equivalence

The linearization Lemma 3.3 allowed us to reduce the set of buckling modes significantly. Yet, even for functions $\boldsymbol{\phi} \in \mathcal{F}_{\text{lin}}(m,n)$ the explicit representation of the functional $\mathfrak{R}_1(h,\boldsymbol{\phi})$ is extremely messy. This can be dealt with by further simplification of the functional via buckling equivalence that permits us to eliminate lower order terms that do not influence the asymptotic behavior of the functional. Our first step is to simplify the denominator in $\mathfrak{R}_1(h,\boldsymbol{\phi})$ by replacing the unknown function $f_r(r)$ in $\phi_r = f_r(r)\cos(mz)e^{in\theta}$ with $f_r(1)$. Here, in order to simplify the formulas we use $m$ in place of $\pi m/L$. Hence, we define a new simplified functional

$$\mathfrak{R}_2(h,\boldsymbol{\phi}) = \frac{\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\int_{\mathcal{C}_h}|\phi_{r,z}(1,\theta,z)|^2 d\boldsymbol{x}}.$$

Lemma **3.5.** *The functionals $\mathfrak{R}_1(h,\boldsymbol{\phi})$ and $\mathfrak{R}_2(h,\boldsymbol{\phi})$ are buckling equivalent.*

*Proof.* We observe that

$$|\phi_{r,z}(r,\theta,z) - \phi_{r,z}(1,\theta,z)| = m\left|\int_1^r \phi_{r,r}(t,\theta,z)\right|.$$

Hence, due to (3.20)

$$\|\phi_{r,z}(r,\theta,z) - \phi_{r,z}(1,\theta,z)\| \le mh\|e(\boldsymbol{\phi})\|.$$

Therefore,

$$\left|\int_{\mathcal{C}_h}|\phi_{r,z}(r,\theta,z)|^2 d\boldsymbol{x} - \int_{\mathcal{C}_h}|\phi_{r,z}(1,\theta,z)|^2 d\boldsymbol{x}\right| \le mh\|e(\boldsymbol{\phi})\|\|\phi_{r,z}\| \le m\sqrt{h}\|e(\boldsymbol{\phi})\|^2,$$

due to Theorem 2.8. Hence,

$$\left|\frac{1}{\mathfrak{R}_1(h,\boldsymbol{\phi})} - \frac{1}{\mathfrak{R}_2(h,\boldsymbol{\phi})}\right| \le Cm\sqrt{h},$$

by coercivity of $\mathsf{L}_0$. For $(m,n) \in \mathcal{M}_h$ we conclude that, due to (2.31) and (3.8),

$$\lim_{h\to 0}\lambda(h)\left|\frac{1}{\mathfrak{R}_1(h,\boldsymbol{\phi})} - \frac{1}{\mathfrak{R}_2(h,\boldsymbol{\phi})}\right| = 0.$$

Theorem 2.13 applies and hence the functionals $\mathfrak{R}_1(h,\boldsymbol{\phi})$ and $\mathfrak{R}_2(h,\boldsymbol{\phi})$ are buckling equivalent. □

We can also simplify the numerator of $\mathfrak{R}_2(h,\boldsymbol{\phi})$ by replacing $r$ with 1 in those places, where it does not affect the asymptotics. The simplification now proceeds at the level of individual components of $e(\boldsymbol{\phi})$. We may, without loss of generality, restrict our attention to $\boldsymbol{\phi} \in \mathcal{F}_{\text{lin}}(m,n)$, such that

$$\phi_r = f_r(r)\cos(n\theta)\cos(mz). \tag{3.28}$$

Of course, choosing $\sin(n\theta)$ instead of $\cos(n\theta)$ in (3.28) works just as well. The choice between $\sin(n\theta)$ and $\cos(n\theta)$ in the remaining components becomes uniquely determined by the requirement that every entry in $e(\boldsymbol{\phi})$ must be made up of terms that have the same kind of trigonometric function in $n\theta$. (We have already taken care of the same requirement in $z$.) Hence, the $\theta$ and $z$ components of $\boldsymbol{\phi} \in \mathcal{F}_{\rm lin}(m,n)$ must have the form

$$\begin{cases} \phi_\theta = (ra_\theta + (r-1)nf_r(1))\sin(n\theta)\cos(mz), \\ \phi_z = (a_z + (r-1)mf_r(1))\cos(n\theta)\sin(mz), \end{cases} \tag{3.29}$$

where $a_\theta$ and $a_z$ are real scalars that determine the amplitude of the Fourier modes. We compute,

$$\begin{cases} e(\boldsymbol{\phi})_{rr} = f_r'(r)\cos(n\theta)\cos(mz), \\ e(\boldsymbol{\phi})_{r\theta} = \dfrac{n(f_r(1)-f_r(r))}{2r}\sin(n\theta)\cos(mz), \\ e(\boldsymbol{\phi})_{rz} = \dfrac{m(f_r(1)-f_r(r))}{2}\cos(n\theta)\sin(mz), \\ e(\boldsymbol{\phi})_{\theta\theta} = \dfrac{n(ra_\theta + (r-1)nf_r(1)) + f_r(r)}{r}\cos(n\theta)\cos(mz), \\ e(\boldsymbol{\phi})_{\theta z} = -\dfrac{mr^2a_\theta + na_z + (r^2-1)mnf_r(1)}{2r}\sin(n\theta)\sin(mz), \\ e(\boldsymbol{\phi})_{rz} = m(a_z + (r-1)mf_r(1))\cos(n\theta)\cos(mz). \end{cases}$$

We can now replace $e(\boldsymbol{\phi})$ with a much simpler matrix $E(\boldsymbol{\phi})$, given by

$$\begin{cases} E(\boldsymbol{\phi})_{rr} = \dfrac{f_r'(r)}{\sqrt{r}}\cos(n\theta)\cos(mz), \\ E(\boldsymbol{\phi})_{r\theta} = 0, \\ E(\boldsymbol{\phi})_{rz} = 0, \\ E(\boldsymbol{\phi})_{\theta\theta} = \dfrac{n(ra_\theta + (r-1)nf_r(1)) + f_r(1)}{\sqrt{r}}\cos(n\theta)\cos(mz), \\ E(\boldsymbol{\phi})_{\theta z} = -\dfrac{mr^2a_\theta + na_z + (r^2-1)mnf_r(1)}{2\sqrt{r}}\sin(n\theta)\sin(mz), \\ E(\boldsymbol{\phi})_{rz} = \dfrac{m(a_z + (r-1)mf_r(1))}{\sqrt{r}}\cos(n\theta)\cos(mz) \end{cases}$$

LEMMA **3.6.** *The functionals* $\mathfrak{R}_2(h,\boldsymbol{\phi})$ *and*

$$\mathfrak{R}_3(h,\boldsymbol{\phi}) = \frac{\int_{\mathcal{C}_h}\langle\widehat{\mathsf{L}}_0 E(\boldsymbol{\phi}), E(\boldsymbol{\phi})\rangle d\boldsymbol{x}}{\int_{\mathcal{C}_h}|\phi_{r,z}(1,\theta,z)|^2 d\boldsymbol{x}} \tag{3.30}$$

*are buckling equivalent.*

*Proof.* Observing that

$$f_r(r) - f_r(1) = \int_1^r f_r'(t)dt,$$

we obtain via (3.20) that

$$\|e(\boldsymbol{\phi})_{r\theta}\|^2 \le Cn^2h^2\|f_r'\|^2 \le Cn^2h^2\|e(\boldsymbol{\phi})_{rr}\|^2.$$

Similarly,

$$\|e(\boldsymbol{\phi})_{rz}\|^2 \le Cm^2h^2\|e(\boldsymbol{\phi})_{rr}\|^2.$$

Hence, for every $(m,n) \in \mathcal{M}_h$ we have

$$\|e(\boldsymbol{\phi})_{r\theta}\|^2 + \|e(\boldsymbol{\phi})_{rz}\|^2 \le Ch\|e(\boldsymbol{\phi})_{rr}\|^2.$$

For the components $(rr)$, $(\theta z)$ and $(zz)$ we have

$$E(\boldsymbol{\phi})_{rr} = \frac{e(\boldsymbol{\phi})_{rr}}{\sqrt{r}}, \quad E(\boldsymbol{\phi})_{\theta z} = \sqrt{r}e(\boldsymbol{\phi})_{\theta z}, \quad E(\boldsymbol{\phi})_{zz} = \frac{e(\boldsymbol{\phi})_{zz}}{\sqrt{r}}.$$

Therefore,

$$|E(\boldsymbol{\phi})_{rr} - e(\boldsymbol{\phi})_{rr}| \le Ch|e(\boldsymbol{\phi})_{rr}|, \quad |E(\boldsymbol{\phi})_{\theta z} - e(\boldsymbol{\phi})_{\theta z}| \le Ch|e(\boldsymbol{\phi})_{\theta z}|, \quad |E(\boldsymbol{\phi})_{zz} - e(\boldsymbol{\phi})_{zz}| \le Ch|e(\boldsymbol{\phi})_{zz}|.$$

Finally we compute

$$E(\boldsymbol{\phi})_{\theta\theta} - e(\boldsymbol{\phi})_{\theta\theta} = (\sqrt{r} - 1)e(\boldsymbol{\phi})_{\theta\theta} - \frac{f_r(r) - f_r(1)}{\sqrt{r}}\cos(n\theta)\cos(mz),$$

which implies

$$\|E(\boldsymbol{\phi})_{\theta\theta} - e(\boldsymbol{\phi})_{\theta\theta}\| \le Ch(\|e(\boldsymbol{\phi})_{\theta\theta}\| + \|e(\boldsymbol{\phi})_{rr}\|).$$

We conclude that that

$$\|E(\boldsymbol{\phi}) - e(\boldsymbol{\phi})\| \le C\sqrt{h}\|e(\boldsymbol{\phi})\|,$$

and thus

$$\left|\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 E(\boldsymbol{\phi}), E(\boldsymbol{\phi})\rangle d\boldsymbol{x} - \int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x}\right| \le C\sqrt{h}\|e(\boldsymbol{\phi})\|^2 \le C\sqrt{h}\int_{\mathcal{C}_h} \langle \widehat{\mathsf{L}}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle d\boldsymbol{x},$$

by coercivity of $\widehat{\mathsf{L}}_0$. It follows that

$$|\mathfrak{R}_3(h,\boldsymbol{\phi}) - \mathfrak{R}_2(h,\boldsymbol{\phi})| \le C\sqrt{h}\mathfrak{R}_2(h,\boldsymbol{\phi}) \le C\sqrt{h}\mathfrak{R}_3(h,\boldsymbol{\phi}) + C\sqrt{h}|\mathfrak{R}_3(h,\boldsymbol{\phi}) - \mathfrak{R}_2(h,\boldsymbol{\phi})|.$$

Thus,

$$|\mathfrak{R}_3(h,\boldsymbol{\phi}) - \mathfrak{R}_2(h,\boldsymbol{\phi})| \le \frac{C\sqrt{h}}{1 - C\sqrt{h}}\mathfrak{R}_3(h,\boldsymbol{\phi}).$$

Dividing this inequality by $\mathfrak{R}_2(h,\boldsymbol{\phi})\mathfrak{R}_3(h,\boldsymbol{\phi})$ we obtain

$$\left|\frac{1}{\mathfrak{R}_2(h,\boldsymbol{\phi})} - \frac{1}{\mathfrak{R}_3(h,\boldsymbol{\phi})}\right| \le \frac{C\sqrt{h}}{\mathfrak{R}_2(h,\boldsymbol{\phi})}.$$

Therefore,

$$\sup_{\phi\in\mathcal{F}_{\rm lin}^h}\tilde{\lambda}(h)\left|\frac{1}{\mathfrak{R}_2(h,\boldsymbol{\phi})}-\frac{1}{\mathfrak{R}_3(h,\boldsymbol{\phi})}\right|\le\frac{C\tilde{\lambda}(h)\sqrt{h}}{\inf_{\phi\in\mathcal{F}_{\rm lin}^h}\mathfrak{R}_2(h,\boldsymbol{\phi})}.$$

It follows that, due to (2.31),

$$\lim_{h\to 0}\sup_{\phi\in\mathcal{F}_{\rm lin}^h}\lambda(h)\left|\frac{1}{\mathfrak{R}_2(h,\boldsymbol{\phi})}-\frac{1}{\mathfrak{R}_3(h,\boldsymbol{\phi})}\right|=0.$$

The application of Theorem 2.13 completes the proof. □

## 3.4 The formula for the buckling load

At this point the strategy for finding the asymptotic formula for the buckling load can be stated as follows. We first compute

$$\lambda_3(h;m,n)=\inf_{\phi\in\mathcal{F}_{\rm lin}(m,n)}\mathfrak{R}_3(h,\boldsymbol{\phi}),\tag{3.31}$$

and then we find $m(h)$ and $n(h)$ as minimizers in

$$\lambda_3(h)=\min_{\substack{m\ge 1\\ n\ge 0}}\lambda_3(h;m,n).\tag{3.32}$$

The goal of the section is to prove that

$$\lim_{h\to 0}\frac{\lambda_3(h)}{\lambda^*(h)}=1,\qquad \lambda^*(h)=\frac{h}{\sqrt{3(1-\nu^2)}}.\tag{3.33}$$

The functional $\mathfrak{R}_3(h,\boldsymbol{\phi})$ given by (3.30) will now be analyzed in its explicit form.

$$\begin{aligned}\mathfrak{R}_3(h,\boldsymbol{\phi})=&\frac{1}{2(\nu+1)hm^2|f_r(1)|^2}\int_{I_h}\{(mr^2a_\theta+na_z+(r^2-1)mnf_r(1))^2+\\&+2(f_r')^2+2(nra_\theta+(r-1)n^2f_r(1)+f_r(1))^2+2m^2(a_z+(r-1)mf_r(1))^2\\&\Lambda(f_r'(r)+nra_\theta+(r-1)n^2f_r(1)+f_r(1)+ma_z+(r-1)m^2f_r(1))^2\}dr,\end{aligned}$$

where $\Lambda=\frac{2\nu}{1-2\nu}$. We minimize the numerator in $f_r(r)$ with prescribed value $f_r(1)$. This can be done by minimizing the numerator in $f_r'(r)$ treating it as a scalar variable for each fixed $r$:

$$f_r'(r)=-\frac{\Lambda}{\Lambda+2}p(r),$$

where

$$p(r)=nra_\theta+(r-1)n^2f_r(1)+f_r(1)+ma_z+(r-1)m^2f_r(1).$$

Thus, we reduce the problem of computing $\lambda_3(h;m,n)$ to finite-dimensional unconstrained minimization:

$$\lambda_3(h;m,n)=\min_{a_\theta,a_z,f_r(1)}\frac{\int_{I_h}\{\frac{2\Lambda}{\Lambda+2}p(r)^2+q(r)\}dr}{2(\nu+1)hm^2|f_r(1)|^2},\tag{3.34}$$

where

$$q(r) = (mr^2a_\theta + na_z + (r^2-1)mnf_r(1))^2 + 2(nra_\theta + (r-1)n^2f_r(1) + f_r(1))^2 + 2m^2(a_z + (r-1)mf_r(1))^2.$$

Since the function to be minimized in (3.34) is homogeneous of degree zero in the vector variable $(a_\theta, a_z, f_r(1))$, we can set $f_r(1) = 1$, without loss of generality. Then, evaluating the integral in $r$ we obtain

$$\lambda_3(h;m,n) = \min_{a_\theta,a_z} \frac{1}{2(\nu+1)m^2}\left\{Q^{(0)}_{m,n}(a_\theta,a_z) + \frac{h^2}{12}Q^{(1)}_{m,n}(a_\theta,a_z) + \frac{h^4}{80}Q^{(2)}_{m,n}(a_\theta,a_z)\right\},$$

where

$$Q^{(0)}_{m,n} = \frac{2\Lambda}{\Lambda+2}(1+na_\theta+ma_z)^2 + 2(na_\theta+1)^2 + 2m^2a_z^2 + (ma_\theta+na_z)^2,$$

$$Q^{(1)}_{m,n} = \frac{2\Lambda}{\Lambda+2}(na_\theta+m^2+n^2)^2 + 2n^2(a_\theta+n)^2 + 2m^4 + 4m^2(a_\theta+n)^2 + 2m(a_\theta+n)(ma_\theta+na_z),$$

$$Q^{(2)}_{m,n} = m^2(a_\theta+n)^2.$$

Let us show that the last term in $Q^{(1)}_{m,n}$, as well as $Q^{(2)}_{m,n}$ can be discarded. Let

$$\tilde{Q}^{(1)}_{m,n}(a_\theta) = \frac{2\Lambda}{\Lambda+2}(na_\theta+m^2+n^2)^2 + 2n^2(a_\theta+n)^2 + 2m^4 + 4m^2(a_\theta+n)^2$$

be the simplified version of $Q^{(1)}_{m,n}$. We observe that

$$2m|(a_\theta+n)(ma_\theta+na_z)| \le hm^2(a_\theta+n)^2 + \frac{1}{h}(ma_\theta+na_z)^2 \le \frac{h}{4}\tilde{Q}^{(1)}_{m,n} + \frac{1}{h}Q^{(0)}_{m,n}.$$

Therefore,

$$\frac{h^4}{80}m^2(a_\theta+n)^2 + \frac{h^2}{6}m|(a_\theta+n)(ma_\theta+na_z)| \le (h^2+h)\left(Q^{(0)}_{m,n} + \frac{h^2}{12}\tilde{Q}^{(1)}_{m,n}\right).$$

Hence,

$$(1-h-h^2)\left(Q^{(0)}_{m,n} + \frac{h^2}{12}\tilde{Q}^{(1)}_{m,n}\right) \le Q^{(0)}_{m,n} + \frac{h^2}{12}Q^{(1)}_{m,n} \le (1+h+h^2)\left(Q^{(0)}_{m,n} + \frac{h^2}{12}\tilde{Q}^{(1)}_{m,n}\right)$$

If we denote

$$\tilde{\lambda}_3(h;m,n) = \min_{a_\theta,a_z} \frac{1}{2(\nu+1)m^2}\left\{Q^{(0)}_{m,n}(a_\theta,a_z) + \frac{h^2}{12}\tilde{Q}^{(1)}_{m,n}(a_\theta)\right\}, \tag{3.35}$$

then

$$(1-h-h^2)\tilde{\lambda}_3(h;m,n) \le \lambda_3(h;m,n) \le (1+h+h^2)\tilde{\lambda}_3(h;m,n),$$

which implies that

$$\lim_{h\to 0}\frac{\tilde{\lambda}_3(h)}{\lambda_3(h)}=1,\qquad \tilde{\lambda}_3(h)=\min_{\substack{m\ge 1\\ n\ge 0}}\tilde{\lambda}_3(h;m,n). \tag{3.36}$$

Minimizing $Q^{(0)}_{m,n}(a_\theta,a_z)$ in $a_z$ we obtain

$$a_z=-\frac{m(2\nu+(\nu+1)na_\theta)}{2m^2+(1-\nu)n^2}. \tag{3.37}$$

The minimization in $a_\theta$ was too tedious to be done by hand. Using computer algebra software (Maple), we have obtained the following expression for $\tilde{\lambda}_3(h;m,n)$:

$$\tilde{\lambda}_3(h;m,n)=\frac{m^2(1-\nu^2)+H(m^2+n^2)^4+Hr_1(m,n)+H^2r_2(m,n)}{(1-\nu^2)m^2(m^2+n^2)^2+Hm^2r_3(m,n)},\quad H=\frac{h^2}{12}, \tag{3.38}$$

where $r_1(m,n)$ is a polynomial in $(m,n)$ of degree 6, $r_2(m,n)$ is a polynomial in $(m,n)$ of degree 8 and $r_3(m,n)$ is a polynomial in $(m,n)$ of degree 4. The minimum was achieved at

$$a_\theta=-\frac{n(n^2+(\nu+2)m^2)+Hs_1(m,n)}{(m^2+n^2)^2+Hs_2(m,n)}, \tag{3.39}$$

where $s_1(m,n)$ is a polynomial in $(m,n)$ of degree 5, and $s_2(m,n)$ is a polynomial in $(m,n)$ of degree 4. Let us show that the terms $r_i(m,n)$ do not affect the asymptotics of $\tilde{\lambda}_3(h)$. Let

$$\lambda_3^*(h;m,n)=\frac{m^4(1-\nu^2)+H(m^2+n^2)^4}{(1-\nu^2)m^2(m^2+n^2)^2},\qquad \lambda_3^*(h)=\min_{\substack{m\ge 1\\ n\ge 0}}\lambda_3^*(h;m,n). \tag{3.40}$$

Lemma **3.7.**

$$\lambda_3^*(h)=\frac{h}{\sqrt{3(1-\nu^2)}}, \tag{3.41}$$

*and is attained on the Koiter circle [10]*

$$\frac{m}{m^2+n^2}=\sqrt{\frac{\lambda_3^*(h)}{2}},\qquad m\ge\frac{\pi}{L},\ n\ge 0. \tag{3.42}$$

*Moreover,*

$$\lim_{h\to 0}\frac{\tilde{\lambda}_3(h)}{\lambda_3^*(h)}=1, \tag{3.43}$$

*Proof.* Formulas (3.41) and (3.42) become obvious, if we observe that

$$\lambda_3^*(h;m,n)=\frac{m^2}{(m^2+n^2)^2}+\frac{H(m^2+n^2)^2}{(1-\nu^2)m^2}.$$

It is also clear from the degrees of polynomials $r_2(m,n)$ and $r_3(m,n)$ that

$$\sup_{\substack{m\ge\pi/L\\ n\ge 0}}\frac{H^2r_2(m,n)}{2m^4(1-\nu^2)+2H(m^2+n^2)^4}\le\sup_{\substack{m\ge\pi/L\\ n\ge 0}}\frac{Hr_2(m,n)}{2(m^2+n^2)^4}\le CH,$$

and

$$\sup_{\substack{m\geq\pi/L\\ n\geq 0}}\frac{Hr_3(m,n)}{(1-\nu)(m^2+n^2)^2}\leq CH,$$

for some constant $C$, independent of $m$, $n$, and $h$.

In order to show that we can also eliminate $Hr_1(m,n)$ from the numerator of $\tilde{\lambda}_3(h;m,n)$ we observe that for any constant $C$

$$\lim_{h\to 0}\min_{m^2+n^2\leq C}\tilde{\lambda}_3(h;m,n)>0.$$

Hence, if $(m(h),n(h))$ is a minimizer in (3.38), then $m(h)^2+n(h)^2\to\infty$, as $h\to 0$. If $(m^*(h),n^*(h))$ denotes a minimizer in (3.40), then formulas (3.41) and (3.42) imply that $m^*(h)^2+n^*(h)^2\to\infty$, as $h\to 0$, and thus

$$\lim_{h\to 0}\frac{Hr_1(m(h),n(h))}{m(h)^2(1-\nu^2)+H(m(h)^2+n(h)^2)^4}=\lim_{h\to 0}\frac{Hr_1(m^*(h),n^*(h))}{m^*(h)^2(1-\nu^2)+H(m^*(h)^2+n^*(h)^2)^4}=0.$$

Therefore,

$$\frac{\tilde{\lambda}_3(h;m(h),n(h))}{\lambda_3^*(h;m(h),n(h))}\leq\frac{\tilde{\lambda}_3(h)}{\lambda_3^*(h)}\leq\frac{\tilde{\lambda}_3(h;m^*(h),n^*(h))}{\lambda_3^*(h;m^*(h),n^*(h))},$$

and (3.43) follows. □

We have now achieved our goal, since (3.33) follows from (3.36) and Lemma 3.7.

## 3.5 Buckling modes

In this section we return to the original boundary conditions and the space $V_h$, defined in (2.10). Let

$$\lambda_1(h)=\inf_{\boldsymbol{\phi}\in\mathcal{A}_h}\mathfrak{R}_1(h,\boldsymbol{\phi}). \tag{3.44}$$

Even though, technically speaking, $V_h$ is not a subspace of $\tilde{V}_h$, it is helpful to think of it as such. Hence, our next lemma is natural (but not entirely obvious).

LEMMA **3.8.** *Let $\lambda_1(h)$ and $\tilde{\lambda}_1(h)$ be given by (3.44) and (3.6), respectively, then*

$$\lambda_1(h)\geq\tilde{\lambda}_1(h). \tag{3.45}$$

*Proof.* In view of Theorem 3.1 it is sufficient to prove the inequality

$$\lambda_1(h)\geq\inf_{\substack{m\geq 1\\ n\geq 0}}\tilde{\lambda}_{m,n}(h).$$

This is done by repeating the arguments in the proof of the analogous inequality in Theorem 3.1. The argument is based on the fact the $2L$-periodic extension of $\boldsymbol{\phi}\in\mathcal{A}_h\subset V_h$, such that $\phi_r$ and $\phi_\theta$ are even and $\phi_z$ is odd, is still of class $H^1$, and the expansion (3.10) is valid. The inequality (3.45) follows from (3.7) and the inequality (3.11), which is valid for each single Fourier mode. □

In order to prove that the asymptotic formula (3.1) holds for $\lambda_1(h)$ (and hence for $\lambda_{\text{crit}}(h)$) it is sufficient to find a test function $\boldsymbol{\phi}^h \in \mathcal{A}_h$ such that

$$\lim_{h\to 0} \frac{\mathfrak{R}_1(h,\boldsymbol{\phi}^h)}{\widetilde{\lambda}_1(h)} = 1. \tag{3.46}$$

Indeed,

$$1 = \lim_{h\to 0} \frac{\mathfrak{R}_1(h,\boldsymbol{\phi}^h)}{\widetilde{\lambda}_1(h)} \geq \overline{\lim_{h\to 0}} \frac{\lambda_1(h)}{\widetilde{\lambda}_1(h)} \geq 1,$$

which proves both that

$$\lim_{h\to 0} \frac{\lambda_1(h)}{\lambda^*(h)} = 1,$$

and that $\boldsymbol{\phi}^h \in \mathcal{A}_h$ is a buckling mode.

We construct the buckling mode $\boldsymbol{\phi}^h \in V_h$ as a 2-term Fourier expansion (3.2). For this purpose we choose $m = m(h) \to \infty$, as $h \to 0$, and $n = n(h)$ to lie on Koiter's circle and

$$\begin{cases} \phi_r^h(r,\theta,z) = \displaystyle\sum_{m\in\{m(h),m(h)+2\}} f_r(r,m,n(h))\cos(n(h)\theta)\cos(\hat{m}z), \\ \phi_\theta^h(r,\theta,z) = \displaystyle\sum_{m\in\{m(h),m(h)+2\}} f_\theta(r,m,n(h))\sin(n(h)\theta)\cos(\hat{m}z), \\ \phi_z^h(r,\theta,z) = \displaystyle\sum_{m\in\{m(h),m(h)+2\}} f_z(r,m,n(h))\cos(n(h)\theta)\sin(\hat{m}z), \end{cases} \tag{3.47}$$

where now, in order to avoid confusion, we distinguish between $m \in \mathbb{Z}$ and

$$\hat{m} = \frac{\pi m}{L}.$$

To ensure that $\boldsymbol{\phi}^h \in V_h$ we require

$$f_\theta(r,m(h)+2,n(h)) = -f_\theta(r,m(h),n(h)).$$

The structure of coefficients $\boldsymbol{f}(r,m,n)$ is determined by optimality at each of the two values of $m$ separately, since the expansion (3.10) is valid for $\boldsymbol{\phi} \in V_h$. In particular, we choose

$$f_\theta(r,m(h),n(h)) = ra_\theta(h) + (r-1)n(h), \qquad a_\theta(h) = -\left.\frac{n(n^2+(\nu+2)\hat{m}^2)}{(\hat{m}^2+n^2)^2}\right|_{\substack{m=m(h)\\ n=n(h)}}.$$

Let

$$F_z(r,m,n,h) = a_z(m,n,h) + (r-1)\hat{m}, \qquad a_z(m,n,h) = -\frac{\hat{m}(2\nu+(\nu+1)na_\theta(h))}{2\hat{m}^2+(1-\nu)n^2},$$

$$F_r(r,m,n,h) = 1 - \frac{\nu(r-1)}{1-\nu}(na_\theta(h)+1+\hat{m}a_z(m,n,h)) - \frac{\nu(r-1)^2}{2(1-\nu)}(na_\theta(h)+n^2+\hat{m}^2).$$

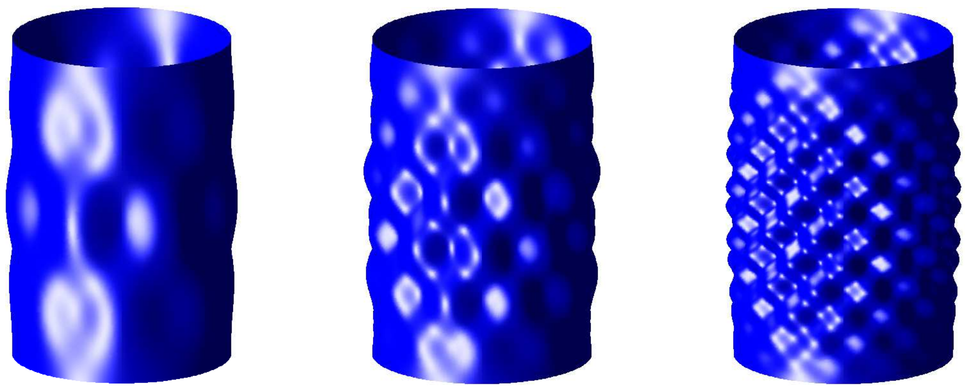

Figure 1: Koiter circle buckling modes corresponding, left to right, to $m(h) \sim h^{-1/8}$, $h^{-1/4}$ and $h^{-3/8}$. Poisson's ratio $\nu = 1/3$.

Then

$$f_r(r, m(h), n(h)) = F_r(r, m(h), n(h), h), \qquad f_r(r, m(h)+2, n(h)) = -F_r(r, m(h)+2, n(h), h),$$

$$f_z(r, m(h), n(h)) = F_z(r, m(h), n(h), h), \qquad f_z(r, m(h)+2, n(h)) = -F_z(r, m(h)+2, n(h), h).$$

Maple calculation verifies that the test function, $\boldsymbol{\phi}^h$ satisfies (3.46). Figure 1 shows buckling modes for

$$\hat{m}(h) = \left(\sqrt{\frac{2}{\lambda^*(h)}}\right)^{\alpha}, \quad \alpha = 1/4,\ 1, 2,\ 3/4.$$

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