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# Analytical Solution of BVPs for Fourth-order Integro-differential Equations by Using Homotopy Analysis Method

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**Abstract:**An analytic technique, the homotopy analysis method (HAM), is applied to obtain the approximate analytical solutions of fourth-order integro-differential equations. The homotopy analysis method (HAM) is one of the most effective method to obtain the exact and approximate solution and provides us with a new way to obtain series solutions of such problems. HAM contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of series solution. It is shown that the solutions obtained by the Adomian decomposition method (ADM) and the homotopy-perturbation method (HPM) are only special cases of the HAM solutions. we have shown that fourth-order boundary value problems can be transformed into a system of differential equations and integro-differential equation, which can be solved by using homotopy analysis method. Several examples are given to illustrate the efficiency and implementation of the method.

**Keywords:**decomposition method; integro-differential equations; homotopy analysis method; homotopy perturbation method

## 1 Introduction

Mathematical modeling of real-life, physics and engineering problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Therefore, they have been of great interest by several authors. The boundary value problems for higher-order integro-differential equations have been investigated by Morchalo[6, 7] and Agarwal[11] among others. Agarwal[11] discussed the existence and uniqueness of the solutions for these problems. In [11], no numerical method was presented.

The present work is motivated by the desire to obtain analytical and numerical solutions to boundary value problems for higher-order integro-differential equations. In 1992, Liao[12] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the Homotopy analysis method (HAM),[3, 4, 12–14]. In recent years, homotopy analysis method has been used in obtaining approximate solutions of a wide class of differential, integral and integro-differential equations. The method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be used directly without using assumptions or transformations. In this work, we aim to implement this reliable technique to integro-differential equations with two point boundary conditions. The boundary conditions will be imposed on various approximants of the obtained series solution to complete the determination of the remaining constants.

The general higher-order integro-differential equation as

$$y^{(m)}(x) = f(x) + \int_0^x K(x,t)F(y(t))dt, \tag{1}$$

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with the boundary conditions

$$\begin{aligned} y^{(j)}(0) &= A_j, \qquad j = 0,1,2,3,\cdots,(r-1), \\ y^{(j)}(b) &= C_j, \qquad j = r,(r+1),(r+2),\cdots,(m-1). \end{aligned}$$

where $y^m(x)$ indicates the $m$th derivative of $y(x)$ and $F(y(x))$ is a nonlinear function. In addition the kernel $k(x,t)$ and $f(x)$ are assumed real, differential for $x \in [0,b]$ and $A_j, 0 \le j \le (r-1)$,$C_j, r \le j \le (m-1)$,are real finite constants.

# 2 The Body of the Article

## 2.1 Basic idea of HAM

We consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{2}$$

where $\mathcal{N}$ is a nonlinear operator, $\tau$ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [14] construct the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\,\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{3}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$ and $\phi(\tau;p)$ is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\phi(\tau;0) = u_0(\tau), \qquad \phi(\tau;1) = u(\tau), \tag{4}$$

respectively. Thus, as $p$ increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to $p$, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{5}$$

where

$$u_m(\tau) = \left[\frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\right]_{p=0}. \tag{6}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p=1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{7}$$

which must be one of solutions of original nonlinear equation, as proved by[14]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\,\mathcal{N}[\phi(\tau;p)] = 0, \tag{8}$$

which is used mostly in the homotopy perturbation method[5], where as the solution obtained directly, without using Taylor series [8]. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\overrightarrow{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}.$$

Differentiating equation (2) $m$ times with respect to the embedding parameter $p$ and then setting $p=0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\,\mathcal{H}(\tau)\mathcal{R}_m(\overrightarrow{u}_{m-1}), \tag{9}$$

where

$$\mathcal{R}_m(\overrightarrow{u}_{m-1}) = \left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}\right]_{p=0}. \tag{10}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{11}$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [14]. If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

## 2.2 Applications

In order to assess the advantages and the accuracy of homotopy analysis method for solving Hight-Order Integro-Differential Equations, we will consider the following two example.

**Example 1.** Consider the linear boundary value problem for the fourth-order integro differential equation

$$y^{(i\nu)}(x) = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt, \qquad 0 < x < 1$$

with the boundary conditions

$$y(0) = 1, \qquad y'(0) = 1, \qquad y(1) = 1 + e, \qquad y'(1) = 2e.$$

The above boundary value problem can be transformed as

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{df}{dx} = z(x), & \frac{dz}{dx} = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt. \end{cases} \tag{12}$$

with the boundary conditions

$$y(0) = 1, \qquad q(0) = 1, \qquad f(0) = A, \qquad z(0) = B. \tag{13}$$

The exact solution of the above boundary value problem is $y(x) = 1 + xe^x$[1].

To solve the Eqs. (12) and (13) by means of homotopy analysis method, we choose the linear oprators

$$\mathcal{L}_i[\phi_i(x;p)] = \frac{\partial \phi_i(x;p)}{\partial x}, \quad i = 1, 2, 3, 4. \tag{14}$$

The inverse operator $\mathcal{L}_i^{-1}$ is given by

$$\mathcal{L}_i^{-1}(\cdot) = \int_0^x (\cdot)ds, \qquad i = 1, 2, 3, 4. \tag{15}$$

We now define a nonlinear operators as

$$\begin{aligned}
\mathcal{N}_1[\phi_1, \phi_2] &= \frac{\partial \phi_1(x;p)}{\partial x} - \phi_2(x;p), \quad \mathcal{N}_2[\phi_2, \phi_3] = \frac{\partial \phi_2(x;p)}{\partial x} - \phi_3(x;p), \\
\mathcal{N}_3[\phi_3, \phi_4] &= \frac{\partial \phi_3(x;p)}{\partial x} - \phi_4(x;p), \\
\mathcal{N}_4[\phi_4, \phi_1] &= \frac{\partial \phi_4(x;p)}{\partial x} - x(1+e^x) - 3e^x - \phi_1(x;p) + \int_0^x \phi_1(t;p)dt.
\end{aligned} \tag{16}$$

Using above definition, we construct the zeroth-order deformation equations

$$\begin{aligned}
(1-p)\mathcal{L}_1[\phi_1(x;p) - y_0(x)] &= p\, \hbar_1 \mathcal{H}_1(x)\mathcal{N}_1[\phi_1, \phi_2], \\
(1-p)\mathcal{L}_2[\phi_2(x;p) - q_0(x)] &= p\, \hbar_2 \mathcal{H}_2(x)\mathcal{N}_2[\phi_2, \phi_3], \\
(1-p)\mathcal{L}_3[\phi_3(x;p) - f_0(x)] &= p\, \hbar_3 \mathcal{H}_3(x)\mathcal{N}_3[\phi_3, \phi_4], \\
(1-p)\mathcal{L}_4[\phi_4(x;p) - z_0(x)] &= p\, \hbar_4 \mathcal{H}_4(x)\mathcal{N}_4[\phi_4, \phi_1].
\end{aligned} \tag{17}$$

Thus, we obtain the $m$th-order $(m \geq 1)$deformation equations

$$\begin{aligned}
&\mathcal{L}_1[y_m(x) - \chi_m y_{m-1}(x)] = \hbar_1 \mathcal{H}_1(x)\mathcal{R}_{1,m}(\vec{y}_{m-1}),\\
&\mathcal{L}_2[q_m(x) - \chi_m q_{m-1}(x)] = \hbar_2 \mathcal{H}_2(x)\mathcal{R}_{2,m}(\vec{q}_{m-1})\\
&\mathcal{L}_3[f_m(x) - \chi_m f_{m-1}(x)] = \hbar_3 \mathcal{H}_3(x)\mathcal{R}_{3,m}(\vec{f}_{m-1}),\\
&\mathcal{L}_4[z_m(x) - \chi_m z_{m-1}(x)] = \hbar_4 \mathcal{H}_4(x)\mathcal{R}_{4,m}(\vec{z}_{m-1}),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\mathcal{R}_{1,m}(\vec{y}_{m-1}) &= \frac{\partial y_{m-1}}{\partial x} - q_{m-1},\\
\mathcal{R}_{2,m}(\vec{q}_{m-1}) &= \frac{\partial q_{m-1}}{\partial x} - f_{m-1},\\
\mathcal{R}_{3,m}(\vec{f}_{m-1}) &= \frac{\partial f_{m-1}}{\partial x} - z_{m-1},\\
\mathcal{R}_{4,m}(\vec{z}_{m-1}) &= \frac{\partial z_{m-1}}{\partial x} - (1-\chi_m)(x(1+e^x) + 3e^x) + \int_0^x y_{m-1} dt
\end{aligned}$$

Now the solution of the $m$th-order$(m \geq 1)$ deformation equations(18)

$$\begin{aligned}
y_m(x) &= \chi_m y_{m-1}(x) + \hbar_1 \int_0^x [\mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{y}_{m-1})]dt,\\
q_m(x) &= \chi_m q_{m-1}(x) + \hbar_2 \int_0^x [\mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{q}_{m-1})]dt,\\
f_m(x) &= \chi_m f_{m-1}(x) + \hbar_3 \int_0^x [\mathcal{H}_3(t)\mathcal{R}_{3,m}(\vec{f}_{m-1})]dt,\\
z_m(x) &= \chi_m z_{m-1}(x) + \hbar_4 \int_0^x [\mathcal{H}_4(t)\mathcal{R}_{4,m}(\vec{z}_{m-1})]dt.
\end{aligned}$$

By start with an initial approximation $y_0(x) = 1$, $q_0(x) = 1$,$f_0(x) = A$, $z_0(x) = B$ and by choose $h_i = -1$ and $\mathcal{H}_i = 1$, $(i = 1, 2, 3, 4)$ we can obtain directly the other components as

$$\begin{cases}
y_1(x) = x,\\
q_1(x) = Ax,\\
f_1(x) = Bx,\\
z_1(x) = x + 2e^x + xe^x - 2.
\end{cases}$$

$$\begin{cases}
y_2(x) = \frac{1}{2}Ax^2,\\
q_2(x) = \frac{1}{2}Bx^2,\\
f_2(x) = \frac{1}{2}x^2 + xe^x - e^x - 2x - 1,\\
z_2(x) = \frac{1}{2}x^2 - \frac{1}{6}x^3.
\end{cases}$$

$$\begin{cases}
y_3(x) = \frac{1}{6}Bx^3,\\
q_3(x) = \frac{1}{6}x(x^2 + 6e^x - 6x - 6),\\
f_3(x) = \frac{1}{6}x^3 - \frac{1}{24}x^4,\\
z_3(x) = -\frac{1}{24}Ax^4 + \frac{1}{6}Ax^3.
\end{cases}$$

$$\vdots$$

In view of the above components, the series solution as

$$y(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{24}x^4 + xe^x - e^x - \frac{1}{6}x^3 - \frac{1}{2}x^2 + 1 + \cdots \tag{19}$$

The approximants must satisfy the boundary conditions. Thus by imposing the boundary conditions at $x = 1$ and solving the obtained systems gives the following sequences

$$S_A = \{1.9565, 1.9815, 1.9949, 1.9990, \cdots\}$$
$$S_B = \{3.1903, 3.0736, 3.0190, 3.0037, \cdots\}$$

for approximations of $A$ and $B$, respectively. It is easily seen that the increasing sequence $S_A$ leads to

$$\lim_{n\to\infty} A = 2. \tag{20}$$

While the decreasing sequence $S_B$ leads to

$$\lim_{n\to\infty} B = 3. \tag{21}$$

which is exactly the same as obtained in[1] by modified Adomians decomposition method.
Substituting (20) and (21) into (19) and by taking the truncated Taylor expansions for the exponential term in (19): e.g. $e^x \approx 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4$, $e^x \approx 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5$ and $e^x \approx 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6$ yields the series solution for different value of $m$ as follows

$$y(x) = \sum_{k=0}^{5} y_k(x) = 1 + x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{180}x^5 + \frac{1}{840}x^6 - \frac{1}{5760}x^7 + \cdots\right).$$
$$y(x) = \sum_{k=0}^{6} y_k(x) = 1 + x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{1260}x^6 + \frac{1}{5760}x^7 + \cdots\right).$$
$$y(x) = \sum_{k=0}^{7} y_k(x) = 1 + x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{10080}x^7 + \cdots\right).$$

According the above equations, one can conclude that when $m \to +\infty$

$$y(x) = 1 + xe^x \tag{22}$$

**Example 2.** Consider the nonlinear boundary value problem for the fourth-order integro differential equation

$$y^{(i\nu)}(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \qquad 0 < x < 1$$

with the boundary conditions

$$y(0) = 1, \qquad y'(0) = 1, \qquad y(1) = e, \qquad y'(1) = e.$$

The above boundary value problem can be transformed as

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{df}{dx} = z(x), & \frac{dz}{dx} = 1 + \int_0^x e^{-t}y^2(t)dt. \end{cases} \tag{23}$$

with the boundary conditions

$$y(0) = 1, \qquad q(0) = 1, \qquad f(0) = A, \qquad z(0) = B. \tag{24}$$

The exact solution of the above boundary value problem is $y(x) = e^x$[2].
To solve the Eqs. (23) and (24) by means of homotopy analysis method, we choose the linear oprators

$$\mathcal{L}_i[\phi_i(x;p)] = \frac{\partial \phi_i(x;p)}{\partial x}, \quad i = 1, 2, 3, 4. \tag{25}$$

The inverse operator $\mathcal{L}_i^{-1}$ is given by

$$\mathcal{L}_i^{-1}(\cdot) = \int_0^x (\cdot)ds, \qquad i = 1,2,3,4. \tag{26}$$

We now define a nonlinear operators as

$$\begin{aligned}
\mathcal{N}_1[\phi_1,\phi_2] &= \frac{\partial\phi_1(x;p)}{\partial x} - \phi_2(x;p), \quad \mathcal{N}_2[\phi_2,\phi_3] = \frac{\partial\phi_2(x;p)}{\partial x} - \phi_3(x;p),\\
\mathcal{N}_3[\phi_3,\phi_4] &= \frac{\partial\phi_3(x;p)}{\partial x} - \phi_4(x;p),\\
\mathcal{N}_4[\phi_4,\phi_1] &= \frac{\partial\phi_4(x;p)}{\partial x} - 1 - \int_0^x e^{-t}\phi_1^2(t;p)dt.
\end{aligned} \tag{27}$$

Using above definition, we construct the zeroth-order deformation equations

$$\begin{aligned}
(1-p)\mathcal{L}_1[\phi_1(x;p) - y_0(x)] = p\,\hbar_1\mathcal{H}_1(x)\mathcal{N}_1[\phi_1,\phi_2],\\
(1-p)\mathcal{L}_2[\phi_2(x;p) - q_0(x)] = p\,\hbar_2\mathcal{H}_2(x)\mathcal{N}_2[\phi_2,\phi_3],\\
(1-p)\mathcal{L}_3[\phi_3(x;p) - f_0(x)] = p\,\hbar_3\mathcal{H}_3(x)\mathcal{N}_3[\phi_3,\phi_4],\\
(1-p)\mathcal{L}_4[\phi_4(x;p) - z_0(x)] = p\,\hbar_4\mathcal{H}_4(x)\mathcal{N}_4[\phi_4,\phi_1].
\end{aligned} \tag{28}$$

Thus, we obtain the $m$th-order $(m \geq 1)$deformation equations

$$\begin{aligned}
\mathcal{L}_1[y_m(x) - \chi_m y_{m-1}(x)] = \hbar_1\mathcal{H}_1(x)\mathcal{R}_{1,m}(\vec{y}_{m-1}),\\
\mathcal{L}_2[q_m(x) - \chi_m q_{m-1}(x)] = \hbar_2\mathcal{H}_2(x)\mathcal{R}_{2,m}(\vec{q}_{m-1})\\
\mathcal{L}_3[f_m(x) - \chi_m f_{m-1}(x)] = \hbar_3\mathcal{H}_3(x)\mathcal{R}_{3,m}(\vec{f}_{m-1}),\\
\mathcal{L}_4[z_m(x) - \chi_m z_{m-1}(x)] = \hbar_4\mathcal{H}_4(x)\mathcal{R}_{4,m}(\vec{z}_{m-1}),
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\mathcal{R}_{1,m}(\vec{y}_{m-1}) &= \frac{\partial y_{m-1}}{\partial x} - q_{m-1},\\
\mathcal{R}_{2,m}(\vec{q}_{m-1}) &= \frac{\partial q_{m-1}}{\partial x} - f_{m-1},\\
\mathcal{R}_{3,m}(\vec{f}_{m-1}) &= \frac{\partial f_{m-1}}{\partial x} - z_{m-1},\\
\mathcal{R}_{4,m}(\vec{z}_{m-1}) &= \frac{\partial z_{m-1}}{\partial x} - (1-\chi_m)(1) - \int_0^x e^{-t}\left(\sum_{i=0}^{m-1} y_i y_{m-1-i}\right)dt
\end{aligned}$$

Now the solution of the $m$th-order$(m \geq 1)$ deformation equations(29)

$$\begin{aligned}
y_m(x) &= \chi_m y_{m-1}(x) + \hbar_1\int_0^x [\mathcal{H}_1(t)\mathcal{R}_{1,m}(\vec{y}_{m-1})]dt,\\
q_m(x) &= \chi_m q_{m-1}(x) + \hbar_2\int_0^x [\mathcal{H}_2(t)\mathcal{R}_{2,m}(\vec{q}_{m-1})]dt,\\
f_m(x) &= \chi_m f_{m-1}(x) + \hbar_3\int_0^x [\mathcal{H}_3(t)\mathcal{R}_{3,m}(\vec{f}_{m-1})]dt,\\
z_m(x) &= \chi_m z_{m-1}(x) + \hbar_4\int_0^x [\mathcal{H}_4(t)\mathcal{R}_{4,m}(\vec{z}_{m-1})]dt.
\end{aligned}$$

By start with an initial approximation $y_0(x) = 1$, $q_0(x) = 1$,$f_0(x) = A$, $z_0(x) = B$ and by choose $h_i = -1$ and

$\mathcal{H}_i = 1$, $(i = 1, 2, 3, 4)$ we can obtain directly the other components as

$$\begin{cases} y_1(x) = x, \\ f_1(x) = Bx, \\ q_1(x) = Ax, \\ z_1(x) = 2x + e^{-x} - 1. \end{cases}$$

$$\begin{cases} y_2(x) = \frac{1}{2}Ax^2, \\ f_2(x) = x^2 - e^{-x} - x + 1, \\ q_2(x) = \frac{1}{2}Bx^2, \\ z_2(x) = x^2 - 6e^{-x} - 4x + 6 - 2xe^{-x}. \end{cases}$$

$$\begin{cases} y_3(x) = \frac{1}{6}Bx^3, \\ q_3(x) = \frac{1}{3}x^3 + e^{-x} - \frac{1}{2}x^2 + x - 1 \\ f_3(x) = x^2 - 6e^{-x} - 4x + 6 - 2xe^{-x}, \\ z_3(x) = 2x + 6e^{-x} - 6 + 4xe^{-x} + Ax^2e^{-x} + 4Axe^{-x} + 6Ae^{-x} + x^2e^{-x} + 2Ax - 6A \end{cases}$$

$$\vdots$$

In view of the above components, the series solution as

$$y(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{12}x^4 - e^{-x} - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + 1 + \cdots \tag{30}$$

The approximants must satisfy the boundary conditions. Thus by imposing the boundary conditions at $x = 1$ and solving the obtained systems gives the following sequences

$$S_A = \{0.9828, 0.9944, 0.9984, 0.9996, \cdots\}$$
$$S_B = \{1.0685, 1.0211, 1.0057, 1.0013, \cdots\}$$

for approximations of $A$ and $B$, respectively. It is easily seen that the increasing sequence $S_A$ leads to

$$\lim_{n\to\infty} A = 1. \tag{31}$$

While the decreasing sequence $S_B$ leads to

$$\lim_{n\to\infty} B = 1. \tag{32}$$

which is exactly the same as obtained in[2] by using homotopy perturbationmethod.

Substituting (31) and (32) into (30) and by taking the truncated Taylor expansions for the exponential term in (30): e.g. $e^{-x} \approx 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$, $e^{-x} \approx 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$ and $e^{-x} \approx 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6$ yields the series solution for different value of $m$ as follows

$$y(x) = \sum_{k=0}^{5} y_k(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \frac{1}{1680}x^7 + \frac{1}{2520}x^8 + \cdots .$$

$$y(x) = \sum_{k=0}^{6} y_k(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{630}x^8 + \frac{1}{2520}x^9 \cdots .$$

$$\begin{aligned} y(x) = \sum_{k=0}^{7} y_k(x) = \; & 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 - \frac{61}{60480}x^9 \\ & - \frac{17}{60480}x^{10} - \frac{1}{30240}x^{11} \cdots . \end{aligned}$$

According the above equations, one can conclude that when $m \to +\infty$

$$y(x) = e^x \tag{33}$$

# 3 Conclusion

In this paper, we have used the homotopy analysis method for finding the analytic solution of linear and nonlinear hight-order integro-differential equations. The algorithm produced results which are of reasonable accuracy. The method needs much less computational work compared with traditional methods. It is shown that HAM is a very fast convergent, precise and cost efficient tool for solving integro-differential equations in the bounded domains.

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