Applied Mathematics and Physics, 2017, Vol. 5, No. 1, 1-10
Available online at http://pubs.sciepub.com/amp/5/1/1

DOI:10.12691/amp-5-1-1

The Existence and Stability of Inclusion Equations Type of Stochastic Dynamical System Driven by Mixed Fractional Brownian Motion in a Real Separable Hilbert Space

Salah H Abid, Sameer Q Hasan*, Zainab A Khudhur

Department of Mathematics, College of Education Almustansryah University
*Corresponding author: dr.sameer_kasim @yahoo.com

Abstract In this paper we presented The existence and stability of inclusion equations type of stochastic dynamical system driven by mixed fractional Brownian motion in a real separable Hilbert space with an illustrative example.

Keywords: *stochastic dynamical system, mixed fractional Brownian motion, mixed-stochastic mild solution, fractional partial differential equations, Asymptotic Stability, real separable Hilbert space*

Cite This Article: Salah H Abid, Sameer Q Hasan, and Zainab A Khudhur, "The Existence and Stability of Inclusion Equations Type of Stochastic Dynamical System Driven by Mixed Fractional Brownian Motion in a Real Separable Hilbert Space." *Applied Mathematics and Physics*, vol. 5, no. 1 (2017): 1-10. doi: 10.12691/amp-5-1-1.

1. Introduction

The theory of integro-differential equations or inclusions has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology and so on, On can see [1,4,14] and references therein. Several authors have established the existence results of mild solutions for these equations (see [3,7,9,11,14] and references therein). In addition, the nonlinear integro-differential equations with resolvent operators serve as an abstract formulation of partial integro-differential equations that arise in many physical phenomena. One can see [16] and references therein. The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic problems. As the generalization of classic impulsive integro-differential equations or inclusions, impulsive neutral stochastic functional integro-differential equations or inclusions have attracted the researchers great interest. And some works have done on the existence results of mild solutions for these equations (see [12,17] and references therein). To the best of our knowledge, there is no work reported on the existence of mild solutions for the impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions and resolvent operators, and the aim of this paper is to close the gap. In this paper, motivated by the previously mentioned papers, we will study this interesting problem. Sufficient conditions for the existence are given by means of the fixed point theorem for multi-valued mapping due to Dhage [6] and the fractional power of operators. Especially, the known results appeared in [2,8,10,12,15] and [5,6,11,16] are generalized to the stochastic settings. An example is provided to illustrate the theory.

2. Preliminaries

For more details on this section, We refer the reader to Da prato and Zabczyk [13] throughout the paper $(H,\|.\|_H)$ and $(K,\|.\|_K)$ denote two real separable Hilbert spaces. In case without confusion, we just use $\langle .,. \rangle$ for the inner product and $\|.\|$ for the norm.

Let $(\Omega,\mathcal{F},p;F)$ $(F=\{\mathcal{F}(t)\}_{t\geq 0})$ be complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t):\Omega\to H$ and the collection of random variables $S=\{x(t,w):\Omega\to H \setminus t\in J\}$ is called a stochastic process. Generally, we just write *x(t)* instead of $x(t,w)$ and $x(t):J\to H$ in the space of *S*. *Let* $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t):t\geq 0\}$ is a cylindrical K −valued wiener process with a finite trace nuclear covariance operator $Q\geq 0$, denote $Tr(Q)=\sum_{i=1}^{\infty}\lambda_i=\lambda<\infty$, which satisfies that $Qe_i=\lambda_i e_i$. *So,* actually, $w(t)=\sum_{i=1}^{\infty}\sqrt{\lambda_i}\,w_i(t)e_i$, where $\{w_i\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard wiener processes. We assume that $\mathcal{F}_t=\sigma\{w(s):0\leq s\leq t\}$ *is the* σ −algebra generated by w and $\mathcal{F}_T=\mathcal{F}$. Let $\Psi\in L(K,H)$ and define $\|\Psi\|_Q^2=Tr(\Psi Q\Psi^*)=\sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\Psi e_n\right\|^2$.

If $\|\Psi\|_Q<\infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$denote the space of all Q-Hilbert-Schmidt operators $\Psi:K\to H$. *The* completion $L_Q(K,H)$ *of*

$L(K,H)$ with respect to topology induced by the norm $\|.\|_Q$ where $\|\Psi\|_Q^2 = \langle\Psi,\Psi\rangle$ is a Hilbert space with the above norm topology. Let $A: D(A) \to H$ be infinitesimal generator of a compact, analytic resolvent operator $S(t)$, $t \geq 0$. Let $L_2(\Omega, \mathcal{F}_{t,H})$ denote the Hilbert space of all $\mathcal{F}_t -$ measurable square integrable random variables with values in H. Let $L_2^{\mathcal{F}}([0,b],H)$ be the Hilbert space of all square integrable and $\mathcal{F}_t -$ measurable processes with values in H. $\boldsymbol{\beta}$([0,b])=$\{x:[0,b] \to H, x_K \in C(J_K,H)\}$ let $L_2^0([,\Omega,],H)$ denote the family of all $\mathcal{F}_0$ −measurable, β –valued random variables $x(0)$. We use the notations $p_{cl}(H)$ for the family of all subsets of H and denote

$$p_{cl}(H)=\{Y\in p(H): Y \text{ is closed}\},$$
$$p_{cv}(H)=\{Y\in p(H): Y \text{ is convex}\},$$
$$p_{bd}(H)=\{Y\in p(H): Y \text{ is bounded}\},$$
$$p_{cp}(H)=\{Y\in p(H): Y \text{ is compact}\}.$$

In what follows, we briefly introduce some facts on multi-valued analysis. For details, one can see [10]. A multi-valued map $\Gamma: H \to p(H)$ is convex (closed) valued, if $\Gamma(x)$ is convex (closed)for all $x \in H$. $\Gamma(x)$ *is* bounded on bounded sets if $\Gamma(B) = \cup_{x\in B}\Gamma(x)$ is bounded in H, for any bounded set B *of* H, that is, $sup_{x\in B}\, sup\{\|y\| \in \Gamma(x)\} < \infty$. Γ is called upper semi continuous (u.s.c. for short) on H, if for any $x \in H$,the set Γ(x) is a nonempty, closed subset of H, and if for each open set B of H containing Γ(x), there exists an open neighborhood N of x such that Γ(N) ⊆ B. Γ is said to be completely continuous if Γ(B) is relatively compact, for every bounded subset B ⊆ H. If the multi –valued map Γ is completely continuous with nonempty compact values, then Γ is u.s.c. if and only if Γ has a closed graph, i.e., $x_n \to x, y_n \to y, y_n \in \Gamma(x_n)$ *imply* $y \in \Gamma(x)$. Γ has a fixed point if there is $x \in H$ such that $x \in \Gamma(x)$. A multi-valued map $\Gamma: J \to p_{cl}$ is said to be measurable if for each $x \in H$, the mean –square distance between x and $\Gamma(t)$ is measurable.

Definition (1) (following in [5])

Let $(\Omega, F, (F_t)_{t\geq 0}, P)$ be a filtered probability space.

(i) The filtration $(F_t)_{t\geq 0}$ is said to be complete if (Ω, F, P) is a complete and if F_0 contains all the P −null sets.

(ii) The filtration, $(F_t)_{t\geq 0}$ is said to satisfy the usual hypotheses if it is complete and right continuous, that is $F_t = F_t^{\,+}$ where $F_t^{\,+} = \cap_{u>t}.F_u$.

Lemma (1) (following in [1])

Let I be a compact interval and a Hilbert space. Let F be an L^2 −Caratheodory multi-valued map with $N_{F,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(I,H)$ *to* $C(I,H)$. Then, the operator $\Gamma_{\circ} N_F: C(I,H) \to p_{cp,cv}(H), x \mapsto (\Gamma_{\circ} N_F)_{(x)} = \Gamma(N_{F,x})$, is a closed graph operator in $C(I,H) \times C(I,H)$, where $N_{F,x}$ is known as the selectors set from F, is given by $\sigma \in N_{F,x} = \{\sigma \in L^2(L(K,H)): \sigma(t) \in F(t,x)$ for $a.e. t \in J\}$.

Lemma (2)*, (Ito isometry theorem) *(following in [20])

Let $V[0,T]$ be the class of functions such that $f:[0,T]\times\Omega \to R$, f is measurable, $Ft-$ adapted and $E\left[\int_0^T (f(t,\omega))^2 dt\right] \leq \infty$. Then for every $f \in V[0,T]$

$$E\left[\int_0^T f(t,\omega)dB(t)\right]^2 = E\left[\int_0^T (f(t,\omega))^2 dt\right],$$

where B is a wiener process.

Definition (2) (following in [12])

Let H be a constant belonging to (0, 1). A one dimensional fractional Brownian motion $B^H = \{B_{(t)}^H, t \geq 0\}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E\left(B_{(t)}^H B_{(s)}^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

for $t, s \geq 0$.

Remark (1) (following in [12])

Let $B^H = \{B_{(t)}^H, t \geq 0\}$ be a one dimensional fractional Brownian motion then

1) $E\left(B_{(t)}^H - B_{(s)}^H\right)^2 = |t-s|^{2H}$.
2) $R(s,t)$ is a covariance function of fractional Brownian motion $(B_{(t)}^H)_{t\geq 0}$ such that

$$R(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \; for\; t,s \geq 0. \quad (1)$$

3) If $H = \frac{1}{2}$ the covariance function becomes $R(s,t) = \min(s,t)$.

Remark (2) (following in [19])

Let $B^H = \{B_{(t)}^H, t \geq 0\}$ be a one dimensional fractional Brownian motion then for every $t_1 < t_2 < t_3 < t_4$

$$E\left(B_{t_1}^H - B_{t_2}^H\right)\left(B_{t_3}^H - B_{t_4}^H\right) = \frac{1}{2}\begin{bmatrix} |t_2 - t_3|^{2H} + |t_1 - t_4|^{2H} \\ -|t_1 - t_3|^{2H} - |t_2 - t_4|^{2H} \end{bmatrix}$$

1. If $H = \frac{1}{2}$ Then the increments of B^H are non-correlated, and consequently independent. So B^H is a Wiener Process which denoted further by B.
2. If $H \in \left(\frac{1}{2}, 1\right)$ then the increments are positively correlated.
3. If $H \in (0, \frac{1}{2})$ then the increments are negative correlated.

Definition (3) (following in [19])

A stochastic process $X = \{X_{(t)}, t \geq 0\}$ is called b −self similar if $\{X_{(at)}, t \geq 0\}$ and $\{a^b X_{(t)}, t \geq 0\}$ have the same law.

Remark (3) (following in [19])

Fractional Brownian motion B^H of Hurst index H is H −self similar.

In the following sections, we explain the Integration of Deterministic Function with Respect to One Dimensional Fractional Brownian motion

Lemma (3) (following in [12])

The one dimensional fractional Brownian motion $B^H = \{B_{(t)}^H, t \geq 0\}$ has the integral representation

$$B^H_{(t)} = \int_0^t K_H(t,s)dB_{(s)} \quad (2)$$

here, B is a wiener process and the kernel $K_H(t,s)$ defined as

$$K_H(t,s) = cHs^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \quad (3)$$

$$\frac{\partial K}{\partial t}(t,s) = cH\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}} \quad (4)$$

$cH = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$, $t > s$ and β is a beta function.

Definition (4) (following in [19])

Let the general indicator function be given by

$$1_{[a,b]}(t) = \begin{cases} 1, & a \le t < b \\ -1, & b \le t < a \\ 0, & \text{otherwise} \end{cases}.$$

The function f is said to be step function, if there exist a finite number of points $t_k \in R$, $0 \le k \le n-1$, and $a_k \in R$, $1 \le k \le n$, such that $f(t) = \sum_{k=1}^n a_k\, 1_{[t_{k-1},t_k]}(t)$.

Now, We denote by ꜫ the set of step functions on $[0,T]$. If $\Phi \in$ ꜫ then by defined above we can write it by the form $(t) = \sum_{k=1}^n a_k\, 1_{[t_k,t_{k+1}]}(t)$, where $t \in [0,T]$.

The integral of a step function $\Phi \in$ ꜫ with respect to one dimensional fractional Brownian motion is defined $\int_0^T \Phi(t)dB_t^H = \sum_{k=1}^n a_k(B^H_{t_{k+1}} - B^H_{t_k})$, where $a_k \in R$, $0 = t_1 < t_2 < \ldots\ldots < t_{n+1} = T$. $cH = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$, $t > s$ and β is a beta function.

Definition (5) (following in [19])

Let the general indicator function be given by

$$1_{[a,b]}(t) = \begin{cases} 1, & a \le t < b \\ -1, & b \le t < a \\ 0, & \text{otherwise} \end{cases}.$$

The function f is said to be step function, if there exist a finite number of points $t_k \in R$, $0 \le k \le n-1$, and $a_k \in R$, $1 \le k \le n$, such that $f(t) = \sum_{k=1}^n a_k\, 1_{[t_{k-1},t_k]}(t)$.

Now, We denote by ꜫ the set of step functions on $[0,T]$. If $\Phi \in$ ꜫ then by defined above we can write it by the form $\Phi(t) = \sum_{k=1}^n a_k\, 1_{[t_k,t_{k+1}]}(t)$, where $t \in [0,T]$.

The integral of a step function $\Phi \in$ ꜫ with respect to one dimensional fractional Brownian motion is defined $\int_0^T \Phi(t)dB_t^H = \sum_{k=1}^n a_k(B^H_{t_{k+1}} - B^H_{t_k})$, where $a_k \in$ R, $0 = t_1 < t_2 < \ldots\ldots < t_{n+1} = T$.

Let H be the Hilbert space defined as the closure of ꜫ with respect to the scalar product $< 1_{[0,t]}, 1_{[0,s]} > = RH(t,s) = E(B_t^H B_s^H)$ [Nualart, 38].

The mapping $1_{[0,t]} \to \{BH(t), t \in [0,T]\}$ can be extended to an isometry between H and span$^{L^2(\Omega)}$ $\{BH(t), t \in [0,T]\}$. i.e. the mapping $H \to L^2$ (Ω, F, P), $\Phi \to \int_0^T \Phi(t)dB_t^H$ is isometry [22].

Remark (4) (following in [21])

- If $H = \frac{1}{2}$ and $H = L^2([0,T])$ then by using (Ito isometry theorem), we have

$$E\left(\int_0^T \Phi(t)dB\right)^2 = \int_0^T (\Phi(t))^2 dt \quad (5)$$

- If $H > \frac{1}{2}$ from the equation (2.30), we have

$$R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), t,s \ge 0$$

$$\frac{\partial R_H}{dt} = H\left(|t|^{2H-1} - |t-s|^{2H-1}\right) \quad (6)$$

$$\partial R_H^2 = H(2H-1)|t-s|^{2H-2}\, dsdt \quad (7)$$

Lemma (4) (following in [11])

For any functions $\Phi, \varphi \in L^2[0,T] \cap L^1[0,T]$, then

(i) $$E\left(\int_0^T \Phi(t)dB^H_{(t)} \int_0^T \varphi(s)dB^H_{(s)}\right) = H(2H-1)\int_0^T . \int_0^T \Phi(t)\varphi(s)|t-s|^{2H-2}dsdt$$

(ii) $$E\left(dB^H_{(t)}dB^H_{(s)}\right) = \frac{\partial R_H^2}{dsdt} = H(2H-1)|t-s|^{2H-2}\, dsdt \quad (8)$$

From this Lemma above, we obtain

$$E\left(\int_0^T \Phi(t)dB^H_{(t)}\right)^2 = H(2H-1)\int_0^T . \int_0^T \Phi(s)\Phi(t)|t-s|^{2H-2}dsdt. \quad (9)$$

Remark (5) (following in [11])

The space H contains the set of functions $\Phi \in L^2[0,T] \cap L^1[0,T]$, such that $\int_0^T . \int_0^T \Phi(s)\Phi(t)\,|\,t-s\,|^{2H-2}\, dsdt < \infty$, which includes $L^{\frac{1}{H}}([0,T])$.

Now, let Ĥ be the Banach space of all measurable functions on $[0,T]$ such that

$$\Phi^2_{\hat{H}} = H(2H-1)\int_0^T . \int_0^T \Phi(s)\Phi(t)|\,t-s|^{2H-2}dsdt < \infty. \quad (10)$$

Lemma (5) (following in [18])

Let Ĥ be the Banach space of all measurable functions on $[0,T]$ and H be the Hilbert space defined as the closure of the set of step functions on $[0,T]$. If $\frac{1}{2} < H < 1$ then. $L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset \hat{H} \subset H$.

As for regard to Integration of Deterministic Function with Respect to Infinite Dimensional Fractional Brownian motion, Let X and Y be two real separable Hilbert spaces and Let $L(X,Y)$ be the space of all bounded linear operators from X to Y.

Definition (6) (following in [22])

A process $X = \{X_t, t \geq 0\}$ with values in separable Hilbert space Y is called Gaussian if, for every $t_1, t_2, t_4, \ldots\ldots, t_n \in [0, T]$ and $y_1, y_2, y_3, \ldots\ldots, y_n \in Y$ the real random variable $\sum_{i=1}^{n} \langle y_i, X_{t_i} \rangle_Y$ has a normal distribution.

Definition (7) (following in [22])

The $Y-$valued process $(W_t^H)_{t\in[0,T]}$ is said to be an infinite–dimensional fractional Brownian motion or ($Q-$fractional Brownian motion) if W^H is a centered Gaussian process with covariance $COV\ (W_t^H, W_s^H) = R(t,s)Q$, where Q the covariance operator.

Lemma (6) (following in [22])

W^H is a $Q-$fractional Brownian motion if and only if there exists a sequence $(B_n^H)_{n\geq 1}$ of real and independent fractional Brownian motion such that

$$W_{(t)}^H = W_{Q(t)}^H = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n B_{n(t)}^H,$$

where the series converges in $L^2(\Omega; Y)$ and $\{e_n\}_{n=1}^{\infty}$ the orthonormal system in Y.

Rmark (6) (following in [22])

Suppose that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in Y. Let $Q \in L(Y,Y)$ be the operator defined by $Qe_n = \lambda_n e_n$, where $\lambda_n \geq 0\ (n = 1,2, \ldots.)$ are non-negative real numbers with finite trace $Tr\, Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. The infinite dimensional fractional Brownian motion on Y can be defined by using covariance operator Q as $W_{(t)}^H = W_{Q\ (t)}^H = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\ e_n\ B_{n\,(t)}^H$, where $B_{n\,(t)}^H$ are one dimensional fractional Brownian motions mutually independent on (Ω, F, P).

In order to defined stochastic integral with respect to the $Q-$fractional Brownian motion. We introduce the space $L_2^0(Y,X)$ of all $Q-$Hilbert-Schmidt operators that is with the inner product $\langle \Phi, \varphi \rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \Phi e_n, \varphi e_n \rangle$ is a separable Hilbert space.

Lemma (7) (following in [18])

Let $\{\Phi(t)\}_{t\epsilon[0,T]}$ be a family of deterministic functions with values in $L_2^0(Y,X)$ The stochastic integral of Φ with respect to W^H is defined by

$$\int_0^t \Phi(s) dW_{(s)}^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \Phi(s) e_n dB_{n(s)}^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\Phi e_n)(s) dB_{n(s)}^H. \quad (1.14)$$

Lemma (8) (following in [18])

If $\varphi: [0, b] \to L_2^0(Y,X)$ satisfies $\int_0^T \|\varphi(s)\|_{L_2^0}^2 ds < \infty$ then the above sum in lemma (1.13) is well defined as an $X-$ valued random variable and we have

$$E\int_0^t \varphi(s) dW_{(S)}^{H\ 2} \leq 2Ht^{2H-1} \int_0^t \varphi(s)_{L_2^0}^2 ds. \quad (11)$$

3. Problem Formulation

In this section, we study the existence and stability of inclusion equations, type of stochastic dynamical system driven by mixed fractional Brownian motion in a real separable Hilbert space H of the following from:

$$d\left[x'(t) - g\left(t, x(h_1(t)), \int_0^t a(t,s,x(h_2(s)))ds\right)\right] \in A\left[x(t) + \int_0^t f(t-s)x(s)ds\right]dt + F_1(t, x(h_3(t)))dw(t) + F_2(h_3(t))dw^H(t) \quad (12)$$

$$x(0) + h(x(t)) = x_0, \frac{1}{2} < H \leq 1,$$

$$x'(0) + h'(x(t)) = x_1$$

Where $A: X \to X$ is a generator of cosine semigroup on a Hilbert space $(X, \|\cdot\|)$, $\{W(t): t \geq 0\}$ and $\{W^H(t): t \geq 0\}$ are $K-$valued Brownian motion and fractional Brownian motion respectively.

To investigate the existence of the mixed-stochastic mild solution to the system (12), and for the operators A we make the following assumption :

1. A is the infinitesimal generator of a compact, analytic resolve operator $S(t), C(t), t \geq 0$ in the Hilbert space H and there exists constant $M^\wedge, N^\wedge$ and M_1, M_2 such that $\|S(t)\|^2 \leq N^\wedge$, $\|C(t)\|^2 \leq M^\wedge$, $t \in J$ on $I = [0,T]$ and $\|f(t)\|^2 \leq M_1$.
2. There exist constant M_2 such that $g : J \times H \times H \to H$, is a continuous function, satisfies the following Lipchitz condition, that is, for any $s, t \in J, x, y \in H$ such that

$$\|g(s, x_1, y_1) - g(s, x_2, y_2)\|^2 \leq M_2\left[|s-t| + \|x_1 - x_2\|^2 + \|y_1 - y_2\|^2\right].$$

$$M_2^\wedge = \sup_{t\in J} g(t, 0).$$

3. a: $D \times H \to H$, $D = \{(t,s) \in J \times J : t \geq s\}$ is a continuous function and there exists a constant $M_3 > 0$ such that for all $t \in J, x, y \in H$

$$\left\|\int_0^t [a(t, s_1, x) - a(t, s_2, y)]ds\right\|^2 \leq M_3\left(\|s_1 - s_1\|^2 + \|x - y\|^2\right)$$

$$and\ M_3^\wedge = \sup_{t\in J} a(t, 0, 0).$$

4. $h_3 \in C(J, J), F_2 \in L^2(L(K,H))$.
5. For the initial condition there exists a positive constants M_6, and M_7 such that $\sup_{E\|x\|^2 \leq 1} h(x_1) = M_4$, $\sup_{E\|x\|^2 \leq 1} h'(x_2) = M_5$.

$$\|h(x_1(t)) - h(x_2(t))\|^2 \leq M_6 \|x_1(t) - x_2(t)\|^2$$

$$\|h'(x_1(t)) - h'(x_2(t))\|^2 \leq M_7 \|x_1(t) - x_2(t)\|^2$$

6. The function $\sigma: [0,T] \to L_2^0(y\,; x)$ satisfies For every $t \in [0,T]$: $\int_0^t \|\sigma(s)\|_{L_2}^2 ds < \infty$ and there exists $C_1 > 0$ such that $Sup\|\sigma(s)(t)\|_{L_2}^2 \leq C_1$.
7. The multi-valued map $AF_1: J \times H \to P_{bd,cl,cv}(L(k,H))$ is an L^2- Caratheodory function satisfies the following conditions:-

i. For each $t \in J$, the function $AF_1(t,.) : H \times H \to P_{bd,cl,cv}(L(k,H))$ is u. s. c, and for each $x \in H$, the function $AF_1(.,y)$ is measurable and for each fixed $x \in B$, the set $N_{AF_1,x} = \{\sigma \in L^2(L(k,H)) : \sigma(t) \in AF_1(t, x(h_3(t)))$ for $t \in J\}$ is nonempty.
ii. For each positive number $L > 0$, there exists appositive function $\mu(I)$ independent on I such that $Sup_{E\|x\|^2 \le 1} \|AF_1(t,x)\| \le \mu(I)$.
iii. $\|AF_1(t_1,x_1) - AF_1(t_2,x_2)\|^2 \le L_1(|t_1 - t_2| + \|x_1 - x_2\|)$.

8. $$r > \max\left\{ \frac{\left[\begin{array}{l} 6M^{\wedge^2} Ex(0)^2 + 12N^{\wedge^2} HT^{2H-1}C_1 \\ +12M^{\wedge^2} M_2 M_2^{\wedge} + M_3 M_3^{\wedge} \\ +6N^{\wedge^2} \mathrm{Tr}(\mathrm{Q})\mu(L) \\ +12N^{\wedge^2}\left(Ex'(0)^2 + C\right) \end{array}\right]}{1-\left(6N^{\wedge^2} T^2 M_1 + 12M^{\wedge^2} M_2 + M_3\right)}, \frac{M^{\wedge^2}\varnothing_1 - \psi_1 + N^{\wedge^2}\varnothing_2 - \psi_2}{(1-\left(\begin{array}{l} M^{\wedge^2} M_6 + N^{\wedge^2} M_7 + M_2 \\ +T\mathrm{Tr}(\mathrm{Q})N^{\wedge^2} L_1 + L_1 M_1 N^{\wedge^2} T^2 \end{array}\right))} \right\}$$

Where

$$1 > (M^{\wedge} M_6 + N^{\wedge} M_7 + M_2 + T\mathrm{Tr}(\mathrm{Q})N^{\wedge} L_1 + L_1 M_1 N^{\wedge} T^2)$$
$$\text{and } 1 > \left(6N^{\wedge^2} T^2 M_1 + 12M^{\wedge^2} M_2 + M_3\right).$$

Lemma(9)

Let $\{c(t)\}_{t \ge 0}$ be a cosine semigroup and H-valued function $g(s) = C(t-s)\,x(s) + S(t-s)\;[\;x\grave{}(s) - g(s, x(s))]$ then the system (12) has a mixed–stochastic mild solution

$$\begin{aligned} x(t) &= C(t)x(0) \\ &+ S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right] \\ &+ \int_0^t \int_0^s S(s-\tau) Af(\tau,\tau_1) x(\tau_1)\, d\tau_1 d\tau \\ &+ \int_0^t s(t-s) AF_1\left(s, x\left(h_3(s)\right) dw(\mathrm{s})\right) \\ &+ \int_0^t S(t-s) AF_2\left(h_3(\mathrm{t})\right) dw^H(\mathrm{s}) \\ &- \int_0^t C(t-s)\mathrm{g}\left(\mathrm{s}, x\left(h_1(\mathrm{s})\right), \int_0^t \mathrm{a}\left(t,\tau, x\left(h_2(\tau)\right)\right) d\tau\right) ds. \end{aligned}$$

Proof:

$$\begin{aligned} g(s) &= C(t-s)x(s) \\ &+ S(t-s)\left[x'(s) - g\left(t, x\left(h_1(t)\right), \int_0^t a\left(t,s,x\left(h_2(s)\right)\right) ds\right)\right] \end{aligned}$$

different both sides for S and use properties in Lemma (9), we get

$$\frac{dg(s)}{ds} = C(t-s)x\acute{}(s) - AS(t-s)x(s) + S(t-s)$$

$$\begin{aligned} &\frac{d}{dt}\left[x\acute{}(s) - g(s, x\left(h_1(s)\right), \int_0^t a(t,s,x(h_2(s)))ds\right] \\ &- C(t-s)(x\grave{}(s) - g(s, x\left(h_2(s)\right), \int_0^t a(t,s,x(h_2(s)))ds) \\ &= C(t-s)x\acute{}(s) - AS(t-s)x(s) \\ &+ S(t-s)\left(A(x(t) + \int_0^t f(t-s)x(s)ds))dt\right) \\ &+ F_1\left(t, x\left(h_3(t)\right)\right)dw(t) + F_2\left(h_3(t)\right)dw^H(t) \\ &- C(t-s)\left(x\acute{}(s) - g(s, x\left(h_2(s)\right), \int_0^t a(t,s,x(h_2(s)))ds\right). \end{aligned}$$

Integrate both sides, we get

$$\begin{aligned} &= \int_0^t S(t-s) Af(s-\tau)x(\tau)d\tau \\ &+ S(t-s)AF_1\left(t, x\left(h_3(s)\right)\right)dw(s) \\ &+ S(t-s)AF_2\left(h_3(s)\right)dw^H(s) \\ &+ C(t-s)g(s, x\left(h_1(s)\right), \int_0^t a\left(t,s,x\left(h_2(s)\right)\right)ds\right) \end{aligned}$$

$$\begin{aligned} x(t) &= C(t)x(0) + S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right] \\ &+ \int_0^t \int_0^s S(t-s) Af(\tau,\tau_1)x(\tau_1)d\tau_1 d\tau \\ &+ \int_0^t s(t-s) AF_1\left(s, x\left(h_3(s)\right) dw(\mathrm{s})\right) \\ &+ \int_0^t S(t-s) AF_2\left(h_3(\mathrm{t})\right) dw^H(\mathrm{s}) \\ &- \int_0^t C(t-s)\mathrm{g}\left(\mathrm{s}, x\left(h_1(\mathrm{s})\right), \int_0^t \mathrm{a}\left(t,\tau,x\left(h_2(\tau)\right)\right)d\tau\right)ds \end{aligned} \tag{13}$$

Definition (8)

A bounded function $x(t): R \to H$ is called mixed solution of the inclusion system (12) if for any $t \in J$

$$\begin{aligned} x(t) &= C(t)x(0) + S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right] \\ &+ \int_0^t S(t-s)\int_0^s Af(t-s)x(s)dsds \\ &+ \int_0^s s(t-s) AF_1\left(s, x\left(h_3(s)\right) dw(\mathrm{s})\right) \\ &+ \int_0^s S(t-s) A\, F_2\left(h_3(\mathrm{t})\right) dw^H(\mathrm{s}) \\ &- \int_0^t C(t-s)\mathrm{g}\left(\mathrm{s}, x\left(h_1(\mathrm{s})\right), \int_0^t \mathrm{a}\left(t,\tau,x\left(h_2(\tau)\right)\right)d\tau\right)ds. \end{aligned}$$

3.1. Existence of the Fractional Stochastic-Integro Differential Inclusion Driven by Mixed Fractional Brownian Motion

In this section, the existence of mixed-stochastic mild solution in to inclusion problem formulation (12) has been develop.

Theorem(1):

Suppose that conditions(1-8) are hold.

Then for initial value $x(0) + h(x(t)) = x_0$, $x\acute{}(0) + h\acute{}(x(t)) = x_1$, such that the intial value mixed-stochastic inclusion problem(1) has mixed-stochastic mild solution $x \epsilon \beta$.

Proof:

Let the operator $\emptyset: B \to p(B)$ defined by

$$\emptyset(x) = \{x \in B : x(t) = C(t-s)x(0)$$
$$+S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right]$$
$$+\int_0^t S(t-s)\int_0^s Af(t-s)x(\tau_1)d(s)d(s)$$
$$+\int_0^t s(t-s)AF_1\left(s, x\left(h_3(s)\right)\right)dw(s)$$
$$+\int_0^t S(t-s)A\,F_2\left(h_3(\mathrm{s})\right)dw^H(\mathrm{s})$$
$$-\int_0^t C(t-s)\mathrm{g}\left(\mathrm{s}, x\left(h_1(\mathrm{s})\right), \int_0^t \mathrm{a}\left(t,\tau, x\left(h_2(\tau)\right)\right)d\tau\right)ds\Big).$$

It is clear that the fixed points of $\emptyset$ are mild solutions of the system (12). Let

$$\emptyset_1(x) = \{x \in B : x(t)$$
$$= C(t)x(0) + S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right]$$
$$+\int_0^t S(t-s)\int_0^s Af(t-s)x(s)dsds \tag{14}$$
$$-\int_0^t C(t-s)\mathrm{g}\left(\mathrm{s}, x\left(h_1(\mathrm{s})\right), \int_0^t \mathrm{a}\left(t,\tau, x\left(h_2(\tau)\right)\right)d\tau ds\right).$$

$$\emptyset_2(x) = \{x \in B : x(t)$$
$$= \begin{cases} \int_0^t s(t-s)AF_1\left(s, x\left(h_3(s)\right)dw(\mathrm{s})\right) \\ +\int_0^t S(t-s)AF_2\left(h_3(\mathrm{t})\right)dw^H(\mathrm{s}) \end{cases}. \tag{15}$$

We prove that the operators $\emptyset_1$ and $\emptyset_2$ are satisfy all the condition of theorem (1), Let $B_l = \{x \in \beta, \mathrm{E}\|x\|^2 \le l\}$

Step(1):- $\emptyset_1$ is a contraction

Let $x_1, x_2 \in B_l$, from assuming that

$$\emptyset_1(x) = \{C(t)x(0) + S(t)\left[x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right]$$
$$+\int_0^t S(t-s)\int_0^s Af(t-s)x(s)dsds$$
$$-\int_0^t C(t-s)\mathrm{g}\left(s, x\left(h_1(s)\right), \int_0^t \mathrm{a}\left(t,\tau, x\left(h_2(\tau)\right)\right)d\tau ds\right)$$

$$E\left\|\emptyset_1(x_1)(t) - \emptyset_1(x_2)(t)\right\|^2$$
$$\le 5\|C(t)\|^2 E\left\|x_1(0) - x_2(0)\right\|^2$$
$$+5\|S(t)\|^2 E\left\|x_1'(0) - x_2'(0)\right\|^2$$
$$+5\|S(t)\|^2 E\left\|g\left(0, x_1\left(h_1(0)\right), 0\right) - g\left(0, x_2\left(h_1(0)\right), 0\right)\right\|^2$$
$$+5\int_0^t \|s(t-s)\|^2 \int_0^t \|f(t-s)\|^2 E\left\|x_1(s) - x_2(s)\right\|^2 dsds$$
$$+\int_0^t \|C(t-s)\|^2 E\left\| g\left(s, x_1\left(h_1(s)\right), \int_0^t a\left(t,\tau, x_1\left(h_2(\tau)\right)\right)d\tau\right) - g\left(s, x_2\left(h_1(s)\right), \int_0^t a\left(t,\tau, x_2\left(h_2(\tau)\right)\right)d\tau\right)\right\|^2 ds.$$

From the assumptions (1-4), we have

$$\le 5M^{\wedge 2}\mathrm{Sup}_{t\in J} E\left\|x_1(0) - x_2(0)\right\|^2$$
$$+5N^{\wedge 2}\mathrm{Sup}_{t\in J}\left\|x'_1(0) - x'_2(0)\right\|^2$$
$$+5N^{\wedge 2}M_5\left\|x_1(t) - x_2(t)\right\|^2$$
$$+5\int_0^t\int_0^s N^{\wedge 2}M_1\mathrm{Sup}_{t\in J} E\left\|x_1(t) - x_2(t)\right\|^2 dsds$$
$$+5\int_0^t \mathrm{M}^{\wedge 2}\mathrm{M}_5\begin{bmatrix} |s-s| + \left\|x_1\left(h_1(s)\right) - x_2\left(h_1(s)\right)\right\|^2 dt \\ +\int_0^t \left\| \begin{matrix} \mathrm{a}\left(t,\tau, x_1\left(h_2(\tau)\right)\right) \\ -\mathrm{a}\left(t,\tau, x_2\left(h_2(\tau)\right)\right)d\tau \end{matrix} \right\|^2 \end{bmatrix}$$

By using initial condition (5), and taking supremum over $t \in [0, T]$ for both sides, we get

$$\mathrm{E}\left\|\emptyset_1(x_1)(t) - \emptyset_1(x_2)(t)\right\|^2$$
$$\le 5M^{\wedge 2}M_4\left\|h\left(x_1(t)\right) - h\left(x_2(t)\right)\right\|^2$$
$$+5N^{\wedge 2}M_5\left\|h'\left(x_1(t)\right) - h'\left(x_2(t)\right)\right\|^2$$
$$+5N^{\wedge 2}M_2\mathrm{Sup}E\left\|x_1(t) - x_2(t)\right\|^2$$
$$+5b^2N^{\wedge 2}M_1\mathrm{Sup}_{t\in J}E\left\|x_1(t) - x_2(t)\right\|^2$$
$$+5bM^{\wedge 2}M_2\mathrm{Sup}_{t\in J}E\left\|x_1(t) - x_2(t)\right\|^2$$
$$+M_3\mathrm{Sup}_{t\in J}E\left\|x_1(t) - x_2(t)\right\|^2.$$

We can rewrite last inequality in the following from:

$$E\left\|\emptyset_1(x_1)(t) - \emptyset_1(x_2)(t)\right\|^2$$
$$\le \mathrm{Sup}_{t\in J}\, x_1(t) - x_2(t)^2\Big(5M^{\wedge 2}M_4M_6$$
$$+5N^{\wedge 2}M_5M_7 + 5N^{\wedge 2}M_2 + 5b^2N^{\wedge 2}M_1$$
$$+5bM^{\wedge 2}M_2 + M_3\Big)\mathrm{Sup}_{t\in J}E\left\|x_1(t) - x_2(t)\right\|^2$$
$$= L_0\mathrm{Sup}_{t\in J}E\left\|x_1(t) - x_2(t)\right\|^2.$$

Where

$$L_0 = \begin{bmatrix} 5M^{\wedge 2}(M_4M_6 + bM_2) \\ +5N^{\wedge 2}\left(M_5M_7 + M_2 + b^2M_1\right) + M_3 \end{bmatrix} > 0.$$

Hence, we obtain $E\|\emptyset_1(x_1)(t) - \emptyset_1(x_2)(t)\|^2 \le L_0\, E\|\, x_1 - \, x_2\|^2$

Step (2):- $\emptyset_2(x)$ is convex for each $x \in B$ if $u_1, u_2 \in \emptyset_2(x)$, then there exists $\sigma_1, \sigma_2 \in N_{F1,2}$ from condition $(7 - \, i - \, ii)$, we get

$$x_1(t) = \int_0^t \mathrm{S}(\mathrm{t}-\mathrm{s})\sigma_1(s)dw(s)$$
$$+\int_0^t \mathrm{S}(\mathrm{t}-\mathrm{s})A\,F_2\left(h_3(\mathrm{s})\right)dw^H(\mathrm{s}) \tag{16}$$

$$x_2(t)=\int_0^t S(t-s)\sigma_2(s)dw(s) + \int_0^t S(t-s)AF_2(h_3(s))dw^H(s). \quad (17)$$

Let $\lambda \in [0,1]$, then

$$\lambda x_1(t)=\int_0^t S(t-s)\lambda\sigma_1(s)dw(s) + \int_0^t \lambda S(t-s)AF_2(h_3(s))dw^H(s)$$

$$(1-\lambda)x_2(t)=\int_0^t S(t-s)(1-\lambda)\sigma_2(s)dw(s) + \int_0^t (1-\lambda)S(t-s)AF_2(h_3(s))dw^H(s)$$

Since $N_{F1,2}$ is convex (because $F_{1,2}$ has convex values), then, we have

$$\lambda x_1(t)+(1-\lambda)x_2(t)\in\emptyset_2(x).$$

Step (3):- $\emptyset_2$ maps bounded sets in to bounded sets in β. To show that there exists a positive constant $\wedge$ such that for each $u \in \emptyset_2(x), x\in B_l$, we have $E\|u(t)\|^2 \le \wedge$. If $u \in \emptyset_2(x)$, then there exists $\sigma \in N_{AF_1}$ for each $t \in J$, such that

$$E\|u(t)\|^2 \le 2E\left\|\int_0^t S(s-t)AF_1(s,x_1(h_3(s)))dw(s)\right\|^2 + 2E\left\|\int_0^t S(s-t)AF_2(h_3(s))dw^H(s)\right\|^2$$

$$\le 2\|S(s-t)\|^2 E\left\|\int_0^t \sigma(s)\right\|^2 dw(s) + 2\|S(s-t)\|^2 E\left\|\int_0^t AF_2(h_3(s))dw^H(s))\right\|^2.$$

By using Lemma (1.8), Lemma (1.14) and by assumption (1), we obtain

$$E\|u(t)\|^2 \le 2N^{\wedge 2}E\left\|\int_0^t \sigma(s)\right\|^2 ds + 2N^{\wedge 2}\left(2Ht^{2H-1}\left\|\int_0^t AF_2(h_3(s))^2\right\|ds\right).$$

From assumptions (6) and (7-ii) ,we get

$$\le 2N^{\wedge 2}Tr(Q)T\mu(J)+4N^{\wedge 2}Ht^{2H-1}C_1=\Lambda. \quad (18)$$

Step(4):- $\emptyset_2$ maps bounded sets in to equicontinuous set of β, Let $\tau\in(0,b]$ such that $0<\tau_1<\tau_2\le b$, then for $eah\ x\in B_l$, and $u\in\emptyset_2(x)$, then,there exists $\sigma \epsilon N_{F\ 1,2}$ such that for each $t\in J$, we have

$$\emptyset_2(x)=\int_0^t S(t-s)AF_1(s,x(h_3(s)))dw(s) + \int_0^t S(t-s)AF_2(h_3(s))dw^H(s).$$

$$E\|\emptyset_2(x)(\tau_2)-\emptyset_2(x)(\tau_1)\|^2$$

$$\le 6E\left\|\int_0^{\tau_1-\varepsilon}\begin{bmatrix}S(\tau_2-s)\\-S(\tau_1-s)\end{bmatrix}AF_1(s,x(h_3(s)))dw(s)\right\|^2$$

$$+6\left\|E\int_{\tau_1-\varepsilon}^{\tau_1}\begin{bmatrix}S(\tau_2-s)\\-S(\tau_1-s)\end{bmatrix}AF_1(s,x(h_3(s)))dw(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2}S(\tau_2-s)AF_1(s,x(h_3(s)))dw(s)\right\|^2$$

$$+6E\left\|\int_0^{\tau_1-\varepsilon}[S(\tau_2-s)-S(\tau_1-s)]AF_2(h_3(s))dw^H(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]AF_2(h_3(s))dw^H\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2}S(\tau_2-s)AF_2(h_3(s))dw^H\right\|^2$$

$$\le 6Tr(Q)\mu(I)E\left\|\int_0^{\tau_1-\varepsilon}[S(\tau_2-s)-S(\tau_1-s)]ds\right\|^2$$

$$+6Tr(Q)\mu(I)E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]ds\right\|^2$$

$$+6HT^{2H-1}C_1E\left\|\int_0^{\tau_1-\varepsilon}[S(\tau_2-s)-S(\tau_1-s)]ds\right\|^2$$

$$+6Ht^{2H-1}C_1E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}[S(\tau_2-s)-S(\tau_1-s)]\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2}\left[S(\tau_2-s)AF_2(h_3(s))dw^H\right]\right\|^2.$$

When $\tau_2\to\tau_1$ the above inequality tends to zero, since S(t) in the uniform operator topology thus the set $\{\phi_2(x): x\in B_l\}$ is equicontinuous.

Step(5):- Now to prove $(\emptyset_2 B_l)(t)$ is relatively compact in H for each $t\in J$, wehre $(\emptyset_2 B_l)(t)=\{u(t): u\in \emptyset_2 B_l\}$, $t \in J$, the set $(\emptyset_2 B_l)(t)$ is relatively compact in H for each $t=0$. Let $0<t\le b$ and $0<\in<t$, for $x\in B_l$ and $u\in\emptyset_2(x)$, there exists $\sigma\in N_{F,X}$ such that

$$u(t)=\int_0^{t-\in}S(t-s)AF_1(s,x(h_3(s)))dw(s) + \int_{t-\in}^t S(s-t)AF_1(s,x(h_3(s)))dw(s) + \int_0^{t-\in}S(s-t)AF_2(h_3(s))dw^H(s) + \int_{t-\in}^t S(s-t)AF_2(h_3(s))dw^H(s). \quad (19)$$

Now, we define

$$u_\in(t)=\int_0^{t-\in}\begin{pmatrix}S(t-\in)\\-(s-\in)\end{pmatrix}AF_1(s,x(h_3(s)))dw(s) + \int_0^{t-\in}\left(S(t-\in)-(s-\in)\right)AF_2(h_3(s))dw^H(s), \quad (20)$$

for each $0<\in<t$, thus,

$$E\|u(t)-u_\in(t)\|^2$$

$$\le 4E\left\|\int_0^{t-\in}S(t-s)AF_1(s,x(h_3(s)))dw(s)\right.$$

$$+\int_{t-\in}^t S(t-s)AF_1(s,x(h_3(s)))dw(s)$$

$$+\int_0^{t-\in}S(t-s)AF_2(h_3(s))dw^H(s)$$

$$+\int_{t-\in}^{t} S(t-s) AF_2(h_3(s))dw^H(s)$$
$$+\int_0^{t-\in}\left(S(t-\in)-(s-\epsilon)\right) AF_1\left(s, x\left(h_3(s)\right)\right)dw(s)$$
$$-\int_0^{t-\in} S\left((t-\epsilon)-(s-\epsilon)\right) AF_2(h_3(s))dw^H(s)\Big\|^2.$$

Since sine simegroup operators are a continuous, we have

$$E\left\|u(t)-u_{\in}(t)\right\|^2$$
$$\le 4E\left\|\int_{t-\in}^{t} S(t-s) AF_1\left(s, x\left(h_3(s)\right)\right)dw(s)\right\|^2$$
$$+\left\|\int_{t-\in}^{t} S(t-s) AF_2(h_3(s))dw^H(s)\right\|^2.$$

Then, there exist $\sigma \in N_{AF_1,X}$, we get

$$\le 4E\left\|\int_{t-\in}^{t} S(t-s)\sigma(s)\,dw(s)\right\|^2$$
$$+\left\|\int_{t-\in}^{t} S(t-s) AF_2(h_3(s))dw^H(s)\right\|^2.$$

By using Lemmas (2), (8), and assumptions (1), (6), (7-ii), we obtain

$$\le 4Tr(Q)\mu(I)\in +8N^{\wedge^2} Ht^{2H-1}C_1\in.$$

The relative compact sets arbitrarily close to the set $\{u(t): u \in \emptyset_2 B_l\}$ then its relative compact in β thus $\emptyset_2$ is a compact multi-valued closed graph .

Step (6):- Now to show that $\emptyset_2$ has a closed graph .

Let $x_n \to x_*, x_n \epsilon B_L, u_n \in \emptyset_2 \ (x_n)$ and $u_n \to u_*$, we aim to show that $u_* \epsilon \emptyset_2 \ (x_*)$ indeed, $u_n \epsilon \emptyset_2 \ (x_n)$ means that there exists

$$\sigma_n \epsilon N_{AF_1,x}\left(s, x\left(h_3(s)\right)\right)$$
$$u_n(t) = \int_0^t S(t-s) AF_{1,n}\left(s, x\left(h_3(s)\right)\right)dw \qquad (21)$$
$$+\int_0^t S(t-s)\sigma_n(s)dw^H.$$

There exists $\sigma_{1,n} \in N_{F,x}$, thus

$$u_n(t) = \int_0^t S(t-s)\sigma_{1,n}(s)dw + \int_0^t S(t-s)\sigma_n(s)dw^H. \ (22)$$

We must prove that there exists $\sigma_1{}^* \in N_{F,x_*}$ such that

$$u_*(t) == \int_0^t S(t-s)\sigma_1^*(s)dw + \int_0^t S(t-s)\sigma_*(s)dw^H. \ (23)$$

Suppose the liner continuous operator $\Gamma_1: L^2(J,H) \to C\ (J,H)$.

From lemma (1) it follows that $\Gamma_1 N_{AF_1,x}\left(s, x(h_3(s))\right) dw$ is closed graph operator and we have $\left\|u_n(t) - u_*(t) + \int_0^t S(t-s)\left(\sigma_n(s) - \sigma_*(s)\right)dw^H\right\| \to 0$ as $n \to \infty$, thus

$$u_n(t) - \int_0^t S(t-s)\sigma_*(s)dw^H \in \Gamma_1 N_{AF_{1,x}}\left(s, x\left(h_3(s)\right)\right)dw.$$

Since $u_n \to u_*$, it follows from Lemma (1) that,

$$u_*(t) - \int_0^t S(t-s)\sigma_*(s)dw^H \in$$
$$\Gamma_1 N_{AF_{1,x}}\begin{pmatrix}\int_0^t S(t-s)\sigma_*(s)dw^H \in \\ \Gamma_1 N_{AF_{1,x}}\left(s, x\left(h_3(s)\right)\right)dw\end{pmatrix}.$$

That is, there exists a $\sigma_1^* \in N_{F,x}$ such that

$$u_*(t) - \int_0^t S(t-s)\sigma_*(s)dw^H = \int_0^t S(t-s)\sigma_1^*(s)dw.$$

Since Γ be a linear continuous mapping from $L^2(I,H)$to $C(I,H)$, in Lemma (1). Therefore $\emptyset_2$ is a closed graph and $\emptyset_2$ u.s.c.

Step (7):-The operator inclusion $x \in \emptyset_1(x) + x \in \emptyset_2(x)$ has a solution in $B[0,r]$. Define an open ball B(0, r) in β , where r satisfies the inequality given in (20),we need to show that the system (12) has least one mild solution, for $\lambda u \in \emptyset_1 x + \emptyset_2 x$. for some $\lambda > 1$ with $E\|x\|^2 = r$, then, we have

$$x(t) = \lambda^{-1}\left(C(t)x(0)\right) + \lambda^{-1}\left(S(t)\left[x'(0) - g\left(s, x(0)\right)\right]\right)$$
$$+\lambda^{-1}\left(\int_0^t\int_0^s S(t-s) Af(t-s)x(s)dsds\right)$$
$$+\lambda^{-1}\left(\int_0^t S(t-s) AF_1\left(s, x\left(h_3(s)\right)\right)dw(s)\right)$$
$$+\lambda^{-1}\left(\int_0^t S(t-s) AF_2(h_3(t))dw^H(s)\right)$$
$$-\lambda^{-1}\left(\int_0^t C(t-s) - g\left(s, x\left(h_1(s)\right), \int_0^t a(t,\tau, x(h_2(\tau))d\tau\right)ds\right).$$

$$E\|x(t)\|^2 \le 6E\left\|C(t)x(0)\right\|^2$$
$$+12E\left\|S(t)\right\|^2\left[\left\|x'(0) - g\left(0, x\left(h_1(0)\right), 0\right)\right\|^2\right]$$
$$+6E\left\|\left(\int_0^t S(t-s)\int_0^s Af(t-s)x(s)dsds\right)\right\|^2$$
$$+6E\left\|\left(\int_0^t S(t-s) AF_1\left(s, x\left(h_3(s)\right)\right)dw(s)\right)\right\|^2$$
$$+6E\left\|\left(\int_0^t S(t-s) AF_2(h_3(t))dw^H(s)\right)\right\|^2$$
$$+12E\left\|\left(\int_0^t C(t-s)\, g\left(s, x\left(h_1(s)\right), \int_0^t a(t,\tau, x(h_2(\tau))d\tau\right)ds\right)\right\|^2.$$

By using assumptions (1-3)and (7-i) ,we get

$$\le 6M^{\wedge^2} Ex(0)^2 + 12N^{\wedge^2}\left(Ex'(0)^2 + C\right)$$
$$+6N^{\wedge^2} b^2 M_1 Ex(t)^2 + 6N^{\wedge^2}\left\|\left(\int_0^t \sigma(s)dw(s)\right)\right\|^2$$
$$+6N^{\wedge^2}\left(2Ht^{2H-1}\left\|\int_0^t AF_2(h_3(t))\right\|^2 ds\right)$$
$$+12M^{\wedge^2}\left\|\int_0^t\begin{pmatrix} g\left(s, x\left(h_1(s)\right), \int_0^t a\left(t,\tau, x\left(h_2(\tau)\right)d\tau\right)\right)ds \\ -g(s,0,0) + g(s,0,0)\end{pmatrix}\right\|^2$$

$$+\left\|\left(\int_0^t a\left(t,\tau,x\left(h_2(\tau)\right)\right)-\int_0^t a(t,\tau,0)+\int_0^t a(t,\tau,0)\right)ds\right\|^2.$$

From the assumptions(1),(2)and(3),(6) and using Lemmas (2) and (8), we obtain

$$\leq 6M^{\wedge 2}E\|x(0)\|^2+12N^{\wedge 2}\left(E\|x'(0)\|^2+C\right)$$

$$+6N^{\wedge 2}b^2M_1E\|x(t)\|^2+6N^{\wedge 2}E\left\|\left(\int_0^t\sigma(s)\right)\right\|^2 ds$$

$$+6N^{\wedge 2}\left(2Ht^{2H-1}\left\|\int_0^t AF_2\left(h_3(t)\right)\right\|^2 ds\right)$$

$$+12M^{\wedge 2}M_2\left(E\|x(t)\|^2+M_2^{\wedge}\right)+M_3\left(E\|x(t)\|^2+M_3^{\wedge}\right).$$

From assumption (6), we get

$$\begin{aligned}&\leq 6M^{\wedge 2}E\|x(0)\|^2+12N^{\wedge 2}\left(E\|x'(0)\|^2+C\right)\\&+6N^{\wedge 2}b^2M_1E\|x(t)\|^2+6N^{\wedge 2}Tr(Q)\mu(I)\\&+12N^{\wedge 2}Ht^{2H-1}C_1+12M^{\wedge 2}M_2\left(E\|x(t)\|^2+M_2^{\wedge}\right)\\&+M_3\left(E\|x(t)\|^2+M_3^{\wedge}\right).\end{aligned} \tag{24}$$

$$E\|x(t)\|^2 \leq \frac{\begin{bmatrix}6M^{\wedge 2}E\|x(0)\|^2+12N^{\wedge 2}Ht^{2H-1}C_1\\+12M^{\wedge 2}M_2M_2^{\wedge}+M_3M_3^{\wedge}\\+6N^{\wedge 2}Tr(Q)\mu(I)+12N^{\wedge 2}\left(E\|x'(0)\|^2+C\right)\end{bmatrix}}{1-\left(6N^{\wedge 2}t^2M_1+12M^{\wedge 2}M_2+M_3\right)}.$$

Thus

$$r\leq \frac{\begin{bmatrix}6M^{\wedge 2}\|Ex(0)\|^2+12N^{\wedge 2}Ht^{2H-1}C_1\\+12M^{\wedge 2}M_2M_2^{\wedge}+M_3M_3^{\wedge}\\+6N^{\wedge 2}Tr(Q)\mu(I)+12N^{\wedge 2}\left(E\|x'(0)\|^2+C\right)\end{bmatrix}}{1-\left(6N^{\wedge 2}b^2M_1+12M^{\wedge 2}M_2+M_3\right)}.$$

Example (1)

Consider the following fractional differential equations

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}u(t,\tau_1)-\int_0^t b(t,\tau_1)u(sint,\tau_1)d\tau_1\right]\in$$

$$\frac{\partial^2}{\partial\tau^2}[u(t,\tau)+\int_0^t\beta e^{-\alpha(t-s)}Au(s,\zeta)ds]$$

$$+F\left(t,u(t,\tau)\right)dw^H+k\left(t,\frac{\partial}{\partial\tau}u(sint,\tau)\right)dw(t)$$

$$u(t,0)=u(t,\pi)=0$$

$$u(0,\tau)=x_0(\tau)+p\left(u(s,\tau)\right)ds$$

$$\frac{\partial u}{\partial t}(0,\tau)=y_0(\tau)+q\left(u(s,\tau)\right)ds,$$

for $(t,\tau)\in[0,a]\ x\ [0,\pi]$, $-\alpha\leq\beta\leq\alpha$,where, where α,β are a constants.

(1) $X=L^2[0,\pi]$, $x_0,y_0\in X$

(2) $A: D(A)\subseteq X\to X$ by $Au=u''$,

$$D(A)=\{u\in X, u(0)=u(\pi)=0\}$$

(3) A is generator of strongly cosine family $\{c(t)\}_{t\in IR}$ on x.

(4) The eigenvalues of A is $-n^2$, $n\in IN$, and the eigenvectors $Zn(\tau)=(\frac{2}{\pi})^{\frac{1}{2}}\sin(n\tau)$ the set of function $\{Zn: n\in N\}$ is oryhonormal basis of x.

(5) For $Z\in x, c(t)\,Z=\sum_{n=1}^{\infty}\cos(nt)<Z,Zn>Zn$, $S(t)=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}<Z,\quad Zn>Zn$, also $\|C(t)\|=\|S(t)\|=1$, for all $t\in R$. In addition, $A\,Z=-\sum_{n=1}^{\infty}n^2<Z,Zn>Zn$, for $Z\in D(A)$. We assume $x_0\in L^2([0,\pi])$ is F_0 −measurable satisfies $E\|x_0\|^2<\infty$

(6) The function $b(.)$ is of classes C^2 on $I\,x\,J$ and $b(\tau_1,\pi)=b(\tau_1,0)=0$, for each $\tau_1\in I$.

(7) The function $F: Ix[0,\pi]\to IR$ is continuous and there is $L_f>0$ such that $|F(t,\tau_1)-F(t,\tau_2)|\leq L_F|\tau_1-\tau_2|$, $t\in I,\tau\ i\in R$

(8) The function $p_1q: R\to R$ are continuous and there are appositive constants Lp, Lq such that

$$|p(\mu_1)-p(\mu_2)|\leq Lp|\mu_1-\mu_2|,\ \mu\ i\in R$$

$$|q(\mu_1)-q(\mu_2)|\leq Lq|\mu_1-\mu_2|,\ \mu\ i\in R.$$

(9) $k:[0,\pi]x\,[0,\pi]\to IR$ is a continuous function.

3.2. Stability for the Mild Solution of Inclusion Formulation Problem (12)

The following theorem investigate the stability of the inclution equation (12) by using Gron will Bellman inequality via cosine dynamical system.

We need to investigate the definition (8) on the inclusion problem (12).

Definition (9)

The solution $x(t,0,\emptyset,\psi)$ of the system (12) in said to be stable , if for any $\in>0$, there exists a number $\delta=\delta(\in)>0$, such that for any other solution $y(t,0,\psi)$ of the system (12) satisfying $\|\emptyset_1-\psi_1\|=\delta_1$, $\|\emptyset_2-\psi_2\|=\delta_2$, then $\|x(t,0,\emptyset)-y(t,0,\emptyset)\|<\in$, $x(t,0,\emptyset)$ is said to be asymptotically stable if it stable and if there is a constant $\delta_1,\delta_2>0$ such that $\|\emptyset_1-\psi_1\|<\delta_1$, $\|\emptyset_2-\psi_2\|<\delta_2$, then

$$\lim_{t\to\infty}x(t,0,\emptyset)-y(t,0,\emptyset)<r,\ r>0.$$

Theorem (2)

Assume the hypotheses (1-9) are hold, and has an asymptotically mild solution

Proof:

Let $x(t)=x(t,0,\emptyset_1,\emptyset_2)$ and $y(t)=y(t,0,\psi_1,\psi_2)$ be a two solutions of equation (12) such that

$$\begin{aligned} x(t) &= C(t)\left(\emptyset_1 - h(x(t))\right) \\ &+S(t)\left[\left(\emptyset_2 - h'(x(t))\right) - g\left(0, x(h_1(0)),0\right)\right] \\ &+\int_0^t S(t-s)\int_0^s Af(t-s)x(t)dsds \\ &+\int_0^t S(t-s)AF_1\left(s,x(h_3(s))\right)dw(s)] \\ &+\int_0^t S(t-s)AF_2\left(s,x(h_3(s))\right)dw^H(s) \\ &-\int_0^t C(t-s)g(s,x(h_1(s)),\int_0^t a\left(t,\tau,x(h_2(\tau)\right)d\tau)ds \end{aligned} \tag{25}$$

and

$$\begin{aligned} y(t) &= C(t)\left(\psi_1 - h(x(t))\right) \\ &+S(t)\left[\psi_2 - h'(x(t)) - g\left(0, y(h_1(0)),0\right)\right] \\ &+\int_0^t S(s-\tau)\int_0^s Af(t-s)y(s)dsds \\ &+\int_0^t S(t-s)AF_1\left(s,x(h_3(s))\right)dw(s) \\ &+\int_0^t S(t-s)AF_2\left(h_3(s)\right)dw^H(s) \\ &+\int_0^t C(t-s)g(s,y(h_1(s)),\int_0^t a\left(t,\tau,y(h_2(\tau)\right)d\tau)ds. \end{aligned} \tag{26}$$

Thus,

$$\begin{aligned} \|x(t)-y(t)\| &\le \|C(t)\|\left(\|\emptyset_1-\psi_1\| + \|h(x(t))-h(y(t))\|\right) \\ &+\|S(t)\|\left(\|\emptyset_2-\psi_2\| + \|h'(x(t))-h'(y(t))\|\right) \\ &+\left\|g\left(0,x(h_1(0)),0\right) - g\left(0,y(h_1(0)),0\right)\right\| \\ &+\int_0^t\int_0^s S(s-\tau)\|f(\tau,\tau_1)\|\|x(\tau_1)-y(\tau_1)\|d\tau_1 d\tau \\ &+\int_0^t \|S(t-s)\|\left\|AF_1\left(s,x(h_3(s))\right) - AF_1\left(s,y(h_3(s))\right)\right\|dw(s) \\ &\le M^{\wedge 2}\left(\|\emptyset_1-\psi_1\| + M_6\|x(t)-y(t)\|\right) \\ &+N^{\wedge 2}\left(\|\emptyset_2-\psi_2\| + M_7\|x(t)-y(t)\|\right) \\ &+M_2\|x(h_1(0))-y(h_1(0))\| \\ &+\int_0^t TN^{\wedge 2}M_1\|x(t)-y(t)\|d\tau \\ &+TTr(Q)N^{\wedge 2}L_1\|x(h_3(s))-y(h_3(s))\|. \end{aligned}$$

Then,

$$\begin{aligned} &\left(1-\begin{pmatrix} M^{\wedge 2}M_6 + N^{\wedge 2}M_7 + M_2 \\ +TTr(Q)N^{\wedge 2} + L_1M_1N^{\wedge 2}T^2 \end{pmatrix}\right)\|x(t)-y(t)\| \\ &\le M^{\wedge 2}\|\emptyset_1-\psi_1\|N^{\wedge 2} + \|\emptyset_2-\psi_2\| \end{aligned} \tag{27}$$

$$\|x(t)-y(t)\| \le \frac{M^{\wedge 2}\|\emptyset_1-\psi_1\| + N^{\wedge 2}\|\emptyset_2-\psi_2\|}{\left(1-\begin{pmatrix} M^{\wedge 2}M_6 + N^{\wedge 2}M_7 + M_2 \\ +T\mathrm{Tr(Q)}N^{\wedge 2}L_1 + L_1M_1N^{\wedge 2}T^2 \end{pmatrix}\right)},$$

Where $1 > \left(M^{\wedge 2}M_6 + N^{\wedge 2}M_7 + M_2 + TTr(Q)N^{\wedge 2}L_1 + L_1M_1N^{\wedge 2}T^2\right)$.

References

[1] Arnold L., "Stochastic Differential Equations; Theory and Applications", John Wiley and Sons, 1974.

[2] Balakrishnan A. V., "Applications of Mathematics: Applied Functional Analysis", 3rd edition, Springer-Verlag, New York, 1976.

[3] Bierens H. J., "Introduction to Hilbert Spaces", Pennsylvania State University, June 24 2007.

[4] Chen M., "Approximate Solutions of Operator Equations", By World Scientific Publishing, Co. Pte. Ltd., 1997.

[5] Coculescu D. and Nikeghbali A., "Filtrations", 2000 Mathematics Subject Classification arXiv:0712.0622v1 [math.PR] , 2007.

[6] Conway, John B., "A course in functional analysis", 2' ed., Springer-Verlag, New York, 1990.

[7] Diagana, T., "An Introduction to Classical and *p*-ADIC Theory of Linear Operators and Applications", Nova Science Publishers, 2006.

[8] Dhage. B.C.,Multi-valued mappings and fixed points II,Tamkang J.Math.37(2006). 27-46.

[9] Erwin, K., "Introduction Functional Analysis with Application", By John Wiley and Sons, 1978.

[10] Einsiedler M. and Ward T., "Functional Analysis Notes", Draft July 2, 2012.

[11] Grippenberg, G. and Norros I., "On The Prediction of Fractional Brownian Motion", Journal of Applied Probability, Vol. 33, No. 2, PP: 400-410, 1996.

[12] Gani J., Heyde C.C., Jagers P. and Kurtz T.G., "Probability and its Applications", Springer-Verlag London Limited, 2008.

[13] Kumlin Peter, "A Note on Fixed Point Theory", TMA 401 / MAN 670 Functional Analysis 2003 /2004.

[14] KressRainer, "Linear Integral Equations",$2'ed$, Springer Science Business Media New York, 1999.

[15] Kisil Vladimir. V "Introduction to Functional Analysis", Courses on Functional Analysis at School of Mathematics of University of Leeds, December 2014 .

[16] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. (2006). Theory and application of fractional differential Equations. Elsevier, Amsterdam.

[17] Lasikcka, I., "Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations", J. Deferential Equation, 47, pp. 246-272, 1983.

[18] Li K., "Stochastic Delay Fractional Evolution Equations Driven by Fractional Brownian Motion", Mathematical Method in the Applied Sciences, 2014.

[19] Mishura Y. S., "Stochastic Calculus for Fractional Brownian Motion and Related Processes", Lect, Notes in Math., 1929, Springer, 2008.

[20] Madsen Henrik, "ito integrals", Aalborg university, Denmark, 2006.

[21] Nualart D., "Fractional Brownian motion: stochastic calculus and Applications", Proceedings of the International Congress of Mathematicians, Madrid, Spain, European Mathematical Society, 2006.

[22] Tudor Ciprian A., "Ito Formula for the Infinite –Dimensional Fractional Brownian Motion", J. Math. Kyoto Univ. (JMKYAZ), Vo. 45, No.3, PP: 531-546, 2005.