The chromatic number of the convex segment disjointness graph*

Dedicat al nostre amic i mestre Ferran Hurtado

Ruy Fabila-Monroy† David R. Wood‡

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Abstract

Let P be a set of n points in general and convex position in the plane. Let D_n be the graph whose vertex set is the set of all line segments with endpoints in P, where disjoint segments are adjacent. The chromatic number of this graph was first studied by Araujo et al. [*CGTA*, 2005]. The previous best bounds are $\frac{3n}{4} \leq \chi(D_n) < n - \sqrt{\frac{n}{2}}$ (ignoring lower order terms). In this paper we improve the lower bound to $\chi(D_n) \geq n - \sqrt{2n}$, to conclude a near-tight bound on $\chi(D_n)$.

1 Introduction

Throughout this paper, P is a set of $n > 3$ points in general and convex position in the plane. The *convex segment disjointness graph*, denoted by D_n, is the graph whose vertex set is the set of all line segments with endpoints in P, where two vertices are adjacent if the corresponding segments are disjoint. Obviously D_n does not depend on the choice of P. This graph and other related graphs, were introduced by Araujo, Dumitrescu, Hurtado, Noy and Urrutia [1], who proved the following bounds on the chromatic number of D_n:

$$2\left\lfloor \tfrac{1}{3}(n+1)\right\rfloor - 1 \leq \chi(D_n) < n - \tfrac{1}{2}\left\lfloor \log n \right\rfloor \quad .$$

Both bounds were improved by Dujmović and Wood [5] to

$$\tfrac{3}{4}(n-2) \leq \chi(D_n) < n - \sqrt{\tfrac{1}{2}n} - \tfrac{1}{2}(\ln n) + 4 \quad .$$

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†Departamento de Matemáticas, Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional, México, D.F., México (ruyfabila@math.cinvestav.edu.mx). Supported by an Endeavour Fellowship from the Department of Education, Employment and Workplace Relations of the Australian Government.

‡Department of Mathematics and Statistics, The Univesity of Melbourne, Melbourne, Australia (woodd@unimelb.edu.au). Supported by a QEII Fellowship from the Australian Research Council.

In this paper we improve the lower bound to conclude near-tight bounds on $\chi(D_n)$.

Theorem 1.

$$n - \sqrt{2n + \tfrac{1}{4}} + \tfrac{1}{2} \leq \chi(D_n) < n - \sqrt{\tfrac{1}{2}n} - \tfrac{1}{2}(\ln n) + 4 \ .$$

The proof of Theorem 1 is based on the observation that eah colour class in a colouring of D_n is a convex thrackle. We then prove that two maximal convex thrackles must share an edge in common. From this we prove a tight upper bound on the number of edges in the union of k maximal convex thrackles. Theorem 1 quickly follows.

2 Convex thrackles

A *convex thrackle* on P is a geometric graph with vertex set P such that every pair of edges intersect; that is, they have a common endpoint or they cross. Observe that a geometric graph H on P is a convex thrackle if and only if $E(H)$ forms an independent set in D_n. A convex thrackle is *maximal* if it is edge-maximal. As illustrated in Figure 1(a), it is well known and easily proved that every maximal convex thrackle T consists of an odd cycle $C(T)$ together with some degree 1 vertices adjacent to vertices of $C(T)$; see [2, 3, 4, 5, 6, 7, 8, 9]. In particular, T has n edges. For each vertex v in $C(T)$, let $W_T(v)$ be the convex wedge with apex v, such that the boundary rays of $W_T(v)$ contain the neighbours of v in $C(T)$. Every degree-1 vertex u of T lies in a unique wedge and the apex of this wedge is the only neighbour of u in T.

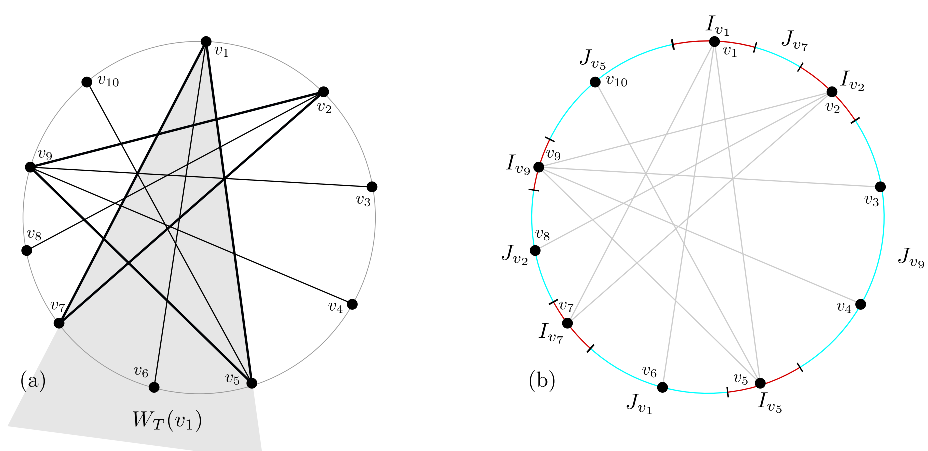

Figure 1: (a) maximal convex thrackle, (b) the intervals pairs (I_u, J_u)

3 Convex thrackles and free $\mathbb{Z}_2$-actions of S^1

A $\mathbb{Z}_2$-*action* on the unit circle S^1 is a homeomorphism $f : S^1 \to S^1$ such that $f(f(x)) = x$ for all $x \in S^1$. We say that f is *free* if $f(x) \neq x$ for all $x \in S^1$.

Lemma 1. *If f and g are free $\mathbb{Z}_2$-actions of S^1, then $f(x) = g(x)$ for some point $x \in S^1$.*

Proof. For points $x, y \in S^1$, let $\overrightarrow{xy}$ be the clockwise arc from x to y in S^1. Let $x_0 \in S^1$. If $f(x_0) = g(x_0)$ then we are done. Now assume that $f(x_0) \neq g(x_0)$. Without loss of generality, $x_0, g(x_0), f(x_0)$ appear in this clockwise order around S^1. Paramaterise $\overrightarrow{x_0g(x_0)}$ with a continuous injective function $p : [0,1] \to \overrightarrow{x_0g(x_0)}$, such that $p(0) = x_0$ and $p(1) = g(x_0)$. Assume that $g(p(t)) \neq f(p(t))$ for all $t \in [0,1]$, otherwise we are done. Since g is free, $p(t) \neq g(p(t))$ for all $t \in [0,1]$. Thus $g(p([0,1])) = \overrightarrow{g(p(0))g(p(1))} = \overrightarrow{g(x_0)x_0}$. Also $f(p([0,1])) = \overrightarrow{f(x_0)f(p(1))}$, as otherwise $g(p(t)) = f(p(t))$ for some $t \in [0,1]$. This implies that $p(t), g(p(t)), f(p(t))$ appear in this clockwise order around S^1. In particular, with $t = 1$, we have $f(p(1)) \in \overrightarrow{x_0g(x_0)}$. Thus $x_0 \in \overrightarrow{f(x_0)f(p(1))}$. Hence $x_0 = f(p(t))$ for some $t \in [0,1]$. Since f is a $\mathbb{Z}_2$-action, $f(x_0) = p(t)$. This is a contradiction since $p(t) \in \overrightarrow{x_0g(x_0)}$ but $f(x_0) \notin \overrightarrow{x_0g(x_0)}$. □

Assume that P lies on S^1. Let T be a maximal convex thrackle on P. As illustrated in Figure 1(b), for each vertex u in $C(T)$, let (I_u, J_u) be a pair of closed intervals of S^1 defined as follows. Interval I_u contains u and bounded by the points of S^1 that are 1/3 of the way towards the first points of P in the clockwise and anticlockwise direction from u. Let v and w be the neighbours of u in $C(T)$, so that v is before w in the clockwise direction from u. Let p be the endpoint of I_v in the clockwise direction from v. Let q be the endpoint of I_w in the anticlockwise direction from w. Then J_u is the interval bounded by p and q and not containing u. Define $f_T : S^1 \longrightarrow S^1$ as follows. For each $v \in C(T)$, map the anticlockwise endpoint of I_v to the anticlockwise endpoint of J_v, map the clockwise endpoint of I_v to the clockwise endpoint of J_v, and extend f_T linearly for the interior points of I_v and J_v, such that $f_T(I_v) = J_v$ and $f_T(J_v) = I_v$. Since the intervals I_v and J_v are disjoint, f_T is a free $\mathbb{Z}_2$-action of S^1.

Lemma 2. *Let T_1 and T_2 be maximal convex thrackles on P, such that $C(T_1) \cap C(T_2) = \emptyset$. Then there is an edge in $T_1 \cap T_2$, with one endpoint in $C(T_1)$ and one endpoint in $C(T_2)$.*

Topological proof. By Lemma 1, there exists $x \in S^1$ such that $f_{T_1}(x) = y = f_{T_2}(x)$. Let $u \in C(T_1)$ and $v \in C(T_2)$ so that $x \in I_u \cup J_u$ and $x \in I_v \cup J_v$, where (I_u, J_u) and (I_v, J_v) are defined with respect to T_1 and T_2 respectively. Since $C(T_1) \cap C(T_2) = \emptyset$, we have $u \neq v$ and $I_u \cap I_v = \emptyset$. Thus $x \notin I_u \cap I_v$. If $x \in J_u \cap J_v$ then $y \in I_u \cap I_v$, implying $u = v$. Thus $x \notin J_u \cap J_v$. Hence $x \in (I_u \cap J_v) \cup (J_u \cap I_v)$. Without loss of generality, $x \in I_u \cap J_v$. Thus $y \in J_u \cap I_v$. If $I_u \cap J_v = \{x\}$ then x is an endpoint of both I_u and J_v, implying $u \in C(T_2)$, which is a contradiction. Thus $I_u \cap J_v$ contains points other than x. It follows that $I_u \subset J_v$ and $I_v \subset J_u$. Therefore the edge uv is in both T_1 and T_2. Moreover one endpoint of uv is in $C(T_1)$ and one endpoint is in $C(T_2)$. □

Combinatorial Proof. Let H be the directed multigraph with vertex set $C(T_1) \cup C(T_2)$, where there is a *blue* arc uv in H if u is in $W_{T_1}(v)$ and there is a *red* arc uv in H if u is in $W_{T_2}(v)$. Since $C(T_1) \cap C(T_2) = \emptyset$, every vertex of H has outdegree 1. Therefore $|E(H)| = |V(H)|$ and there is a cycle Γ in the undirected multigraph underlying H. In fact, since every vertex has outdegree 1, Γ is a directed cycle. By construction, vertices in H are not incident to an incoming and an outgoing edge of the same color. Thus Γ alternates between blue and red arcs. The red edges of Γ form a matching as well as the blue edges, both of which are thrackles. However, there is only one matching thrackle on a set of points in convex position. Therefore Γ is a 2-cycle and the result follows. □

4 Main Results

Theorem 2. *For every set P of n points in convex and general position, the union of k maximal convex thrackles on P has at most $kn - \binom{k}{2}$ edges.*

Proof. For a set $\mathcal{T}$ of k maximal convex thrackles on P, define

$$r(\mathcal{T}) := |\{(v, T_i, T_j) : v \in C(T_i) \cap C(T_j), T_i, T_j \in \mathcal{T} \text{ and } T_i \neq T_j\}| \quad .$$

The proof proceeds by induction on $r(\mathcal{T})$.

Suppose that $r(\mathcal{T}) = 0$. Thus $C(T_i) \cap C(T_j) = \emptyset$ for all distinct $T_i, T_j \in \mathcal{T}$. By Lemma 2, T_i and T_j have an edge in common, with one endpoint in $C(T_i)$ and one endpoint in $C(T_j)$. Hence distinct pairs of thrackles have distinct edges in common. Since every maximal convex thrackle has n edges and we overcount at least one edge for every pair, the total number of edges is at most $kn - \binom{k}{2}$.

Now assume that $r(\mathcal{T}) > 0$. Thus there is a vertex v and a pair of thrackles T_i and T_j, such that $v \in C(T_i) \cap C(T_j)$. As illustrated in Figure 2, replace v by two consecutive vertices v' and v'' on P, where v' replaces v in every thrackle except T_j, and v'' replaces v in T_j. Add one edge to each thrackle so that it is maximal. Let $\mathcal{T}'$ be the resulting set of thrackles. Observe that $r(\mathcal{T}') = r(\mathcal{T}) - 1$, and the number of edges in $\mathcal{T}'$ equals the number of edges in $\mathcal{T}$ plus k. By induction, $\mathcal{T}'$ has at most $k(n+1) - \binom{k}{2}$ edges, implying $\mathcal{T}$ has at most $kn - \binom{k}{2}$ edges. □

We now show that Theorem 2 is best possible for all $n \geq 2k$. Let S be a set of k vertices in P with no two consecutive vertices in S. If $v \in S$ and x, v, y are consecutive in this order in P, then $T_v := \{vw : w \in P \setminus \{v\})\} \cup \{xy\}$ is a maximal convex thrackle, and $\{T_v : v \in S\}$ has exactly $kn - \binom{k}{2}$ edges in total.

Proof of Theorem 1. If $\chi(D_n) = k$ then, there are k convex thrackles whose union is the complete geometric graph on P. Possibly add edges to obtain k maximal convex thrackles with $\binom{n}{2}$ edges in total. By Theorem 2, $\binom{n}{2} \leq kn - \binom{k}{2}$. The quadratic formula implies the result. □

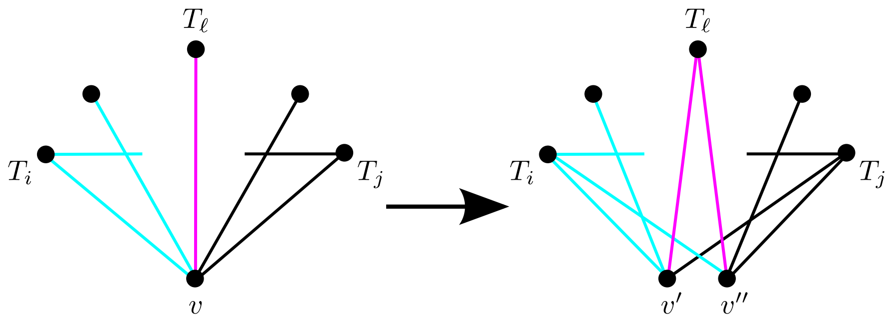

Figure 2: Construction in the proof of Theorem 2.

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