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Optimal Relative Timing of Stance Push-Off and Swing Leg Retraction

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***Abstract*— Swing leg retraction, the backward rotation of the swing leg prior to heel-strike, is known to have several advantages in legged locomotion. To achieve this motion, a hip torque is required at the end of the swing phase to brake the forward rotation of the leg and/or accelerate its backward motion. In walking, pre-emptive push-off of the stance leg also occurs at the end of the swing, so its relative timing with late-swing retracting torque influences gait energetics. To find the best relative timing between the stance leg's push-off force and the swing leg retraction torque, we calculate their work-based energetics in a simple bipedal model using impulsive approximations and with the aid of the so-called overlap parameter that quantifies the relative order and the percentage overlap of the push-off and retraction impulses. By minimizing the energetic cost of the gait, we found that it is energetically favorable to start with the push-off force, and postpone braking the leg swing until completely after the push-off (impulsive force/torque). The implication for the more realistic non-impulsive cases is to apply the retraction torque at the very end of the push-off before heel-strike. We show that the results are valid for many other bipedal models, for both periodic and aperiodic gaits, and regardless of the actuator efficiencies for positive and negative work.**

I. Introduction

Although legged robots might be significantly different from their biological counterparts, both groups are subject to the same mechanics principles governing their motions. Therefore, the insights achieved from the analysis of human and animal locomotion, especially the gait characteristics that are common in many organisms, may help us enhance the performance of legged robots.

A. Swing leg retraction: a common gait characteristic in biological legged locomotion

When walking or running, humans rotate their swing leg forward (protraction) during most of the swing phase, and just before the swing foot hits the ground they apply a hip torque to brake and even reverse the rotation of the swing leg [1]. This late-swing rotation reversal, known as *swing leg retraction*, is also observed in different gaits of numerous terrestrial animals [2]. In fast gaits, *e.g.* running, galloping, or trotting, swing leg retraction can be easily seen. In walking, because of the short duration of retraction (about 20% of the gait period in humans) and small leg angular velocities at the end of the swing, the changes in leg orientation caused by retraction are not large enough to be easily seen by eyes or even on videos. However, swing leg retraction in walking can be clearly seen when hip angle motions are quantified.

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B. The advantages of swing leg retraction

Why is leg retraction a common characteristic among different gaits of many legged organisms? Previous studies using theoretical models and physical robots have revealed several benefits of swing leg retraction.

Seyfarth *et. al.* [3] showed that swing leg retraction improves running stability by substantially increasing the range of parameters (*e.g.* average speed, apex height) for which stable periodic running is possible. For a simple bipedal model walking passively down a shallow slope, Wisse *et. al.* [4] showed that all gaits without a retraction phase are consistently unstable, whereas a mild retraction speed improves the stability by reducing the size of the eigenvalues of the Jacobian matrix of the Poincare map. Furthermore, smaller eigenvalues due to a mild retraction speed can also result in a faster transient response, implying faster recovery from small disturbances [5]. For larger disturbances, Hobbelen and Wisse [5] found that the magnitude of a random step-height disturbance that causes a walking model fall is greatest at mild retraction speeds. A similar result is verified by Karssen *et. al.* [6] for a simple running model.

The latter study also shows that swing leg retraction can potentially reduce the overall energy expenditure of running, the risk of slippage at heel-strike, and the tangential (normal to the leg) ground reaction force at heel-strike. Another benefit of active swing leg retraction is to improve the state estimation by moving the foot more vertical at heel-strike, increasing the accuracy of the predicted time of heel-strike in presence of terrain irregularities or model uncertainties [7].

C. When to apply hip torque to retract the swing leg?

Most of the advantages of swing leg retraction are achieved or maximized at certain ranges or values of retraction speed [6]. Therefore, hip torque is required at the end of the swing phase to brake the leg swing and/or regulate its reversed rotation at an optimal speed. In walking, the pre-emptive push-off, which is a key part of energy effective walking, also takes place at the end of swing phase (single stance). In a multi-body mechanism forces and/or torques at different joints are mechanically coupled, so their relative timing potentially influences the gait energetics. For example, the energetic cost of the step-to-step transition in walking (transition from one stance leg to the other) changes substantially when the order of impulses on leading and trailing legs is changed from (a) heel-strike followed by push-off to (b) push-off then heel-strike [8]. Therefore, the question arises what is the best (energetically optimum) relative timing of the pre-emptive push-off force and the retraction hip torque to achieve a given swing leg retraction

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speed prior to heel-strike? Should the retraction torque be just after, just before, or synchronous with the pre-emptive push-off force?

D. An overview of the solution strategy

We study the problem of optimal relative timing of the push-off force and the retraction torque using different mechanical models probed with exact analytic methods. We start with a very simple model to investigate the underlying principles, and then generalize the results to more elaborate and realistic models. The procedure is as follows: First, we approximate the retraction torque and the push-off force as impulsive torque and force quantified by their impulse. Next, the influence of the retraction and push-off impulses on the joint velocities are determined. Then, the energetics of each impulse is calculated using a work-based energetic cost model and through the so-called 'overlap parameter' [8] that quantifies the order and percentage overlap of the retraction torque and the push-off force. Finally, given the push-off and retraction impulses, the optimal relative timing of push-off and swing retraction is calculated by minimizing the net energetic cost of walking. The result is valid for both periodic and aperiodic gaits.

II. The Simple Bipedal Model

The first model used in this paper is shown in Fig. 1. It consists of two identical rigid legs, each with length ℓ and mass m. The leg mass is arbitrarily distributed with no restrictions on its inertia or the location of its center of mass.

This biped is powered with two actuators: (i) a prismatic actuator on the stance leg (not shown in Fig. 1), applying the extensional force F along the leg, and (ii) a revolute actuator at the hip, applying the torque τ between the legs. In general, these actuators can have many arbitrary force/torque profiles (as long as walking is feasible), where each combination results in a different energetic cost. However, the energy optimality requirement restricts our choice.

A. Energy minimizing actuation profiles

To identify the optimal profiles for leg force and hip torque in our model, consider the following observations from both human experimental data and previously done numerical optimizations.

- Nonlinear numerical optimizations with simpler or even more elaborate models [9], [10], [11] have shown that for an energy efficient walking gait the stance leg should be extended (push-off) before the swing leg hits the ground (heel-strike). When walking, humans start extending their trailing leg's ankle and knee joints at the end of single stance [12]. The peak activity of the resultant push-off force is very close to heel-strike.
- The energy minimizing hip torque has a bang-coast-bang profile, with peak activities at the beginning and end of the swing [11]. Experimental data from human walking show that during swing the leg motion is almost passive except at the start and the end of the swing phase where the hip muscles have burst activities to accelerate and decelerate/retract the swing leg [12].
- For a given task, application of the force/torque in a shorter period of time with a larger magnitude, as an impulsive force/torque, tends to minimize the mechanical work [9], [11].

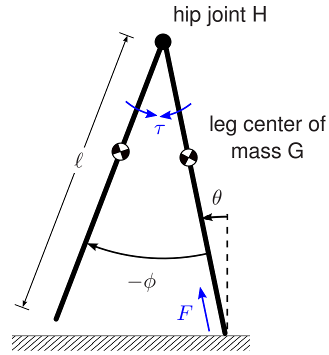

Fig. 1. The simple bipedal model consists of two rigid legs of length ℓ and mass m. The leg mass has an arbitrary distribution with no restriction on the location of its center of mass. The prismatic actuator on the stance leg (not shown in the figure) can apply the extensional force F along the leg and the revolute actuator at the hip can apply torque τ between the legs.

Based on these observations, for a work-based energetic cost the energy optimal walking gait is identified as the one with (i) a burst extensional push-off force applied just prior to heel-strike by the stance leg's prismatic actuator, and (ii) two burst hip torques applied at the beginning of swing (just after the toe-off) and again at the end of the swing (just before heel-strike). The first burst torque is intended to accelerate the swing leg to achieve the desired swing frequency and step length, and the second (referred to as the retraction torque) is aimed to brake and/or reverse the leg swing.

B. Simplification to impulses

Because the burst push-off force and the burst retraction torque are applied over a short period of time, the biped configuration changes a little, or not at all, during their application, whereas velocity changes are more noticeable. For the sake of simplicity, we approximate the burst push-off force and the burst retraction torque as theoretical impulsive (infinitesimal duration with infinite magnitude) force and torque that cause discontinuous velocity jumps in an exactly fixed biped configuration. Although this approximation is unrealistic, the insights achieved from the analysis of the resulting simplified models are an important step in improving our understanding of effective legged locomotion. For example, the impulsive models have been previously used to explore the energetic benefits of pre-emptive push-off in bipedal walking [8], [13] and the foot sequencing in animal gaits [8], and the energetic consequence of step-to-step transitions in human walking [14].

Fig. 2a shows the biped at the end of swing phase (single stance) when the impulsive push-off force and the impulsive retraction torque are applied on the biped during an infinitesimal period just before heel-strike. The impulsive push-off

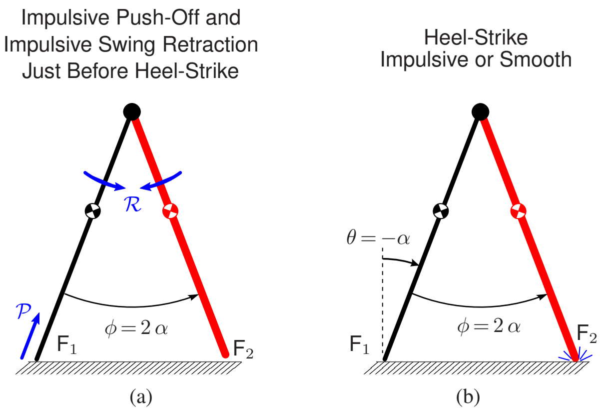

Fig. 2. The biped configuration and the impulsive actuations at the end of swing. The leg transitioning from swing to stance is thick red. **a)** At the end of single stance, when $\phi = 2\,\alpha$, an impulsive force is applied along the stance leg by the leg prismatic actuator (not shown in the figure), and an impulsive retracting (contracting the hip angle) torque is applied by the hip actuator. $\mathcal{P}$ and $\mathcal{R}$ are the impulse intensity (time integral) of these force and torque. **b)** The swing foot (F_2) touches the ground immediately after push-off impulse $\mathcal{P}$ and retraction impulse $\mathcal{R}$ are completed.

force and the impulsive retraction torque are quantified by their net impulse denoted by $\mathcal{P}$ and $\mathcal{R}$. If we denote the time instant just before both impulsive actions by $t^-_{\rm pr}$ and the time instant just before heel-strike (just after both impulsive actions are completed) by $t^-_{\rm h}$, the push-off impulse $\mathcal{P}$ and the retraction impulse $\mathcal{R}$ are given by

$$\mathcal{P} = \int_{t^-_{\rm pr}}^{t^-_{\rm h}} F(t)\,dt \tag{1}$$

$$\mathcal{R} = -\int_{t^-_{\rm pr}}^{t^-_{\rm h}} \tau(t)\,dt. \tag{2}$$

We assume that the sign of the push-off force and the retraction torque does not change during their application. Therefore, based on our convention in Fig. 1 for the positive directions, the push-off force F is always positive, and the retraction torque τ is always negative. Hence, $\mathcal{P} \geqslant 0$ and $\mathcal{R} \geqslant 0$. Although the impulsive push-off and retraction actions take place at the same biped configuration (Fig. 2a), we treat $\mathcal{P}$ and $\mathcal{R}$ as isolated in time (one after the other) or as having some (specified) overlap with each other (to be discussed later in detail). We will show in section IV that the relative timing of these two impulses can change the energetics of the gait.

Immediately after both impulsive actions are completed the swing foot hits the ground (Fig. 2b). Our analysis in this paper does not depend on whether heel-strike is assumed to be impulsive or non-impulsive.

III. Details of The Dynamics

Our focus in this section is on the final moment of the swing phase, when the impulsive push-off force and the impulsive retraction torque are applied on the biped just before heel-strike. We use $t^-_{\rm p}$ and $t^+_{\rm p}$ to denote the time instants just before and just after the impulsive push-off $\mathcal{P}$, and $t^-_{\rm r}$ and $t^+_{\rm r}$ to denote the time instants just before and just after the impulsive retraction $\mathcal{R}$. Because these impulsive actions take place during the infinitesimal interval $(t^-_{\rm pr}, t^-_{\rm h})$, we have $t^-_{\rm pr} = \min(t^-_{\rm p}, t^-_{\rm r})$, and $t^-_{\rm h} = \max(t^+_{\rm r}, t^+_{\rm p})$.

During the impulsive actions the stance foot is free to move along the leg, but not orthogonal to the leg, respecting the prismatic-actuator model. Due to the impulsive push-off $\mathcal{P}$, the stance leg's extension rate is positive after the push-off, *i.e.* $\dot{\ell} > 0$. To keep the prismatic assumption that the tangential (orthogonal to the leg) foot velocity after push-off is zero, an induced tangential constraint impulse is applied on the foot at the time of push-off.

Although the magnitude of the push-off force is infinite, the biped does not take off from the ground if a new foot contact is made by the leading leg just after the impulsive push-off and retraction, *i.e.* at $t^-_{\rm h}$. This is possible if the retraction impulse $\mathcal{R}$ is large enough to push the leg backwards so that the leading foot is moving downwards at $t^-_{\rm h}$.

A. Joint velocities just before heel-strike

To find the joint velocities at $t^-_{\rm h}$ (just after both impulsive actions) in terms of the kinematic variables at $t^-_{\rm pr}$ and the impulse magnitudes $\mathcal{P}$ and $\mathcal{R}$, we integrate the biped's equations of motion (EoM) over the infinitesimal interval $(t^-_{\rm pr}, t^-_{\rm h})$. Throughout the impulsive push-off and retraction the model dynamics follow the EoM in single stance (Fig. 1) which can be obtained from (i) linear momentum balance equation of the whole mechanism along the stance leg, (ii) angular momentum balance (AMB) equation of the swing leg about the hip joint, and (iii) AMB equation of the whole mechanism about the stance foot. After rearrangement, these three equations can be written in the following standard form:

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} F \\ \tau \\ 0 \end{bmatrix}, \tag{3}$$

where the vector $\mathbf{q}(t) = [\,\ell(t),\ \phi(t),\ \theta(t)\,]^{\rm T}$ determines the biped configuration, $\mathbf{M}$ is the symmetric mass matrix, the vector $\mathbf{h}$ includes the Coriolis, centrifugal and gravity terms, and the hip torque τ and the stance leg force F have impulsive profiles in $t^-_{\rm pr} \leqslant t \leqslant t^-_{\rm h}$ with impulse magnitudes given by (1) and (2). Because velocities are always bounded by assumption, and the configuration vector $\mathbf{q}$ remains unchanged between $t^-_{\rm pr}$ and $t^-_{\rm h}$, integrating both sides of (3) over the infinitesimal interval $(t^-_{\rm pr}, t^-_{\rm h})$ results in

$$\mathbf{M}_{t^-_{\rm pr}} \cdot \left(\dot{\mathbf{q}}_{t^-_{\rm h}} - \dot{\mathbf{q}}_{t^-_{\rm pr}}\right) = \begin{bmatrix} \mathcal{P} \\ -\mathcal{R} \\ 0 \end{bmatrix}, \tag{4}$$

where $\dot{\mathbf{q}}_{t^-_{\rm pr}} = \dot{\mathbf{q}}(t^-_{\rm pr})$, $\dot{\mathbf{q}}_{t^-_{\rm h}} = \dot{\mathbf{q}}(t^-_{\rm h})$, and $\mathbf{M}_{t^-_{\rm pr}} = \mathbf{M}(\mathbf{q}_{t^-_{\rm pr}})$. Therefore, given the push-off impulse $\mathcal{P}$ and the retraction impulse $\mathcal{R}$, the velocities just before heel-strike are given by

$$\dot{\mathbf{q}}_{t^-_{\rm h}} = \dot{\mathbf{q}}_{t^-_{\rm pr}} + \mathbf{M}^{-1}_{t^-_{\rm pr}} \begin{bmatrix} \mathcal{P} \\ -\mathcal{R} \\ 0 \end{bmatrix}. \tag{5}$$

Our main interest here is how the joint velocities are influenced by the push-off and retraction impulses. Thus, we expand and simplify (5) as below, where for the sake of simplicity the leg extension rate $\dot{\ell}$ and the hip angular velocity $\dot{\phi}$ are denoted by v, and ω, correspondingly.

$$\begin{bmatrix} v_{t_{\mathrm{h}}^-} \\ \omega_{t_{\mathrm{h}}^-} \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_{t_{\mathrm{pr}}^-} \end{bmatrix} + \begin{bmatrix} J_{v/P} & J_{v/R} \\ J_{\omega/P} & J_{\omega/R} \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{R} \end{bmatrix} \tag{6}$$

In this equation, the 2×2 impulse-influence matrix is obtained from $\mathbf{M}_{t_{\mathrm{pr}}^-}^{-1}$ after dropping its last row and last column and inverting the sign of its second column. The properties of this velocity map will be discussed in section III-C.

B. Joint velocities during the impulsive actions

In section IV we are going to calculate the work done by the impulsive push-off force and the impulsive retraction torque, so we need to find how the joint velocities change *during* their application period. The velocity mapping in (6) does not provide all the information required, because it gives the joint velocities only at t_{h}^-, when both the impulsive actions are completed, whereas we need to find the joint velocities in all intermediate time instants from the beginning to the end of the applied force/torque. For example, to calculate the work done by the retraction impulse $\mathcal{R}$, we need to calculate the hip angular velocity $\omega(t)$ for all t inside the infinitesimal interval $(t_{\mathrm{r}}^-, t_{\mathrm{r}}^+)$.

To find the joint velocities during the application of the push-off impulse $\mathcal{P}$ and the retraction impulse $\mathcal{R}$, we adopt the concept of *partial impulses* from [8], and define the partial push-off impulse $\tilde{\mathcal{P}}(t)$ and the partial retraction impulse $\tilde{\mathcal{R}}(t)$ as

$$\tilde{\mathcal{P}}(t) = \int_{t_{\mathrm{pr}}^-}^{t} F(t')\,dt' = p(t)\,\mathcal{P}, \tag{7}$$

$$\tilde{\mathcal{R}}(t) = -\int_{t_{\mathrm{pr}}^-}^{t} \tau(t')\,dt' = r(t)\,\mathcal{R}, \tag{8}$$

where the integration upper limit t satisfies $t_{\mathrm{pr}}^- \leqslant t \leqslant t_{\mathrm{h}}^-$, and the time-dependent parameters p and r express the degree of impulse completeness and satisfy $0 \leqslant p \leqslant 1$ and $0 \leqslant r \leqslant 1$. For example, at t_{p}^-, when the impulsive push-off force is going to start, we have $p(t_{\mathrm{p}}^-) = 0 = \tilde{\mathcal{P}}$, and at t_{p}^+, when the impulsive push-off force has completed, we have $p(t_{\mathrm{p}}^+) = 1$ and $\tilde{\mathcal{P}} = \mathcal{P}$. Fig. 3 shows the partial push-off and retraction impulses as the partial area under the force/torque curves in an arbitrary scenario. In this figure the infinitesimal intervals $(t_{\mathrm{r}}^-, t_{\mathrm{r}}^+)$ and $(t_{\mathrm{p}}^-, t_{\mathrm{p}}^+)$ are exaggerated for clarity of illustration.

Now, following the method we used to derive the velocity map (6), we integrate the EoM in (3) over the infinitesimal interval (t_{pr}^-, t) to get the intermediate joint velocities as

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_{t_{\mathrm{pr}}^-} \end{bmatrix} + \begin{bmatrix} J_{v/P} & J_{v/R} \\ J_{\omega/P} & J_{\omega/R} \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{P}}(t) \\ \tilde{\mathcal{R}}(t) \end{bmatrix} \tag{9}$$

The 2×2 impulse-influence matrix in the above equation is the same as that in (6). Thus, if $\tilde{\mathcal{P}} = \mathcal{P}$ and $\tilde{\mathcal{R}} = \mathcal{R}$, or equivalently $t = t_{\mathrm{h}}^-$, both velocity maps in (6) and (9) become equal.

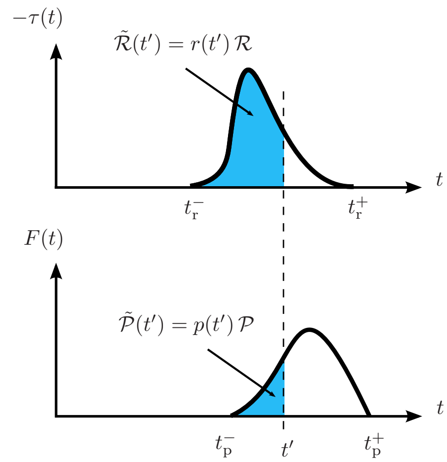

Fig. 3. The partial retraction impulse $\tilde{\mathcal{R}}$ and the partial push-off impulse $\tilde{\mathcal{P}}$ as the partial area under the force/torque curves for the arbitrary profiles of the impulsive retraction torque and the impulsive push-off force. The timing between the two impulses is also arbitrary. The length of the periods over which the push-off force and the retraction torque are applied is infinitesimal, but is exaggerated here for clarity of illustration.

C. Properties of the push-off-retraction velocity map

1) The joint velocities just before heel-strike are independent of the relative timing of the impulsive push-off and retraction. Given the step angle α, the push-off impulse $\mathcal{P}^*$, the retraction impulse $\mathcal{R}^*$, and the initial hip angular velocity $\omega_{t_{\mathrm{pr}}^-}$ (before both impulses), no matter when the impulsive push-off force is applied relative to the impulsive retraction torque (*i.e.* completely before, completely after, or with any arbitrary overlap), the final joint velocities just before heel-strike are always the same, calculated by substituting $\mathcal{P} = \mathcal{P}^*$ and $\mathcal{R} = \mathcal{R}^*$ in (6).

2) The relative timing of the impulsive push-off force and retraction torque changes the instantaneous joint velocities during their application. Given the step angle α, the push-off impulse $\mathcal{P}^*$, the retraction impulse $\mathcal{R}^*$, and the initial hip angular velocity $\omega_{t_{\mathrm{pr}}^-}$ (before both impulses), if the impulsive push-off force is applied completely before the impulsive retraction torque (*i.e.* $t_{\mathrm{p}}^+ = t_{\mathrm{r}}^-$), the joint velocities just before and just after push-off are calculated by evaluating (9) with (i) $\tilde{\mathcal{P}} = \tilde{\mathcal{R}} = 0$, and (ii) $\tilde{\mathcal{P}} = \mathcal{P}^*$, $\tilde{\mathcal{R}} = 0$, respectively. For instance, for the leg extension rate, these substitutions give $v_{t_{\mathrm{p}}^-} = 0$ and $v_{t_{\mathrm{p}}^+} = J_{v/P}\,\mathcal{P}^*$. Now, if the impulsive push-off force is applied completely after the impulsive retraction torque, the joint velocities just before and just after push-off are calculated respectively by evaluating (9) with (i) $\tilde{\mathcal{R}} = \mathcal{R}^*$, $\tilde{\mathcal{P}} = 0$, and (ii) $\tilde{\mathcal{P}} = \mathcal{P}^*$, $\tilde{\mathcal{R}} = \mathcal{R}^*$. For instance, with these substitutions we get $v_{t_{\mathrm{p}}^-} = J_{v/R}\,\mathcal{R}^*$ and $v_{t_{\mathrm{p}}^+} = J_{v/P}\,\mathcal{P}^* + J_{v/R}\,\mathcal{R}^*$. Comparing these results shows that unlike the final joint velocities just before heel-strike, the joint velocities during the impulsive push-off force and the impulsive retraction torque *do change* with the relative timing of push-off and retraction. This has consequences for the work done by each impulse, which will be discussed in the next section.

3) Both the push-off and retraction impulses increase the stance leg's extension rate and decrease the hip joint's angular velocity. The push-off force increases the leg extension rate v, and the retraction torque tends to decrease/reverse the hip angular velocity ω (contracting the leg splay angle). Therefore, $J_{v/P} > 0$ and $J_{\omega/R} < 0$. Also, the push-off impulse $\mathcal{P}$ accelerates the hip along the stance leg, applying a clockwise (negative) torque on the swing leg which, in turn, tends to decrease/reverse ω. This fact together with the symmetry of the mass matrix $\mathbf{M}$ imply $J_{v/R} = -J_{\omega/P} > 0$. Thus, during the application of push-off force and retraction torque the leg extension rate $v(t)$, given by (9), is always positive and increasing, while the hip angular velocity $\omega(t)$ is continuously decreasing, with only one possible zero-crossing. These properties will be used in the following sections to find the optimal relative timing of push-off and retraction.

IV. Energetic Cost of Push-Off and Retraction

We use a work-based energetic cost in which the cost of doing mechanical work by an actuator is equal to the energy consumed by that actuator during the action. This, in turn, depends on the efficiency of the corresponding actuator. In general, an actuator can have different efficiencies for doing positive and negative work. In most actuators negative mechanical work on the actuator output acts as an energy source that partially or entirely compensates for the actuator's internal losses (*e.g.* resistive, frictional). In this case less energy needs to be supplied to the actuator compared to the case when the same amount of positive work is done by the actuator (the input energy to the actuator should cover both the output mechanical work and the losses). Therefore, the efficiency of negative work is normally more than that of positive work. For example, in human muscles the average positive and negative work efficiencies are respectively 25% and -120% [8], [15].

Based on the above discussion, the total energetic cost associated with doing W^+ positive work and W^- negative work ($W^- < 0$) by an actuator is defined as:

$$E = e_1\, W^+ - e_2\, W^-, \qquad (10)$$

where the positive coefficients e_1 and e_2 are, respectively, the *average* energetic cost of unit positive and negative work ($1/e_1$ and $1/e_2$ are the corresponding *average* efficiencies).

A. Work of the push-off and retraction impulses

Our analysis in this paper is based on the assumption that the sign of the push-off force and the retraction torque does not change during their application, *i.e.* for all $t_{\rm pr}^- \leqslant t \leqslant t_{\rm h}^-$, the push-off force $F(t) \geqslant 0$, and the retraction torque $-\tau(t) \geqslant 0$. On the other hand, during push-off and retraction the leg extension rate $v(t)$ is always positive and increasing, while the hip angular velocity $\omega(t)$ is always decreasing and can have at most one zero-crossing (see section III-C). Therefore, the push-off force does only positive work (accelerating the center of mass), whereas the retraction torque can do both positive and negative work. If at the beginning of the retraction torque the hip angle is decreasing, *i.e.* $\omega_{t_{\rm r}^-} < 0$, the retraction torque does only positive work (accelerating the swing leg). However, when $\omega_{t_{\rm r}^-} > 0$, the retraction torque starts with doing negative work (decelerating the swing leg), and then switches to positive work (accelerating the swing leg) when ω becomes negative.

When the impulsive push-off force and retraction torque are isolated, the net work done by each impulse is given by the change in total kinetic energy of the system throughout the application of the corresponding force/torque. However, when the push-off force and the retraction torque overlap, the change in kinetic energy of the system does not give the work done by each impulsive action but the net work done by both. In these cases a technique introduced by Ruina *et. al.* [8] can be used to calculate the work done by each of the overlapping impulsive push-off force and retraction torque. The details are explained below.

Given the retraction impulse $\mathcal{R}$, we can calculate the net work done by the impulsive retraction torque as

$$W_{\mathcal{R}} = \int_{t_{\rm r}^-}^{t_{\rm r}^+} \omega(t)\, \tau(t)\, dt = -\int_0^{\mathcal{R}} \omega(t)\, d\tilde{\mathcal{R}}, \qquad (11)$$

where the second equality is obtained using $d\tilde{\mathcal{R}} = -\tau(t)\, dt$ from the definition of the partial retraction impulse in (8). Now, given the hip angular velocity $\omega_{t_{\rm pr}^-}$ at the beginning of impulsive actions, the net retraction work $W_{\mathcal{R}}$ in (11) can be simplified by substituting for $\omega(t)$ from (9):

$$\begin{aligned} W_{\mathcal{R}} &= -\int_0^{\mathcal{R}} \Big(\omega_{t_{\rm pr}^-} + J_{\omega/R}\, \tilde{\mathcal{R}} + J_{\omega/P}\, \tilde{\mathcal{P}}\Big)\, d\tilde{\mathcal{R}} \\ &= -\int_0^{1} \Big(\omega_{t_{\rm pr}^-} + J_{\omega/R}\, r\, \mathcal{R} + J_{\omega/P}\, p\, \mathcal{P}\Big)\, \mathcal{R}\, dr \\ &= -\omega_{t_{\rm pr}^-}\, \mathcal{R} - \frac{1}{2}\, J_{\omega/R}\, \mathcal{R}^2 - s\, J_{\omega/P}\, \mathcal{R}\, \mathcal{P}, \end{aligned} \qquad (12)$$

where

$$s = \int_0^1 p\, dr. \qquad (13)$$

The parameter s satisfies $0 \leqslant s \leqslant 1$, and quantifies the percentage overlap between the impulsive push-off force and retraction torque. In Fig. 4b the overlap parameter s corresponds to the area under the cross-plots of p *vs.* r for different episodes of the impulsive push-off force and retraction torque shown in Fig. 4a. When $s = 0$, the impulsive retraction torque occurs completely before the impulsive push-off force, whereas $s = 1$ corresponds to the case where the impulsive retraction torque starts completely after the impulsive push-off force. In both of these cases the push-off and retraction impulses are isolated. Two impulses are 'synchronous' if they are proportional (episode iv), resulting in $p = r$ and $s = 0.5$.

Following the same method used in (11) and (12), the net work done by a given push-off impulse $\mathcal{P}$ is calculated as

$$W_{\mathcal{P}} = \frac{1}{2}\, J_{v/P}\, \mathcal{P}^2 + (1-s)\, J_{v/R}\, \mathcal{R}\, \mathcal{P}, \qquad (14)$$

where s is given by (13).

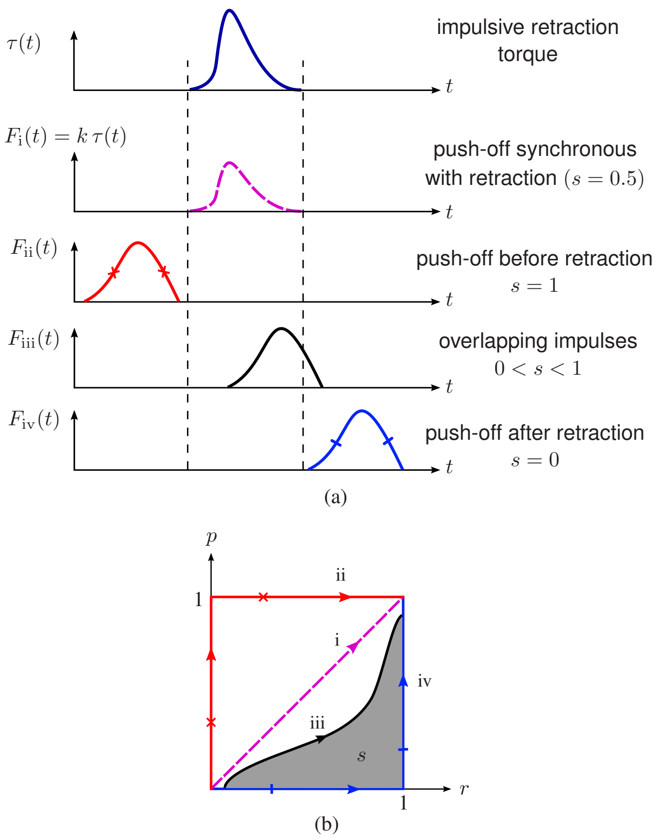

Fig. 4. Visualization of the overlap parameter s. (**a**) As the impulsive push-off force F moves relative to the impulsive retraction torque τ, the overlap parameter s changes from 0 to 1. (**b**) The overlap parameter s can be considered as the area under the cross-plot of p *vs.* r. Different paths in (**b**) correspond to different episodes in (**a**). Impulsive F and τ are 'synchronous' if they are proportional, and thus $s = 0.5$.

For the push-off force the positive work and the net work $W_{\mathcal{P}}$ are always equal. If the hip angular velocity ω has no zero-crossing during the retraction torque, then $W_{\mathcal{R}}$ in (12) gives the positive or negative work done by $\mathcal{R}$, otherwise the individual positive and negative work of the retraction torque should be calculated separately (see the Appendix).

V. Optimal Order of the Push-off Force and the Retraction Torque

Given the push-off impulse $\mathcal{P}$, the retraction impulse $\mathcal{R}$, the initial hip angular velocity $\omega_{t_{\mathrm{pr}}^-}$ before both the impulsive actions, and the unit positive and negative work costs e_1 and e_2, the *optimal* overlap between the impulsive push-off force and the impulsive retraction torque, quantified by s^*, is the one that minimizes the energy expenditure of the gait. Because the joint velocities just before heel-strike (at t_{h}^-) and thereafter are independent of the overlap parameter s, the mechanical work done during the rest of the gait is not influenced by the relative timing of the impulsive push-off force and retraction torque. Therefore, the optimal s is determined by the net energetic cost of the two impulses, *i.e.* $E_{\mathcal{P}} + E_{\mathcal{R}} = E_{\mathcal{PR}}$, where $E_{\mathcal{P}}$ and $E_{\mathcal{R}}$ are given by (10).

Based on the work done by a given retraction impulse $\mathcal{R}$, we can have the following three possible cases:

1) The impulsive retraction torque does only negative work (braking the leg swing): In this case $E_{\mathcal{PR}} = e_1\,W_{\mathcal{P}} - e_2\,W_{\mathcal{R}}$, where $W_{\mathcal{R}}$ and $W_{\mathcal{P}}$ are given by (12) and (14). The fact that $J_{v/R} = -J_{\omega/P} > 0$ (section III-C) results in $\partial E_{\mathcal{PR}}/\partial s = -(e_1 + e_2)\,J_{v/R}\,\mathcal{P}\,\mathcal{R} \leqslant 0$. Thus, the energetic cost $E_{\mathcal{PR}}$ monotonically decreases with the overlap parameter s, except for the trivial case $\mathcal{R} = 0$. Energy expenditure is minimized when s is maximized, *i.e.* $s^* = 1$, which is the case when the impulsive retraction torque is applied completely after the impulsive push-off force. Note that in this case the individual retraction work in (12) and the push-off work in (14) are also minimized.

2) The impulsive retraction torque does only positive work (accelerating the swing leg): For this case $E_{\mathcal{PR}} = e_1\,(W_{\mathcal{P}} + W_{\mathcal{R}})$, where $W_{\mathcal{R}}$ and $W_{\mathcal{P}}$ are given by (12) and (14). Because $J_{v/R} = -J_{\omega/P}$, the individual variations of the push-off and retraction work due to change in s cancel each other, and the energetic cost does not depend on the overlap parameter s. However, the condition that the retraction torque does only positive work, while the hip angular velocity is always decreasing, implies that ω is negative at the start of the retraction torque, *i.e.* $\omega_{t_{\mathrm{r}}^-} = \omega_{t_{\mathrm{pr}}^-} + p_{t_{\mathrm{r}}^-}\,\mathcal{P}\,J_{\omega/P} \leqslant 0$, or equivalently $p_{t_{\mathrm{r}}^-} \geqslant -\omega_{t_{\mathrm{pr}}^-}/(J_{\omega/P}\,\mathcal{P})$. Because $J_{\omega/P} < 0$, if we have $\omega_{t_{\mathrm{pr}}^-} \leqslant 0$, the impulsive push-off force and retraction torque can be applied with any relative timing (*i.e.* $p_{t_{\mathrm{r}}^-} \geqslant 0$ or $0 \leqslant s^* \leqslant 1$). However, when $\omega_{t_{\mathrm{pr}}^-} \geqslant 0$ we must have $p_{t_{\mathrm{r}}^-} > 0$, implying that the impulsive push-off force should *start* first to reverse the hip angular velocity before the impulsive retraction torque. Note that in this case the push-off force does not have to be completed before the retraction torque.

3) The impulsive retraction torque does both positive and negative work: Here, the retraction torque starts with doing negative work, and according to case 1 above, to simultaneously minimize the individual push-off work and negative retraction work, the impulsive retraction torque must be applied completely after the impulsive push-off force. In this case, according to case 2 above, the increase in the positive retraction work will be cancelled by the partial decrease in the push-off work. Thus, in this case the optimal overlap parameter is $s^* = 1$. In the Appendix this result is obtained using a more direct method.

Summarizing all three cases above, if the hip angular velocity just before the late-swing impulsive actions is negative, *i.e.* $\omega_{t_{\mathrm{pr}}^-} \leqslant 0$, the relative timing of the given push-off impulse $\mathcal{P}$ and the retraction impulse $\mathcal{R}$ has no influence on the energetic cost. However, when $\omega_{t_{\mathrm{pr}}^-} > 0$, the minimum cost is achieved when the impulsive push-off force *starts* first. In this case if the push-off impulse $\mathcal{P}$ is big enough to reverse the hip angular velocity without the retraction impulse, the impulsive retraction torque can start during the push-off force after the zero-crossing of ω, otherwise the impulsive retraction torque should be applied completely after the impulsive push-off force to minimize the cost.

VI. Discussion and The Future Work

We have used a simple bipedal model to find the energetically efficient relative timing of push-off and swing retraction in walking. The push-off force and the retraction torque are approximated as the impulsive force and torque. We showed that the optimal relative timing is the one that simultaneously minimizes the individual push-off work and the negative retraction work by applying the push-off force before the retraction torque. Interestingly, this result is valid for both periodic and aperiodic gaits and regardless of the of the actuator efficiencies for positive and negative work (e_1 and e_2 constants).

The resulting optimal relative timing is achieved based on the fact that $J_{\omega/P} = -J_{v/R} < 0$. The equality relation here is guaranteed by the symmetry of the mass matrix, and the inequality comes from the fact that the push-off force tends to retract the swing leg. Interestingly, these conditions are satisfied for even more complex models, implying that the same optimal relative timing of impulsive push-off and retraction holds for a large range of bipedal models. For example, consider the more realistic models in Fig. 5. In these models the retraction impulse $\mathcal{R}$ is applied by the swing hip actuator (both models have two hip actuators acting between the torso and the corresponding leg) and pushes the swing leg toward the stance leg (decreasing the swing hip angle ϕ). In the model with straight legs the push-off impulse $\mathcal{P}$ is directly applied by the prismatic actuator along the stance leg, whereas in the model with articulated legs the equivalent push-off impulse is provided by the knee impulse $\mathcal{K}$ and the ankle impulse $\mathcal{A}$ that extend the corresponding joint angles. Following the procedure we used in (3)-(9), the push-off-retraction velocity maps in these models can be related to the mass matrix. Thus, the symmetry of the mass matrix guarantees $J_{\dot{\phi}/\mathcal{P}} = -J_{\dot{l}/\mathcal{R}}$ for the straight-leg biped, and $J_{\dot{\phi}/\mathcal{K}} = -J_{\dot{\psi}/\mathcal{R}}$ and $J_{\dot{\phi}/\mathcal{A}} = -J_{\dot{\theta}/\mathcal{R}}$ for the articulated-leg biped. On the other hand, in both models the (equivalent) push-off impulse accelerates the hip forward, which in a normal configuration (upright torso) tends to decrease the swing hip angle ϕ. This Implies $J_{\dot{\phi}/\mathcal{P}} < 0$ for the straight-leg model and $J_{\dot{\phi}/\mathcal{K}} < 0$ and $J_{\dot{\phi}/\mathcal{A}} < 0$ for the articulated-leg model. Therefore, the optimality conditions are satisfied for both of these models, so the same optimal relative timing of push-off and retraction holds for both.

Although we have used impulsive functions to simplify the analysis, arguably the results are still applicable to many practical cases in which the (optimal) gait includes burst forces/torques. Although realistic forces/torques have bounded magnitudes and are applied over an extended period of time, the duration of the burst forces/torque is relatively short, so the biped configuration does not change much during the application of the burst force/torque. Therefore, these burst forces/torques effectively change the velocities while the biped configuration is approximately constant, similar to the effect of theoretically impulsive actions.

The analysis in the previous section shows that for given push-off and retraction impulses, the minimum cost is obtained when the overlap parameter s is maximized. In the case of impulsive forces/torques this can be achieved when the retraction torque is completely isolated and applied between the push-off and heel-strike ($s = 1$). However, for practical non-impulsive cases, this isolation of the actions will degrade the gait efficiency by inserting a non-infinitesimal delay between the push-off and heel-strike (for efficient walking push-off should be applied just before heel-strike [12]). Maximizing the overall energy efficiency in practical cases can be achieved by postponing the retracting hip torque completely until the *end* of the push-off before heel-strike. Verification of this strategy in human walking and in practical robots is part of our future work.

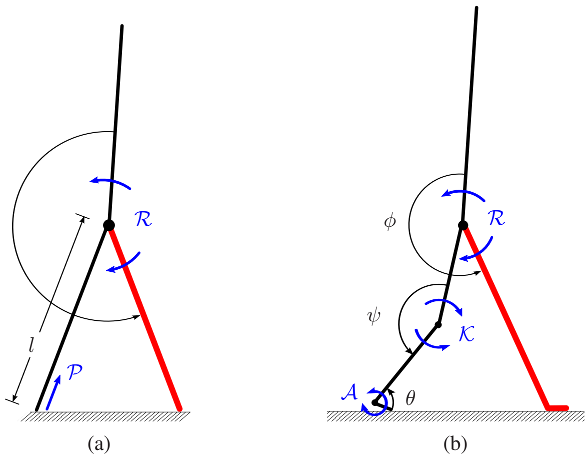

Fig. 5. Two bipedal models with torso. There are two hip actuators, each acting between the torso and the corresponding thigh. The swing retraction torque, quantified by the impulse $\mathcal{R}$, decreases ϕ and pushes the swing leg toward the stance leg. (**a**) The straight-leg model: the push-off impulse $\mathcal{P}$ is provided by a prismatic actuator along the stance leg. (**b**) The articulated-leg model: push-off is provided by the knee and ankle torques, quantified by their impulses $\mathcal{K}$ and $\mathcal{A}$, which tend to extend the corresponding angles (consequently extending the leg).

Previous analytical studies of walking do not take leg mass into account, and thus do not show the mechanism through which swing leg retraction influences gait energetics. The analysis here opens a new perspective on the study of optimum retraction impulse for maximum efficiency.

VII. Conclusion

To achieve or maximize different benefits of swing leg retraction, a hip torque is required before heel-strike to brake the leg swing and/or accelerate its backward rotation to some optimal speed. In walking, the retracting hip torque and the pre-emptive push-off appear at the same phase of the gait cycle. Thus, due to mechanical coupling, their relative timing influences the gait energetics even if the resulting pre-heel-strike velocities are fixed. We use an analytical approach to study the optimal relative timing of the retracting torque and the push-off force. The analysis is done using a simple bipedal model, and then generalized to more complex models. Impulsive forces/torques are used to simplify the analysis. Given the push-off and retraction impulses and the hip angular velocity just before the impulsive actions, we minimized the net energetic cost of the gait, and found that the optimal relative timing is the one that eliminates or minimizes the negative retraction work. In other words, if the hip joint is extending at the end of swing, it is energetically optimal not to apply the retraction torque until

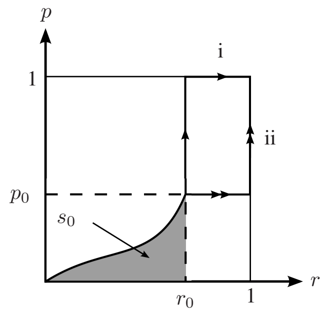

Fig. 6. The partial overlap parameter s_0. The path i and ii determine the upper and lower bounds of the total overlap parameter s, respectively.

the hip extension is stopped by the push-off force, and if the push-off impulse is not big enough to reverse the hip angular velocity by itself, then the optimal sequence is to apply the impulsive braking hip torque completely after the impulsive push-off force. These results are valid for both periodic and aperiodic gaits and regardless of the actuator efficiencies for positive and negative work.

APPENDIX

A. The work of the retraction impulse $\mathcal{R}$ when it does both positive and negative work

Assume that the only zero-crossing of the hip angular velocity ω occurs at $t = t_0$. Then, there exist $r_0 = r(t_0)$ and $p_0 = p(t_0)$ where $0 \leqslant r_0 \leqslant 1$ and $0 \leqslant p_0 \leqslant 1$, and

$$\omega(t_0) = \omega_{t_{\rm pr}^-} + r_0\, J_{\omega/R}\, \mathcal{R} + p_0\, J_{\omega/P}\, \mathcal{P} = 0. \tag{15}$$

The retraction impulse $\mathcal{R}$ does only negative work before t_0 (ω is decreasing between $t_{\rm pr}^-$ and $t_{\rm h}^-$). To calculate this negative work, denoted by $W_{\mathcal{R}1}$, we just need to change the upper bound of the integrations in (11) and (12) from $t_{\rm r}^+$ to t_0, from $\mathcal{R}$ to $r_0\mathcal{R}$, and from $r = 1$ to $r = r_0$. The result is

$$W_{\mathcal{R}1} = -\omega_{t_{\rm pr}^-}\, r_0\, \mathcal{R} - \frac{1}{2}\, J_{\omega/R}\, r_0^2\, \mathcal{R}^2 - s_0\, J_{\omega/P}\, \mathcal{R}\, \mathcal{P}, \tag{16}$$

where

$$s_0 = \int_0^{r_0} p\, dr, \tag{17}$$

quantifies the overlap between $\mathcal{R}$ and $\mathcal{P}$ before t_0. Fig. 6 shows the area in r-p plane that corresponds to s_0 for an arbitrary scenario. From this figure and also from (17) it is clear that in all cases $0 \leqslant s_0 \leqslant r_0\, p_0$.

Finally, using $W_{\mathcal{R}}$ from (12), the positive retraction work done after t_0 is given by

$$W_{\mathcal{R}2} = W_{\mathcal{R}} - W_{\mathcal{R}1}. \tag{18}$$

B. The optimal overlap parameter s^* when a given retraction impulse $\mathcal{R}$ does both positive and negative work

The partial overlap parameter s_0 limits the total overlap parameter s by constraining it between the following lower and upper bounds, corresponding respectively to the paths i and ii in Fig. 6.

$$s_0 + (1 - r_0)\, p_0 \leqslant s \leqslant s_0 + 1 - r_0 \tag{19}$$

Using the negative and positive retraction work in (16) and (18) and the push-off work in (14), the net energetic cost of $\mathcal{P}$ and $\mathcal{R}$ is given by $E_{\mathcal{PR}} = e_1\,(W_{\mathcal{P}} + W_{\mathcal{R}2}) - e_2\, W_{\mathcal{R}1}$. It is a simple practice to verify that $E_{\mathcal{PR}}$ in this case is not a direct function of s. Now, because $J_{\omega/P} < 0$ we get

$$\frac{\partial E_{\mathcal{PR}}}{\partial s_0} = (e_1 + e_2)\, J_{\omega/P}\, \mathcal{R}\, \mathcal{P} \leqslant 0. \tag{20}$$

Thus, $E_{\mathcal{PR}}$ decreases with increasing s_0, and becomes minimum when s_0 takes its allowed maximum, *i.e.*

$$s_0^* = p_0\, r_0. \tag{21}$$

On the other hand,

$$\frac{\partial E_{\mathcal{PR}}}{\partial r_0} = -\,(e_1 + e_2)\; p_0\, J_{\omega/P}\, \mathcal{R}\, \mathcal{P} \geqslant 0 \tag{22}$$

implying that at the optimal point, r_0 should be minimum. Now, according to (15) the minimum r_0 is achieved when $p_0 = 1$. Finally, substituting the latter and (21) into (19) results in $s^* = 1$.

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