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On a bi-layer shallow-water problem

María Luz Muñoz-Ruiz[a,*], François Joseph Chatelon[b], Pierre Orenga[b]

[a] *Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, s/n, 29071 Málaga, Spain*

[b] *Faculté des Sciences et Techniques, Université de Corse, Campus Grossetti, B.P. 54, 20250 Corte, Corse, France*

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Abstract

In this paper, we prove an existence and uniqueness result for a bi-layer shallow water model in depth-mean velocity formulation. Some smoothness results for the solution are also obtained. In a previous work we proved the same results for a one-layer problem. Now the difficulty arises from the terms coupling the two layers. In order to obtain the energy estimate, we use a special basis which allows us to bound these terms.

Keywords: Shallow-water equations; Bi-layer model; Nonlinear partial differential equations; A priori estimates; Sobolev and Orlicz spaces

1. Introduction

The problem that gave rise to our study is the modelling of the dynamics of water masses in the Alboran Sea and the Strait of Gibraltar (the western-most part of the Mediterranean Sea). In this sea, two layers of water can be distinguished: the surface Atlantic water penetrating into the Mediterranean through the Strait of Gibraltar, and the deeper, denser Mediterranean water flowing into the Atlantic. The observation of this simplified picture shows that, if a bi-dimensional model is going to be used to simulate the flow in this region, it is necessary to consider, at least, a two-layer model.

Here, a model is proposed that considers sea water as composed of two immiscible layers of different constant densities. In such a model, waves appear not only on the surface but also at the interface between the layers. It will be assumed that phenomena

* Corresponding author. Fax: +34-952131894.

E-mail address: munoz@anamat.cie.uma.es (M.L. Muñoz-Ruiz).

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to be modelled have wavelengths large enough to make an appropriate shallow water approximation in each layer. Therefore, the partial differential equations system to be studied is a coupled system of shallow water equations.

1.1. Positioning the problem

Let Ω be a fixed bounded, smooth, and simply connected open domain of $\mathbb{R}^2$ with boundary Γ. Physically, Ω is the domain corresponding to the surface of the sea assumed to be at rest. We denote by $x=(x_1,x_2)$ a point in Ω and by n the exterior unit normal vector to Ω on Γ. Let Q be equal to $\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$.

If $v=(v_1,v_2)$ is a vector function from Ω into $\mathbb{R}^2$ and q is a scalar function from Ω into $\mathbb{R}$, we define the operators α, Curl and curl as follows:

$$\alpha(v)=\begin{pmatrix}-v_2\\ v_1\end{pmatrix},\qquad \operatorname{Curl} q=\begin{pmatrix}\dfrac{\partial q}{\partial x_2}\\ -\dfrac{\partial q}{\partial x_1}\end{pmatrix},\qquad \operatorname{curl} v=\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}.$$

We consider a system composed by two layers of superposed fluids with densities ρ_1 and ρ_2 ($\rho_2<\rho_1$). In what follows, index 1 makes reference to the deeper layer and index 2 to the upper layer of the fluid. Let A_1 and A_2 be the respective coefficients of viscosity for each layer and g the acceleration of gravity. We denote by u_1 and u_2 the velocity vector fields defined in Q and by h_1 and h_2 the thickness of the lower and upper layer, respectively.

The problem we study is the following [8]:

$$(\mathscr{P})\quad\begin{cases}\dfrac{\partial u_1}{\partial t}-A_1\Delta u_1+\dfrac{1}{2}\nabla u_1^2+\operatorname{curl} u_1\alpha(u_1)+g\nabla h_1+g\dfrac{\rho_2}{\rho_1}\nabla h_2=0 & \text{in } Q,\\ u_1\cdot n=0 & \text{on } \Sigma,\\ \operatorname{curl} u_1=0 & \text{on } \Sigma,\\ u_1(t=0)=u_{1,0} & \text{in } \Omega,\\ \dfrac{\partial h_1}{\partial t}+\operatorname{div}(u_1h_1)=0 & \text{in } Q,\\ h_1(t=0)=h_{1,0} & \text{in } \Omega,\\ \dfrac{\partial u_2}{\partial t}-A_2\Delta u_2+\dfrac{1}{2}\nabla u_2^2+\operatorname{curl} u_2\alpha(u_2)+g\nabla h_2+g\nabla h_1=0 & \text{in } Q,\\ u_2\cdot n=0 & \text{on } \Sigma,\\ \operatorname{curl} u_2=0 & \text{on } \Sigma,\\ u_2(t=0)=u_{2,0} & \text{in } \Omega,\\ \dfrac{\partial h_2}{\partial t}+\operatorname{div}(u_2h_2)=0 & \text{in } Q,\\ h_2(t=0)=h_{2,0} & \text{in } \Omega,\end{cases}$$

where $u_{1,0}$, $u_{2,0}$ and $h_{1,0}$, $h_{2,0}$ are the initial conditions for velocities and depths, respectively. In order to simplify this, we consider homogeneous momentum equations and homogeneous boundary conditions.

1.2. Weak formulation

We will denote by $(.,.)$ the scalar product of $L^2(\Omega)$ and $L^2(\Omega)^2$ and by $\|.\|_{W^{m,p}}$ the usual norm in the spaces $W^{m,p}(\Omega)$ and $W^{m,p}(\Omega)^2$.

Let V be the space

$$V = \{u \in L^2(\Omega)^2, \operatorname{div} u \in L^2(\Omega), \operatorname{curl} u \in L^2(\Omega), u \cdot n = 0 \text{ on } \Gamma\}.$$

This is a Banach space with the norm $\|u\|_V^2 = \|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2$. The following results concerning V [4] will be used:

If Ω is smooth enough, V is algebraically and topologically included in the space $\{u \in H^1(\Omega)^2, u \cdot n = 0 \text{ on } \Gamma\}$.

If Ω is simply connected, the previous norm is equivalent to that given by $\|u\|_V^2 = \|\operatorname{div} u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2$, and the bilinear form

$$a(u,v) = (\operatorname{div} u, \operatorname{div} v) + (\operatorname{curl} u, \operatorname{curl} v)$$

is elliptic. [1]

Let us consider problem $(\mathscr{P})$ under the following weak formulation:

Find (u_1,h_1) and (u_2,h_2) in $[L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;V)] \times [L^\infty(0,T;L^1(\Omega)) \cap L^2(Q)]$ such that $h_1 > 0$, $h_2 > 0$ and

$$(\mathscr{V}) \quad \begin{cases} \left(\dfrac{\partial u_1}{\partial t}, v\right) + A_1 a(u_1,v) - \dfrac{1}{2}(u_1^2, \operatorname{div} v) + (\operatorname{curl} u_1 \alpha(u_1), v) \\ \quad -g(h_1, \operatorname{div} v) - g\dfrac{\rho_2}{\rho_1}(h_2, \operatorname{div} v) = 0 \quad \forall v \in V, \\ \left(\dfrac{\partial u_2}{\partial t}, v\right) + A_2 a(u_2,v) - \dfrac{1}{2}(u_2^2, \operatorname{div} v) + (\operatorname{curl} u_2 \alpha(u_2), v) \\ \quad -g(h_2, \operatorname{div} v) - g(h_1, \operatorname{div} v) = 0 \quad \forall v \in V, \\ \dfrac{\partial h_1}{\partial t} + \operatorname{div}(u_1 h_1) = 0 \quad \text{in } L^1(0,T;W^{-1,p}(\Omega)), \quad p < 2, \\ \dfrac{\partial h_2}{\partial t} + \operatorname{div}(u_2 h_2) = 0 \quad \text{in } L^1(0,T;W^{-1,p}(\Omega)), \quad p < 2, \\ u_1(t=0) = u_{1,0} \in V, \qquad u_2(t=0) = u_{2,0} \in V, \\ h_1(t=0) = h_{1,0} > 0 \in L^2(\Omega), \qquad h_2(t=0) = h_{2,0} > 0 \in L^2(\Omega). \end{cases}$$

The following orthogonal decomposition of $L^2(\Omega)^2$ in a sum of gradient vectors and curl vectors holds [4,9]:

$$L^2(\Omega)^2 = \nabla H^1(\Omega) \oplus \operatorname{Curl} H_0^1(\Omega).$$

This decomposition will be used to look for u_1 and u_2 in V under the form

$$u_1 = u_{p,1} + u_{q,1} = \nabla p_1 + \operatorname{Curl} q_1, \quad u_2 = u_{p,2} + u_{q,2} = \nabla p_2 + \operatorname{Curl} q_2,$$

[1] Notice that $\Delta u = \nabla \operatorname{div} u - \operatorname{Curl} \operatorname{curl} u$ for explaining the choice of V and a.

with p_1, q_1 and p_2, q_2 solutions of the scalar problems

$$\begin{cases} \Delta p_1 = \operatorname{div} u_1 & \text{in } \Omega, \\ \dfrac{\partial p_1}{\partial n} = u_1 \cdot n = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta q_1 = \operatorname{curl} u_1 & \text{in } \Omega, \\ q_1 = 0 & \text{on } \Gamma \end{cases}$$

and

$$\begin{cases} \Delta p_2 = \operatorname{div} u_2 & \text{in } \Omega, \\ \dfrac{\partial p_2}{\partial n} = u_2 \cdot n = 0 & \text{on } \Gamma, \end{cases} \qquad \begin{cases} -\Delta q_2 = \operatorname{curl} u_2 & \text{in } \Omega, \\ q_2 = 0 & \text{on } \Gamma. \end{cases}$$

The functions p_i can be chosen such that $\int_\Omega p_i = 0$. We must also remember that $\operatorname{curl} u_{p,i} = 0$ and $\operatorname{div} u_{q,i} = 0$, $i = 1, 2$.

We are going to prove the existence and uniqueness of the solution of problem $(\mathscr{V})$ and some smoothness results. Those results have been already proved in the single-layer case [3]. In the two-layer model, we find the same difficulties together with a new one due to the appearance of coupling terms between the layers.

In order to simplify the notations, we only consider the simply connected case. The additional difficulty appearing in the multiply connected domain is solved taking into account a dissipation condition at the bottom [10]. In this case, some new functions must be added to the decomposition: the functions $\operatorname{curl} r$ with r solutions of the following m problems:

$$-\Delta r_i = 0 \quad \text{in } \Omega, \qquad r_i = 1 \quad \text{on } \Gamma_i, \qquad r_i = 0 \quad \text{on } \Gamma_j, \; j \neq i,$$

where $i = 1, \dots, m$ and $j = 0, \dots, m$. We have assumed that Γ has a finite number of connected components Γ_i, $i = 0, \dots, m$, Γ_0 indicating the boundary of the infinite connected component of the complementary of Ω in $\mathbb{R}^2$.

2. An existence theorem

In this section, we present a global existence result with controlled data.

2.1. Theorem

2.1.1. Preliminaries

Let C be the best constant associated with Gagliardo–Nirenberg's inequality:

$$\|u\|_{L^4}^2 \leqslant C\|u\|_{L^2}\|u\|_V \quad \forall u \in V.$$

Let C' be the injection constant of $\{\Theta \in W^{1,1}(\Omega) : \int_\Omega \Theta = 0\}$ into $L^2(\Omega)$:

$$\|\Theta\|_{L^2} \leqslant C'\|\nabla\Theta\|_{L^1} \quad \forall \Theta \in W^{1,1}(\Omega) : \int_\Omega \Theta = 0.$$

We consider the N-function

$$\Phi(x) = e^{x^2} - 1$$

and the associated Orlicz space $L_\Phi(\Omega)$, which is a Banach space with the Orlicz norm, denoted by $\|.\|_{L_\Phi}$.

The Sobolev space $H^1(\Omega)$ is embedded in $L_\Phi(\Omega)$ [1]. Let k be the injection constant:

$$\|p\|_{L_\Phi} \leqslant k\|p\|_{H^1} \quad \forall p \in H^1(\Omega).$$

It is not possible to give an analytical expression for Ψ, the complementary N-function to Φ. But it can be shown[2] that Ψ is equivalent to $\tilde{\Psi}$, with

$$\tilde{\Psi}(x) = x\sqrt{\log^+ x}.$$

Let $\|.\|_{L_\Psi}$ and $\|.\|_{L_{\tilde{\Psi}}}$ be the Orlicz norms in the Orlicz spaces $L_\Psi(\Omega)$ and $L_{\tilde{\Psi}}(\Omega)$, respectively. The equivalence relation between the N-functions Ψ and $\tilde{\Psi}$ implies the equivalence between the norms $\|.\|_{L_\Psi}$ and $\|.\|_{L_{\tilde{\Psi}}}$ and allows us to identify the spaces $L_\Psi(\Omega)$ and $L_{\tilde{\Psi}}(\Omega)$. Let k' be the best constant such that

$$\|h\|_{L_\Psi} \leqslant k'\|h\|_{L_{\tilde{\Psi}}} \quad \forall h \in L_\Psi(\Omega) = L_{\tilde{\Psi}}(\Omega).$$

Finally, denote by k'' the best constant such that

$$\|p\|_{H^1} \leqslant k''\|\nabla p\|_{L^2} \quad \forall p \in H^1(\Omega) : \int_\Omega p = 0.$$

Now we can define

$$K = kk'k''.$$

2.1.2. Conditions of the theorem

Let λ and μ be positive numbers such that

$$g > \frac{3}{4}\lambda + g\frac{\mu}{2}. \tag{1}$$

We define

$$C_\lambda = \frac{C^2 + 2C'^2}{4\lambda}$$

and

$$C_1 = \frac{\|h_{1,0}\|_{L^1}}{2\,\mathrm{meas}(\Omega)}, \qquad C_1' = \frac{gT}{\mathrm{meas}(\Omega)}\|h_{1,0}\|_{L^1}\left(\|h_{1,0}\|_{L^1} + \frac{\rho_2}{\rho_1}\|h_{2,0}\|_{L^1}\right),$$

$$C_2 = \frac{\|h_{2,0}\|_{L^1}}{2\,\mathrm{meas}(\Omega)}, \qquad C_2' = \frac{gT}{\mathrm{meas}(\Omega)}\|h_{2,0}\|_{L^1}(\|h_{1,0}\|_{L^1} + \|h_{2,0}\|_{L^1}).$$

Let us assume that the eddy viscosities A_1 and A_2 satisfy

$$A_1 > C_1 + g\frac{\rho_2}{\rho_1}\frac{1}{2\mu}, \qquad A_2 > C_2 + g\frac{1}{2\mu}. \tag{2}$$

[2] By proving that $\lim_{x\to\infty} \Psi(x)/\tilde{\Psi}(x) = 1$.

Then

$$B_1 = A_1 - C_1 - g\frac{\rho_2}{\rho_1}\frac{1}{2\mu}$$

and

$$B_2 = A_2 - C_2 - g\frac{1}{2\mu}$$

are strictly positive.

We assume that the data are small in the following sense:

$$\sum_{i=1}^{2}\left[\frac{1}{2}\|u_{i,0}\|_{L^2}^2 + (g+A_i)\int_\Omega h_{i,0}\log h_{i,0} + \frac{g}{e}\text{meas}(\Omega) - A_i\overline{h_{i,0}}\log\overline{h_{i,0}}\,\text{meas}(\Omega)\right.$$
$$\left. + C_i' + \int_\Omega p_i(0)h_{i,0} + 2K^2\right] < \theta\min_{i=1,2}\left(\frac{\sqrt{C^2+16B_iC_\lambda}-C}{16C_\lambda}\right)^2, \qquad (3)$$

where $\theta \in (0,1)$ and $\overline{h_{i,0}}$ represent the averaged water elevation at the initial instant, $i = 1, 2$.

Theorem 1. *Let Ω be a simply connected bounded smooth open domain of $\mathbb{R}^2$. Let $u_{i,0} \in V$, $h_{i,0} \in L^2(\Omega)$ such that $h_{i,0} > 0$, $i = 1, 2$, satisfying the following conditions*:

$$\|h_{i,0}\|_{L^1} < \frac{g}{2K^2}, \qquad (4)$$

$$\|u_{i,0}\|_{L^2} < \frac{\sqrt{C^2+16B_iC_\lambda}-C}{4C_\lambda}. \qquad (5)$$

If the previous hypothesis on the data is satisfied, then the weak problem $(\mathscr{V})$ has a solution $\{(u_1,h_1),(u_2,h_2)\}$ that satisfies the following estimate:

$$\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + \|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + \sup_t\int_\Omega h_1\log h_1 + \sup_t\int_\Omega h_2\log h_2$$
$$+ \|u_1\|_{L^2(0,T;V)}^2 + \|u_2\|_{L^2(0,T;V)}^2 + \|h_1\|_{L^2(Q)}^2 + \|h_2\|_{L^2(Q)}^2 \leqslant \mathrm{C}, \qquad (6)$$

where $\mathrm{C} > 0$ depends on the initial data.

As in the one-layer case, the proof of this theorem is split into several steps: first we give some a priori estimates, then we build a sequence of approximated solutions that satisfy these estimates, and, finally, we pass to the limit into the continuity and momentum equations as in [11]. In this case, the main difficulty is to obtain an a priori estimate because of the coupled terms.

2.2. A priori estimates

Lemma 1. *If $\{(u_1,h_1),(u_2,h_2)\}$ is a classical solution of problem $(\mathscr{V})$ and if relations (1)–(5) are satisfied, then we have*

$$h_1 > 0, \qquad h_2 > 0, \qquad (7)$$

$$\int_\Omega h_1 = \int_\Omega h_{1,0}, \qquad \int_\Omega h_2 = \int_\Omega h_{2,0}, \tag{8}$$

$$\begin{aligned}
-\frac{2g}{e}\text{meas}(\Omega) \leqslant\ & \frac{1}{4}\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \frac{1}{4}\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \\
& + (g - 2K^2\|h_{1,0}\|_{L^1}) \sup_t \int_\Omega h_1 \log^+ h_1 - g \sup_t \int_\Omega h_1 \log^- h_1 \\
& + (g - 2K^2\|h_{2,0}\|_{L^1}) \sup_t \int_\Omega h_2 \log^+ h_2 - g \sup_t \int_\Omega h_2 \log^- h_2 \\
& + \left(B_1 - \frac{C}{2}\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda \|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_1\|^2_{L^2(0,T;V)} \\
& + \left(B_2 - \frac{C}{2}\|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda \|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_2\|^2_{L^2(0,T;V)} \\
& + \left(g - \frac{3}{4}\lambda - g\frac{\mu}{2}\right)\|h_1\|^2_{L^2(Q)} + \left(g - \frac{3}{4}\lambda - g\frac{\rho_2}{\rho_1}\frac{\mu}{2}\right)\|h_2\|^2_{L^2(Q)} \\
\leqslant\ & \sum_{i=1}^{2}\left[\frac{1}{2}\|u_{i,0}\|^2_{L^2} + (g + A_i)\int_\Omega h_{i,0}\log h_{i,0} - A_i \overline{h_{i,0}} \log \overline{h_{i,0}}\, \text{meas}(\Omega)\right. \\
& \left. + C_i' + \int_\Omega p_i(0) h_{i,0} + 2K^2\right].
\end{aligned} \tag{9}$$

$$\begin{aligned}
& B_1 - \frac{C}{2}\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} > 0, \\
& B_2 - \frac{C}{2}\|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)} > 0.
\end{aligned} \tag{10}$$

Proof. Inequalities (7) are easily deduced from the identity:

$$h_i(X_i(t),t) = h_i(x_0,0)\, \mathrm{e}^{-\int_0^t \mathrm{div} u_i(X_i(s),s)\,\mathrm{d}s}, \tag{11}$$

where $X_i(t)$ is the solution of the problem

$$\begin{cases} \dfrac{\mathrm{d}X_i}{\mathrm{d}t} = u_i(X_i(t),t), \\ X_i(0) = x_0, \end{cases}$$

for $i = 1,2$.

Relations (8) are obtained by integration over Ω of the respective continuity equations.

The main difficulties that appear in obtaining estimate (9) arise from the fact that $\mathrm{div}\, u_i \neq 0$. Due to this, the nonlinear terms and the pressure terms do not vanish in the energy inequalities, as is the case of the incompressible Navier–Stokes equations [7].

In order to estimate the nonlinear terms $(u_i^2, \mathrm{div}\, u_i)$ we must build a stability space [11], using Gagliardo–Nirenberg's inequality.

The estimate of the terms $(h_i, \operatorname{div} u_i)$ is overcome with an additional estimate of $h_i \log h_i$ in $L^1(\Omega)$ [11].

The two previous difficulties also appear in the case of a single layer [2]. But there is a third one that does not appear in the one-layer system. This is related to the presence of the terms $(h_1, \operatorname{div} u_2)$ and $(h_2, \operatorname{div} u_1)$: an estimation of h_i in $L^2(Q)$ is required to handle these.

To obtain these estimates we adapt the techniques used in [3]. The difficulty in this case comes from the fact that the L^2-estimates for h_i and the existence of a solution must be obtained at the same time, while in [3], the L^2-estimate is obtained once the existence and boundedness of the solution is known.

The first step is to obtain the energy inequalities, by taking $v = u_1$ in $(\mathscr{V})_1$ and $v = u_2$ in $(\mathscr{V})_2$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1\|_{L^2}^2 + A_1\|u_1\|_V^2 - \frac{1}{2}(u_1^2, \operatorname{div} u_1) - g(h_1, \operatorname{div} u_1) - g\frac{\rho_2}{\rho_1}(h_2, \operatorname{div} u_1) = 0, \quad (12)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|_{L^2}^2 + A_2\|u_2\|_V^2 - \frac{1}{2}(u_2^2, \operatorname{div} u_2) - g(h_2, \operatorname{div} u_2) - g(h_1, \operatorname{div} u_2) = 0. \quad (13)$$

To estimate the nonlinear terms $(u_1^2, \operatorname{div} u_1)$ and $(u_2^2, \operatorname{div} u_2)$, we use Gagliardo–Nirenberg's inequality

$$(u_i^2, \operatorname{div} u_i) \leqslant \|u_i\|_{L^4}^2\|u_i\|_V \leqslant C\|u_i\|_{L^2}\|u_i\|_V^2, \quad i = 1, 2.$$

Next, we estimate the terms $(h_1, \operatorname{div} u_1)$ and $(h_2, \operatorname{div} u_2)$ by formally writing

$$-(h_i, \operatorname{div} u_i) = (\nabla h_i, u_i) = \left(\frac{\nabla h_i}{h_i}, u_i h_i\right) = (\nabla \log h_i, u_i h_i) = -(\log h_i, \operatorname{div}(u_i h_i)).$$

Using the continuity equations and (8) we have

$$-(h_i, \operatorname{div} u_i) = \left(\log h_i, \frac{\partial h_i}{\partial t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(h_i \log h_i - h_i, 1) = \frac{\mathrm{d}}{\mathrm{d}t}(h_i \log h_i, 1), \quad i = 1, 2.$$

Then, adding (12) and (13), and integrating in $(0, t)$, we get

$$\begin{aligned}
&\frac{1}{2}\|u_1\|_{L^2}^2 + \frac{1}{2}\|u_2\|_{L^2}^2 + g\int_\Omega h_1 \log h_1 + g\int_\Omega h_2 \log h_2 \\
&\quad + \left(A_1 - \frac{C}{2}\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_1\|_{L^2(0,t;V)}^2 \\
&\quad + \left(A_2 - \frac{C}{2}\|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_2\|_{L^2(0,t;V)}^2 \\
&\quad \leqslant \frac{1}{2}\|u_{1,0}\|_{L^2}^2 + \frac{1}{2}\|u_{2,0}\|_{L^2}^2 + g\int_\Omega h_{1,0} \log h_{1,0} + g\int_\Omega h_{2,0} \log h_{2,0} \\
&\quad + g\frac{\rho_2}{\rho_1}\int_0^T |(h_2, \operatorname{div} u_1)| + g\int_0^T |(h_1, \operatorname{div} u_2)|. \qquad (14)
\end{aligned}$$

The second step consists in obtaining estimates for h_1 and h_2 in $L^2(Q)$. To do this, we consider the L^2-projection of equations $(\mathscr{P})_1$ and $(\mathscr{P})_7$ on the gradient vectors field [3]:

$$\int_\Omega \left(\frac{\partial u_1}{\partial t} - A_1\Delta u_1 + \frac{1}{2}\nabla u_1^2 + \operatorname{curl} u_1 \alpha(u_1) + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2\right)\nabla P\,\mathrm{d}x = 0,$$

$$\int_\Omega \left(\frac{\partial u_2}{\partial t} - A_2\Delta u_2 + \frac{1}{2}\nabla u_2^2 + \operatorname{curl} u_2 \alpha(u_2) + g\nabla h_2 + g\nabla h_1\right)\nabla P\,\mathrm{d}x = 0,$$

where $P \in H^1(\Omega)$.

Setting $u_1 = u_{p,1} + u_{q,1}$ and $u_2 = u_{p,2} + u_{q,2}$, we have

$$\int_\Omega \left(\frac{\partial u_{p,1}}{\partial t} - A_1\Delta u_{p,1} + \frac{1}{2}\nabla u_1^2 + \operatorname{curl} u_{q,1} \alpha(u_1) + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2\right)\nabla P\,\mathrm{d}x = 0,$$

$$\int_\Omega \left(\frac{\partial u_{p,2}}{\partial t} - A_2\Delta u_{p,2} + \frac{1}{2}\nabla u_2^2 + \operatorname{curl} u_{q,2} \alpha(u_2) + g\nabla h_2 + g\nabla h_1\right)\nabla P\,\mathrm{d}x = 0.$$

Here, we have considered that the projections on the space $\nabla H^1(\Omega)$ of $\partial u_{q,i}/\partial t$ and $\operatorname{Curl}\operatorname{curl} u_{q,i}$ are zero. Recalling that $\operatorname{div} u_{p,i} = \Delta p_i$, we arrive at

$$\nabla\left(\frac{\partial p_1}{\partial t} - A_1\Delta p_1 + \frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right) = 0,$$

$$\nabla\left(\frac{\partial p_2}{\partial t} - A_2\Delta p_2 + \frac{1}{2}u_2^2 + \Theta_2 + gh_2 + gh_1\right) = 0,$$

where $\nabla\Theta_1$ and $\nabla\Theta_2$ are the respective projections for the $L^2(\Omega)^2$ scalar product of $\operatorname{curl} u_{q,1}\alpha(u_1)$ and $\operatorname{curl} u_{q,2}\alpha(u_2)$ on the space $\nabla H^1(\Omega)$. Then

$$\frac{\partial p_1}{\partial t} - A_1\Delta p_1 + \frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2 = \zeta_1, \tag{15}$$

$$\frac{\partial p_2}{\partial t} - A_2\Delta p_2 + \frac{1}{2}u_2^2 + \Theta_2 + gh_2 + gh_1 = \zeta_2, \tag{16}$$

where ζ_1 and ζ_2 are functions that only depend on time.

Now we define, for $\delta \in [0, T]$, the function

$$\xi_\delta(t) = \begin{cases} 1, & 0 \leqslant t \leqslant T-\delta, \\ \frac{T-t}{\delta}, & T-\delta \leqslant t \leqslant T. \end{cases}$$

Multiplying (15) by $\xi_\delta h_1$ and (16) by $\xi_\delta h_2$ and integrating over Q, we obtain

$$g\int_Q \xi_\delta h_1^2 - A_1\int_Q \Delta p_1 \xi_\delta h_1 + \frac{1}{2}\int_Q u_1^2\xi_\delta h_1 + g\frac{\rho_2}{\rho_1}\int_Q \xi_\delta h_1 h_2$$
$$= \int_Q \zeta_1\xi_\delta h_1 - \int_Q \frac{\partial p_1}{\partial t}\xi_\delta h_1 - \int_Q \Theta_1\xi_\delta h_1, \tag{17}$$

$$g\int_Q \xi_\delta h_2^2 - A_2\int_Q \Delta p_2 \xi_\delta h_2 + \frac{1}{2}\int_Q u_2^2\xi_\delta h_2 + g\int_Q \xi_\delta h_1 h_2$$
$$= \int_Q \zeta_2\xi_\delta h_2 - \int_Q \frac{\partial p_2}{\partial t}\xi_\delta h_2 - \int_Q \Theta_2\xi_\delta h_2. \tag{18}$$

The L^2-estimates for h_1 and h_2 will be performed by estimating the terms in these equations and then, passing to the limit when $\delta \to 0$. First, notice that $\int_Q \xi_\delta h_1 h_2 \geqslant 0$ because $h_1, h_2 > 0$.

Then, the second terms on the left-hand side are treated by using

$$-\int_\Omega h_i \Delta p_i = -\int_\Omega h_i \operatorname{div} u_i = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h_i \log h_i$$

as follows

$$\begin{aligned}
-A_i \int_Q \Delta p_i \xi_\delta h_i &= -A_i \int_0^T \xi_\delta \int_\Omega h_i \operatorname{div} u_i = A_i \int_0^T \xi_\delta \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega h_i \log h_i \\
&= A_i \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \xi_\delta h_i \log h_i - A_i \int_0^T \frac{\partial \xi_\delta}{\partial t} \int_\Omega h_i \log h_i \\
&= -A_i \int_\Omega h_{i,0} \log h_{i,0} + \frac{A_i}{\delta} \int_{T-\delta}^T \int_\Omega h_i \log h_i.
\end{aligned}$$

Using the convexity inequality

$$h_i \log h_i \geqslant \bar{h}_i \log \bar{h}_i + (\log \bar{h}_i + 1)(h_i - \bar{h}_i)$$

and (8), we have

$$\int_\Omega h_i \log h_i \geqslant \bar{h}_i \log \bar{h}_i \operatorname{meas}(\Omega) = \overline{h_{i,0}} \log \overline{h_{i,0}} \operatorname{meas}(\Omega). \tag{19}$$

Then

$$\frac{A_i}{\delta} \int_{T-\delta}^T \int_\Omega h_i \log h_i \geqslant \frac{A_i}{\delta} \int_{T-\delta}^T \overline{h_{i,0}} \log \overline{h_{i,0}} \operatorname{meas}(\Omega) = A_i \overline{h_{i,0}} \log \overline{h_{i,0}} \operatorname{meas}(\Omega),$$

and

$$-A_i \int_Q \Delta p_i \xi_\delta h_i \geqslant -A_i \int_\Omega h_{i,0} \log h_{i,0} + A_i \overline{h_{i,0}} \log \overline{h_{i,0}} \operatorname{meas}(\Omega), \qquad i = 1, 2.$$

The term $\frac{1}{2}\int_Q u_i^2 \xi_\delta h_i$ is split as follows:

$$\frac{1}{2}\int_Q u_i^2 \xi_\delta h_i = \frac{1}{2}\int_Q u_{p,i}^2 \xi_\delta h_i + \int_Q u_{p,i} u_{q,i} \xi_\delta h_i + \frac{1}{2}\int_Q u_{q,i}^2 \xi_\delta h_i, \tag{20}$$

for $i = 1, 2$.

To estimate the first terms on the right-hand side we look for an expression for ζ_1 and ζ_2, integrating (15) and (16) over Ω. We use $\int_\Omega p_i = 0$ and $\int_\Omega \Delta p_i = \int_\Omega \operatorname{div} u_i = 0$ and choose Θ_i such that $\int_\Omega \Theta_i = 0$, $i = 1, 2$. We arrive at

$$\zeta_1 = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \left(\frac{1}{2} u_1^2 + g h_1 + g \frac{\rho_2}{\rho_1} h_2 \right) = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \left(\frac{1}{2} u_1^2 + g h_{1,0} + g \frac{\rho_2}{\rho_1} h_{2,0} \right),$$

$$\zeta_2 = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \left(\frac{1}{2} u_2^2 + g h_2 + g h_1 \right) = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \left(\frac{1}{2} u_2^2 + g h_{2,0} + g h_{1,0} \right).$$

Then

$$\int_Q \zeta_i \xi_\delta h_i \leqslant C_i \|u_i\|^2_{L^2(0,T;V)} + C_i',$$

with C_i and C_i' as previously defined, $i = 1, 2$.

We consider next the second terms on the right-hand side. We first integrate by parts:

$$-\int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = -\int_Q \frac{\partial}{\partial t}(p_i \xi_\delta h_i) + \int_Q p_i \frac{\partial}{\partial t}(\xi_\delta h_i)$$
$$= \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{\partial \xi_\delta}{\partial t} h_i + \int_Q p_i \xi_\delta \frac{\partial h_i}{\partial t}$$

and then, we use the continuity equation

$$-\int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{\partial \xi_\delta}{\partial t} h_i - \int_Q p_i \xi_\delta \operatorname{div}(u_i h_i)$$
$$= \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{\partial \xi_\delta}{\partial t} h_i + \int_Q \nabla p_i \xi_\delta u_i h_i.$$

Finally, the decomposition $u_i = u_{p,i} + u_{q,i}$ is used

$$-\int_Q \frac{\partial p_i}{\partial t} \xi_\delta h_i = \int_\Omega p_i(0) h_{i,0} + \int_Q p_i \frac{\partial \xi_\delta}{\partial t} h_i + \int_Q u^2_{p,i} \xi_\delta h_i + \int_Q u_{p,i} u_{q,i} \xi_\delta h_i, \quad (21)$$

for $i = 1, 2$.

The difficulty is to bound the term

$$\int_Q p_i \frac{\partial \xi_\delta}{\partial t} h_i = \int_0^T \frac{\partial \xi_\delta}{\partial t} \int_\Omega p_i h_i.$$

For this, we must use the theory involving N-functions and Orlicz spaces [5,6]:

Using the extension of the Hölder inequality for Orlicz spaces we have

$$\int_\Omega p_i h_i \leqslant \|p_i\|_{L_\Phi} \|h_i\|_{L_\Psi}.$$

Then

$$\int_\Omega p_i h_i \leqslant kk' \|p_i\|_{H^1} \|h_i\|_{L_{\tilde{\Psi}}}.$$

And the norm $\|.\|_{L_{\tilde{\Psi}}}$ verifies

$$\|h_i\|_{L_{\tilde{\Psi}}} \leqslant 1 + \int_\Omega \tilde{\Psi}(h_i).$$

Then we have

$$\int_\Omega p_i h_i \leqslant kk'k'' \|\nabla p_i\|_{L^2} \left(1 + \int_\Omega \tilde{\Psi}(h_i)\right)$$
$$\leqslant K \|u_i\|_{L^2} \left(1 + \int_\Omega h_i \sqrt{\log^+ h_i}\right)$$

$$= K\|u_i\|_{L^2} + K\|u_i\|_{L^2}\int_\Omega \sqrt{h_i}\sqrt{h_i\log^+ h_i}$$
$$\leqslant \frac{1}{8}\|u_i\|_{L^2}^2 + 2K^2 + \frac{1}{8}\|u_i\|_{L^2}^2 + 2K^2\left(\int_\Omega \sqrt{h_i}\sqrt{h_i\log^+ h_i}\right)^2.$$

Using the Hölder inequality

$$\int_\Omega \sqrt{h_i}\sqrt{h_i\log^+ h_i} \leqslant \left(\int_\Omega h_i\right)^{1/2}\left(\int_\Omega h_i\log^+ h_i\right)^{1/2}$$

we can write

$$\int_\Omega p_i h_i \leqslant \frac{1}{4}\|u_i\|_{L^2}^2 + 2K^2 + 2K^2\|h_{i,0}\|_{L^1}\int_\Omega h_i\log^+ h_i.$$

Recalling that $\partial\xi_\delta/\partial t = 0$ over $(0, T-\delta)$ and $\partial\xi_\delta/\partial t = -1/\delta$ over $(T-\delta, T)$, we can conclude that

$$\int_Q \frac{\partial\xi_\delta}{\partial t}p_i h_i \leqslant \frac{1}{4}\|u_i\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + 2K^2 + 2K^2\|h_{i,0}\|_{L^1}\sup_t\int_\Omega h_i\log^+ h_i.$$

To estimate the terms

$$\int_Q u_{p,i}^2\xi_\delta h_i \quad \text{and} \quad \int_Q u_{p,i}u_{q,i}\xi_\delta h_i$$

in (21) we take into account the presence of (20) on the left-hand side of the equations.

Then we must bound

$$\frac{1}{2}\int_Q u_{p,i}^2\xi_\delta h_i - \frac{1}{2}\int_Q u_{q,i}^2\xi_\delta h_i.$$

To do this, we use Gagliardo–Nirenberg's inequality

$$\frac{1}{2}\int_Q u_{p,i}^2\xi_\delta h_i - \frac{1}{2}\int_Q u_{q,i}^2\xi_\delta h_i \leqslant \frac{1}{2}\int_Q u_{p,i}^2\xi_\delta h_i$$
$$\leqslant \frac{1}{2}\left(\frac{\lambda}{2}\int_Q \xi_\delta h_i^2 + \frac{1}{2\lambda}\int_0^T \|u_{p,i}\|_{L^4}^4\right)$$
$$\leqslant \frac{\lambda}{4}\int_Q \xi_\delta h_i^2 + \frac{C^2}{4\lambda}\int_0^T \|u_i\|_{L^2}^2\|u_i\|_V^2.$$

The estimate of the terms on the right-hand side of (17) and (18) concludes with the estimate of $-\int_Q \Theta_i\xi_\delta h_i$:

$$-\int_Q \Theta_i\xi_\delta h_i \leqslant \frac{\lambda}{2}\int_Q \xi_\delta h_i^2 + \frac{1}{2\lambda}\int_0^T \|\Theta_i\|_{L^2}^2$$
$$\leqslant \frac{\lambda}{2}\int_Q \xi_\delta h_i^2 + \frac{C'^2}{2\lambda}\int_0^T \|\nabla\Theta_i\|_{L^1}^2$$
$$\leqslant \frac{\lambda}{2}\int_Q \xi_\delta h_i^2 + \frac{C'^2}{2\lambda}\int_0^T \|\operatorname{curl} u_{q,i}\|_{L^2}^2\|u_i\|_{L^2}^2$$
$$\leqslant \frac{\lambda}{2}\int_Q \xi_\delta h_i^2 + \frac{C'^2}{2\lambda}\int_0^T \|u_i\|_{L^2}^2\|u_i\|_V^2.$$

Then, (17) and (18) yield respectively,

$$\left(g-\frac{3}{4}\lambda\right)\int_Q \xi_\delta h_1^2 \leqslant A_1\int_\Omega h_{1,0}\log h_{1,0} - A_1\overline{h_{1,0}}\log\overline{h_{1,0}}\,\mathrm{meas}(\Omega)$$
$$+C_1\|u_1\|^2_{L^2(0,T;V)} + C_1' + \int_\Omega p_1(0)h_{1,0}$$
$$+\frac{1}{4}\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + 2K^2 + 2K^2\|h_{1,0}\|_{L^1}\sup_t\int_\Omega h_1\log^+ h_1$$
$$+C_\lambda\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\|u_1\|^2_{L^2(0,T;V)} \tag{22}$$

and

$$\left(g-\frac{3}{4}\lambda\right)\int_Q \xi_\delta h_2^2 \leqslant A_2\int_\Omega h_{2,0}\log h_{2,0} - A_2\overline{h_{2,0}}\log\overline{h_{2,0}}\,\mathrm{meas}(\Omega)$$
$$+C_2\|u_2\|^2_{L^2(0,T;V)} + C_2' + \int_\Omega p_2(0)h_{2,0}$$
$$+\frac{1}{4}\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + 2K^2 + 2K^2\|h_{2,0}\|_{L^1}\sup_t\int_\Omega h_2\log^+ h_2$$
$$+C_\lambda\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\|u_2\|^2_{L^2(0,T;V)}. \tag{23}$$

The next step is adding Eq. (14) to Eqs. (22) and (23). Before this, we take the supremum in the terms of the left-hand side of (14) and make δ tend to zero in (22) and (23).

Note that we can split the term $\int_\Omega h_i\log h_i$ into

$$\int_\Omega h_i\log h_i = \int_\Omega h_i\log^+ h_i - \int_\Omega h_i\log^- h_i.$$

To take the supremum on the left-hand side of (14) we need a lower bound for the nonnegative term $-g\int_\Omega h_i\log^- h_i$. This is obtained by using the convexity inequality

$$h_i\log h_i \geqslant -\frac{1}{e},$$

that implies

$$-g\int_\Omega h_i\log^- h_i = g\int_{\{x\in\Omega:h_i(x,t)<1\}} h_i\log h_i$$
$$\geqslant -\frac{g}{e}\mathrm{meas}(\{x\in\Omega : h_i(x,t)<1\}) \geqslant -\frac{g}{e}\mathrm{meas}(\Omega).$$

Thus, we have

$$-\frac{2g}{e}\mathrm{meas}(\Omega) \leqslant \frac{1}{4}\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \frac{1}{4}\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}$$
$$+(g-2K^2\|h_{1,0}\|_{L^1})\sup_t\int_\Omega h_1\log^+ h_1 - g\sup_t\int_\Omega h_1\log^- h_1$$
$$+(g-2K^2\|h_{2,0}\|_{L^1})\sup_t\int_\Omega h_2\log^+ h_2 - g\sup_t\int_\Omega h_2\log^- h_2$$

$$
\begin{aligned}
&+\left(A_1-C_1-\frac{C}{2}\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)}-C_\lambda\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_1\|^2_{L^2(0,T;V)}\\
&+\left(A_2-C_2-\frac{C}{2}\|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)}-C_\lambda\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_2\|^2_{L^2(0,T;V)}\\
&+\left(g-\frac{3}{4}\lambda\right)\|h_1\|^2_{L^2(Q)}+\left(g-\frac{3}{4}\lambda\right)\|h_2\|^2_{L^2(Q)}\\
\leqslant&\sum_{i=1}^{2}\left[\frac{1}{2}\|u_{i,0}\|^2_{L^2}+(g+A_i)\int_\Omega h_{i,0}\log h_{i,0}-A_i\overline{h_{i,0}}\log\overline{h_{i,0}}\,\mathrm{meas}(\Omega)\right.\\
&\left.+C_i'+\int_\Omega p_i(0)h_{i,0}+2K^2\right]+g\frac{\rho_2}{\rho_1}\int_0^T|(h_2,\operatorname{div}u_1)|+g\int_0^T|(h_1,\operatorname{div}u_2)|.
\end{aligned}
$$

Using that

$$
g\frac{\rho_2}{\rho_1}\int_0^T|(h_2,\operatorname{div}u_1)|\leqslant g\frac{\rho_2}{\rho_1}\frac{\mu}{2}\|h_2\|^2_{L^2(Q)}+g\frac{\rho_2}{\rho_1}\frac{1}{2\mu}\|u_1\|^2_{L^2(0,T;V)}
$$

and

$$
g\int_0^T|(h_1,\operatorname{div}u_2)|\leqslant g\frac{\mu}{2}\|h_1\|^2_{L^2(Q)}+g\frac{1}{2\mu}\|u_2\|^2_{L^2(0,T;V)}
$$

we obtain

$$
\begin{aligned}
-\frac{2g}{e}\mathrm{meas}(\Omega)\leqslant&\frac{1}{4}\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}+\frac{1}{4}\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\\
&+(g-2K^2\|h_{1,0}\|_{L^1})\sup_t\int_\Omega h_1\log^+h_1-g\sup_t\int_\Omega h_1\log^-h_1\\
&+(g-2K^2\|h_{2,0}\|_{L^1})\sup_t\int_\Omega h_2\log^+h_2-g\sup_t\int_\Omega h_2\log^-h_2\\
&+\left(B_1-\frac{C}{2}\|u_1\|_{L^\infty(0,T;L^2(\Omega)^2)}-C_\lambda\|u_1\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_1\|^2_{L^2(0,T;V)}\\
&+\left(B_2-\frac{C}{2}\|u_2\|_{L^\infty(0,T;L^2(\Omega)^2)}-C_\lambda\|u_2\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\right)\|u_2\|^2_{L^2(0,T;V)}\\
&+\left(g-\frac{3}{4}\lambda-g\frac{\mu}{2}\right)\|h_1\|^2_{L^2(Q)}+\left(g-\frac{3}{4}\lambda-g\frac{\rho_2}{\rho_1}\frac{\mu}{2}\right)\|h_2\|^2_{L^2(Q)}\\
\leqslant&\sum_{i=1}^{2}\left[\frac{1}{2}\|u_{i,0}\|^2_{L^2}+(g+A_i)\int_\Omega h_{i,0}\log h_{i,0}-A_i\overline{h_{i,0}}\log\overline{h_{i,0}}\,\mathrm{meas}(\Omega)\right.\\
&\left.+C_i'+\int_\Omega p_i(0)h_{i,0}+2K^2\right].
\end{aligned}
$$

Now, to obtain estimates for u_i and h_i we only[3] need to prove the positivity of

$$B_i - \frac{C}{2}\|u_i\|_{L^\infty(0,T;L^2(\Omega)^2)} - C_\lambda\|u_i\|^2_{L^\infty(0,T;L^2(\Omega)^2)}, \quad i = 1,2.$$

This can be done using the small data hypothesis:

Let us assume that u_1 and u_2 are continuous from $[0,T]$ into $L^2(\Omega)^2$. As

$$\|u_{i,0}\|_{L^2} < \frac{\sqrt{C^2 + 16B_iC_\lambda} - C}{4C_\lambda},$$

there exists t' such that

$$\|u_i(t)\|_{L^2} < \frac{\sqrt{C^2 + 16B_iC_\lambda} - C}{4C_\lambda} \quad \text{in } [0,t'),$$

for $i = 1,2$. Suppose that

$$\|u_1(t')\|_{L^2} = \frac{\sqrt{C^2 + 16B_1C_\lambda} - C}{4C_\lambda}$$

and

$$\|u_2(t')\|_{L^2} \leqslant \frac{\sqrt{C^2 + 16B_2C_\lambda} - C}{4C_\lambda},$$

for instance. Then, (9) implies

$$\left(\frac{\sqrt{C^2+16B_1C_\lambda}-C}{16C_\lambda}\right)^2 \leqslant \sum_{i=1}^{2}\left[\frac{1}{2}\|u_{i,0}\|^2_{L^2}+(g+A_i)\int_\Omega h_{i,0}\log h_{i,0}+\frac{g}{e}\text{meas}(\Omega)\right.$$
$$\left. -A_i\overline{h_{i,0}}\log\overline{h_{i,0}}\,\text{meas}(\Omega) + C_i'+\int_\Omega p_i(0)h_{i,0}+2K^2\right],$$

which contradicts (3). The same contradiction holds if

$$\|u_2(t')\|_{L^2} = \frac{\sqrt{C^2 + 16B_2C_\lambda} - C}{4C_\lambda}$$

and

$$\|u_1(t')\|_{L^2} \leqslant \frac{\sqrt{C^2 + 16B_1C_\lambda} - C}{4C_\lambda}.$$

Therefore, (10) is proved.

[3] Notice that condition (1) implies

$$g > \frac{3}{4}\lambda + g\frac{\rho_2}{\rho_1}\frac{\mu}{2},$$

because $\rho_1 > \rho_2$.

2.3. Approximated solutions

Let us introduce a basis for V denoted by $\{v_1,\dots,v_n,\dots\}$, whose elements belong to $H^4(\Omega)^2$. Let V_n be the set of linear combinations of the n first elements of the basis. We consider the problem:

Find $(u_{1,n},h_{1,n})$ and $(u_{2,n},h_{2,n})$ in $[L^\infty(0,T;L^2(\Omega)^2)\cap L^2(0,T;V_n)]\times\mathscr{C}^1(\bar{Q})$ such that:

$$(\mathscr{V}_n)\quad\begin{cases}\left(\dfrac{\partial u_{1,n}}{\partial t},v\right)+A_1a(u_{1,n},v)-\dfrac{1}{2}(u_{1,n}^2,\operatorname{div}v)+(\operatorname{curl}u_{1,n}\alpha(u_{1,n}),v)\\ \quad -g(h_{1,n},\operatorname{div}v)-g\dfrac{\rho_2}{\rho_1}(h_{2,n},\operatorname{div}v)=0\quad\forall v\in V_n,\\ \left(\dfrac{\partial u_{2,n}}{\partial t},v\right)+A_2a(u_{2,n},v)-\dfrac{1}{2}(u_{2,n}^2,\operatorname{div}v)+(\operatorname{curl}u_{2,n}\alpha(u_{2,n}),v)\\ \quad -g(h_{2,n},\operatorname{div}v)-g(h_{1,n},\operatorname{div}v)=0\quad\forall v\in V_n,\\ \dfrac{\partial h_{1,n}}{\partial t}+\operatorname{div}(u_{1,n}h_{1,n})=0,\\ \dfrac{\partial h_{2,n}}{\partial t}+\operatorname{div}(u_{2,n}h_{2,n})=0,\\ u_{1,n}(t=0)=u_{1,0,n}\in V_n,\qquad u_{2,n}(t=0)=u_{2,0,n}\in V_n,\\ h_{1,n}(t=0)=h_{1,0,n}\in\mathscr{C}^1(\Omega),\qquad h_{2,n}(t=0)=h_{2,0,n}\in\mathscr{C}^1(\Omega),\end{cases}$$

where the data and the constants satisfy the conditions of Theorem 1. Then we have:

Lemma 2. *Problem* $(\mathscr{V}_n)$ *has a solution* $\{(u_{1,n},h_{1,n}),(u_{2,n},h_{2,n})\}$ *in*

$$[[L^\infty(0,T;L^2(\Omega)^2)\cap L^2(0,T;V_n)]\times\mathscr{C}^1(\bar{Q})]^2,$$

which satisfies

$$\|u_{1,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}+\|u_{2,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}+\sup_t\int_\Omega h_{1,n}\log h_{1,n}+\sup_t\int_\Omega h_{2,n}\log h_{2,n}$$
$$+\|u_{1,n}\|^2_{L^2(0,T;V)}+\|u_{2,n}\|^2_{L^2(0,T;V)}+\|h_{1,n}\|^2_{L^2(Q)}+\|h_{2,n}\|^2_{L^2(Q)}\leqslant C.\qquad(24)$$

Proof. To prove this lemma we apply the second Schauder fixed point theorem [13] as in [2]. We obtain approximated solutions that satisfy the a priori estimates.

In fact, due to the regularity of the basis, we have: $u_{1,n},u_{2,n}\in H^1(0,T;H^4(\Omega)^2)$. Therefore $u_{1,n},u_{2,n}\in\mathscr{C}^0([0,T];\mathscr{C}^2(\bar{\Omega})^2)$ and, using (11) and the positivity of initial data $h_{1,0},h_{2,0}$, we have: $h_{1,n},h_{2,n}\in\mathscr{C}^1(\bar{Q})$ and $h_{1,n},h_{2,n}>0$.

2.4. Passage to the limit

In this section, we present a lemma that is used to pass to the limit in the approximated equations and to conclude the proof of the theorem. The passage to the limit is done by adapting the procedure developed in [2] for a one-layer problem. In that case, the most difficult point was to pass to the limit in the continuity equation. Now, this can be done in an easier way, because we have obtained an additional estimate for h_i in $L^2(Q)$.

Lemma 3. *For each* $n \in \mathbb{N}$, *let*

$$\{(u_{1,n}, h_{1,n}), (u_{2,n}, h_{2,n})\} \in [[L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;V_n)] \times \mathscr{C}^1(\bar{Q})]^2$$

be the solution of $(\mathscr{V}_n)$ *given by Lemma* 2, *that satisfies*

$$\|u_{1,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|u_{2,n}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \sup_t \int_\Omega h_{1,n} \log h_{1,n} + \sup_t \int_\Omega h_{2,n} \log h_{2,n}$$
$$+ \|u_{1,n}\|^2_{L^2(0,T;V)} + \|u_{2,n}\|^2_{L^2(0,T;V)} + \|h_{1,n}\|^2_{L^2(Q)} + \|h_{2,n}\|^2_{L^2(Q)} \leqslant \mathrm{C}.$$

Then we have, for $i = 1, 2$:

$$u_{i,n}h_{i,n} \text{ is bounded in } L^2(0,T;L^1(\Omega)^2), \tag{25}$$

$$\frac{\partial u_{i,n}}{\partial t} \text{ is bounded in } L^{4/3}(0,T;H^{-1}(\Omega)^2) \tag{26}$$

and we can extract from $u_{i,n}$ *and* $h_{i,n}$ *subsequences still denoted* $u_{i,n}$ *and* $h_{i,n}$ *such that*

$$u_{i,n} \to u_i \quad \text{in } L^2(0,T;V) \text{ weakly}, \tag{27}$$

$$u_{i,n} \to u_i \quad \text{in } L^\infty(0,T;L^2(\Omega)^2) \text{ weakly star}, \tag{28}$$

$$h_{i,n} \to h_i \quad \text{in } L^2(Q) \text{ weakly}, \tag{29}$$

$$u_{i,n}h_{i,n} \to u_i h_i \quad \text{in } L^{4/3}(Q)^2 \text{ weakly}, \tag{30}$$

$$\operatorname{curl} u_{i,n}\alpha(u_{i,n}) \to \operatorname{curl} u_i\alpha(u_i) \quad \text{in } L^{4/3}(Q)^2 \text{ weakly}, \tag{31}$$

$$\nabla u_{i,n}^2 \to \nabla u_i^2 \quad \text{in } L^{4/3}(Q)^2 \text{ weakly}. \tag{32}$$

Proof. Results (25), (27)–(29) are a direct consequence of (24).

Gagliardo–Nirenberg's inequality implies that $u_{i,n}$ is bounded in $L^4(Q)^2$. Then $u_{i,n}h_{i,n}$, $\operatorname{curl} u_{i,n}\alpha(u_{i,n})$ and $\nabla u_{i,n}^2$ are bounded in $L^{4/3}(Q)^2$, and we can extract subsequences from $u_{i,n}$ and $h_{i,n}$ such that

$$u_{i,n}h_{i,n} \to \kappa_i \quad \text{in } L^{4/3}(Q)^2 \text{ weakly},$$

$$\operatorname{curl} u_{i,n}\alpha(u_{i,n}) \to \theta_i \quad \text{in } L^{4/3}(Q)^2 \text{ weakly}$$

and

$$\nabla u_{i,n}^2 \to \vartheta_i \quad \text{in } L^{4/3}(Q)^2 \text{ weakly}.$$

In order to obtain $\kappa_i = u_i h_i$, $\theta_i = \operatorname{curl} u_i\alpha(u_i)$ and $\vartheta_i = \nabla u_i^2$ we need an estimate for $\partial u_{i,n}/\partial t$ in $L^{4/3}(0,T;H^{-1}(\Omega)^2)$:

Notice that $\nabla h_{i,n}$ is bounded in $L^2(0,T;H^{-1}(\Omega)^2)$. By estimating the other terms in the momentum equations we get (26).

Now, using Aubin's compacity theorem [7] with

$$A_0 = V, \quad A_1 = L^2(\Omega)^2, \qquad A_2 = H^{-1}(\Omega)^2,$$

$$p = 2, \qquad q = 4/3,$$

we have

$$u_{i,n} \to u_i \quad \text{in } L^2(Q)^2 \text{ and a.e. in } Q. \tag{33}$$

This will allow us to prove (30)–(32):

Let $\varphi \in \mathscr{D}(Q)^2$. Then

$$\begin{aligned} |(u_{i,n}h_{i,n} - u_i h_i, \varphi)| &\leqslant |(u_{i,n}h_{i,n} - u_i h_{i,n}, \varphi)| + |(u_i h_{i,n} - u_i h_i, \varphi)| \\ &\leqslant \|u_{i,n} - u_i\|_{L^2(Q)} \|h_{i,n}\|_{L^2(Q)} \|\varphi\|_{L^\infty(Q)} + |(h_{i,n} - h_i, u_i\varphi)| \end{aligned}$$

and so,

$$u_{i,n}h_{i,n} \to u_i h_i \quad \text{in } \mathscr{D}'(Q)^2,$$

which implies that $\kappa_i = u_i h_i$.

Equally, it can be proved that $\theta_i = \operatorname{curl} u_i \alpha(u_i)$ and $\vartheta_i = \nabla u_i^2$, $i = 1, 2$.

2.5. Proof of the theorem

Let $u_{1,0}, u_{2,0}$ and $h_{1,0}, h_{2,0}$ be the initial conditions of problem $(\mathscr{P})$.

Let $\{u_{1,0,n}\}$ and $\{u_{2,0,n}\}$ be two sequences with elements $u_{i,0,n} \in V_n$ such that

$$u_{1,0,n} \to u_{1,0} \quad \text{and} \quad u_{2,0,n} \to u_{2,0} \ \text{ in } V.$$

Also, let $\{h_{1,0,n}\}$ and $\{h_{2,0,n}\}$ be two sequences in $\mathscr{C}^1(\Omega)$ such that

$$h_{1,0,n} \to h_{1,0} \quad \text{and} \quad h_{2,0,n} \to h_{2,0} \ \text{ in } L^2(\Omega).$$

For each $n \in \mathbb{N}$, set

$$\{(u_{1,n}, h_{1,n}), (u_{2,n}, h_{2,n})\} \in [[L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; V_n)] \times \mathscr{C}^1(\bar{Q})]^2$$

a solution of $(\mathscr{V}_n)$ given by Lemma 2. This satisfies estimate (24).

Using Lemma 3, we can extract two subsequences to $\{u_{1,n}\}$ and $\{u_{2,n}\}$, also denoted by $\{u_{1,n}\}$ and $\{u_{2,n}\}$, such that

$$u_{1,n} \to u_1 \quad \text{and} \quad u_{2,n} \to u_2 \ \text{ in } L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; V) \text{ weakly star}$$

and two subsequences to $h_{1,n}$ and $h_{2,n}$, also denoted by $h_{1,n}$ and $h_{2,n}$, such that

$$h_{1,n} \to h_1 \quad \text{and} \quad h_{2,n} \to h_2 \ \text{ in } L^2(Q).$$

Then,

$$u_{1,n}h_{1,n} \to u_1 h_1 \quad \text{and} \quad u_{2,n}h_{2,n} \to u_2 h_2 \ \text{ in } L^{4/3}(Q)^2 \text{ weakly.}$$

Now we can deduce from the previous results that $\operatorname{div}(u_i h_i)$ belong to $L^{4/3}(0, T; W^{-1,4/3}(\Omega))$ and so $h_{i,t}$. We also have $h_i(t = 0) = h_{i,0}$.

So we can pass to the limit in momentum equations and obtain $u_i(t = 0) = u_{i,0}$.

This concludes the proof that $\{(u_1,h_1),(u_2,h_2)\}$ is a solution of the weak problem $(\mathscr{V})$.

Having shown the existence of solutions to problem $(\mathscr{V})$, we are going to prove the uniqueness of the solution. In order to do this, first we have to prove some smoothness results for $(u_i,h_i),\ i=1,2$.

3. Some smoothness and uniqueness results

3.1. Smoothness for the curl velocity

We consider the L^2-projection of equations $(\mathscr{P})_1$ and $(\mathscr{P})_7$ on the curl vectors field:

$$\int_\Omega \left(\frac{\partial u_1}{\partial t} - A_1\Delta u_1 + \frac{1}{2}\nabla u_1^2 + \operatorname{curl} u_1 \alpha(u_1) + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2\right)\operatorname{Curl} Q = 0,$$

$$\int_\Omega \left(\frac{\partial u_2}{\partial t} - A_2\Delta u_2 + \frac{1}{2}\nabla u_2^2 + \operatorname{curl} u_2 \alpha(u_2) + g\nabla h_2 + g\nabla h_1\right)\operatorname{Curl} Q = 0,$$

where $Q \in H_0^1(\Omega)$.

Using the orthogonal decomposition $u_1 = u_{p,1} + u_{q,1}$ and $u_2 = u_{p,2} + u_{q,2}$ we arrive at

$$P\left(\frac{\partial u_{q,1}}{\partial t} + A_1 \operatorname{Curl}(\operatorname{curl} u_{q,1}) + \operatorname{curl} u_{q,1}\alpha(u_1)\right) = 0, \tag{34}$$

$$P\left(\frac{\partial u_{q,2}}{\partial t} + A_2 \operatorname{Curl}(\operatorname{curl} u_{q,2}) + \operatorname{curl} u_{q,2}\alpha(u_2)\right) = 0, \tag{35}$$

where P denotes the L^2-projection operator on the curl vectors field.

The results stated in this section were proved in [3] for the one-layer case. It is easy to adapt those results and their proofs to the two-layer case.

Lemma 4. *If $u_{i,0} \in H^1(\Omega)^2$, then we have*

$$u_{q,i} \in L^4(0,T;W^{1,4}(\Omega)^2), \tag{36}$$

$$\frac{\partial u_{q,i}}{\partial t} \in L^2(Q)^2, \tag{37}$$

for $i=1,2$.

Lemma 5. *If $u_{i,0} \in H^2(\Omega)^2$ and if $\partial u_{p,i}/\partial t \in L^2(Q)^2$, then*

$$\frac{\partial u_{q,i}}{\partial t} \in L^\infty(0,T;L^2(\Omega)^2), \tag{38}$$

$$\operatorname{curl}\left(\frac{\partial u_{q,i}}{\partial t}\right) \in L^2(Q) \tag{39}$$

and

$$u_{q,i} \in W^{1,4}(Q)^2, \tag{40}$$

for $i=1,2$.

Lemma 6. *If* $u_{i,0} \in H^2(\Omega)^2$ *and if* $u_{p,i} \in L^2(0,T;W^{1,4}(\Omega)^2)$ *we have*

$$u_{q,i} \in L^4(0,T;W^{2,4}(\Omega)^2), \tag{41}$$

for $i = 1,2$.

Remark 1. In the proof of (36), the following smoothness for $u_{q,i}$ is obtained

$$u_{q,i} \in L^\infty(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2), \tag{42}$$

$i = 1,2$.

3.2. A crucial estimate

First, we recall the L^2-projection of equations $(\mathscr{P})_1$ and $(\mathscr{P})_7$ on the gradient vectors field previously obtained:

$$\frac{\partial p_1}{\partial t} - A_1 \Delta p_1 + \frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2 = \zeta_1, \tag{43}$$

$$\frac{\partial p_2}{\partial t} - A_2 \Delta p_2 + \frac{1}{2}u_2^2 + \Theta_2 + gh_2 + gh_1 = \zeta_2, \tag{44}$$

where the functions ζ_1 and ζ_2 depend only on time, and $\nabla\Theta_1$ and $\nabla\Theta_2$ are, respectively, the projections for the $L^2(\Omega)$ scalar product of $\operatorname{curl} u_{q,1}\alpha(u_1)$ and $\operatorname{curl} u_{q,2}\alpha(u_2)$ on the space $\nabla H^1(\Omega)$.

We were able to choose Θ_i such that $\int_\Omega \Theta_i = 0$. Then we have

$$\|\nabla\Theta_i\|_{L^2} \leqslant k\|\operatorname{curl} u_{q,i}\alpha(u_i)\|_{L^2}$$

and the norm $\|\Theta_i\|_{H^1}$ is equivalent to $\|\nabla\Theta_i\|_{L^2}$, $i = 1,2$.

The following results concerning Θ_i and ζ_i are easily proved as in [3].

Lemma 7. *The functions* Θ_i *previously defined verify*:

$$\Theta_i \in L^4(Q), \tag{45}$$

$$\|\nabla\Theta_{i,t}\|_{L^1} \leqslant C\left(1 + \left\|\frac{\partial u_{p,i}}{\partial t}\right\|_{L^2}\right), \tag{46}$$

for $i = 1,2$.

Lemma 8. *The functions* ζ_i *verify*:

$$\zeta_i \in L^\infty(0,T), \tag{47}$$

$$\zeta_{i,t} = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u_i u_{i,t}, \tag{48}$$

for $i = 1,2$.

Now we give a lemma that will allow us to prove our smoothness theorem.

Lemma 9. *Let Q_t be equal to $\Omega \times (0,t)$, for any $t \in [0,T]$.*

If $h_{i,0} \in L^3(\Omega)$, then the relation

$$\|h_i\|^3_{L^\infty(0,t;L^3(\Omega)^2)} + \|h_i\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(1 + \|p_{i,t}\|^4_{L^4(Q_t)}) \tag{49}$$

holds for $i \in \{1,2\}$, and also the relation

$$\|\Delta p_i\|^4_{L^4(Q_t)} \leqslant \mathrm{C}\left(1 + \sum_{j=1}^{2} \|p_{j,t}\|^4_{L^4(Q_t)}\right). \tag{50}$$

In both cases, $\mathrm{C} > 0$ is a constant that depends only on the data.

Proof. To prove (49) we first multiply the continuity equation $(\mathscr{P})_5$ by $3h_1^2$ to obtain

$$\frac{\partial h_1^3}{\partial t} + \operatorname{div}(u_1 h_1^3) + 2h_1^3 \Delta p_1 = 0.$$

Replacing Δp_1 with its value given by Eq. (43) we have

$$A_1 \frac{\partial h_1^3}{\partial t} + A_1 \operatorname{div}(u_1 h_1^3) + u_1^2 h_1^3 + 2g h_1^4$$
$$+ 2g \frac{\rho_2}{\rho_1} h_1^3 h_2 = 2h_1^3 \zeta_1 - 2h_1^3 \frac{\partial p_1}{\partial t} - 2h_1^3 \Theta_1. \tag{51}$$

Then, integrating (51) over Q_t we obtain

$$A_1 \|h_1(t)\|^3_{L^3} - A_1 \|h_{1,0}\|^3_{L^3} + A_1 \int_{Q_t} \operatorname{div}(u_1 h_1^3)$$
$$+ \int_{Q_t} u_1^2 h_1^3 + 2g \|h_1\|^4_{L^4(Q_t)} + 2g \frac{\rho_2}{\rho_1} \int_{Q_t} h_1^3 h_2$$
$$\leqslant 2 \int_{Q_t} |h_1|^3 |\zeta_1| + 2 \int_{Q_t} |h_1|^3 \left|\frac{\partial p_1}{\partial t}\right| + 2 \int_{Q_t} |h_1|^3 |\Theta_1|. \tag{52}$$

We know that $\int_{Q_t} \operatorname{div}(u_1 h_1^3) = 0$, $\int_{Q_t} u_1^2 h_1^3 \geqslant 0$ and $\int_{Q_t} h_1^3 h_2 \geqslant 0$. To bound the terms on the right-hand side of (52) we use Young's inequality as follows

$$\int_{Q_t} |h_1|^3 |\zeta_1| \leqslant \frac{\varepsilon}{3} \|h_1\|^4_{L^4(Q_t)} + C_\varepsilon \|\zeta_1\|^4_{L^4(Q_t)},$$
$$\int_{Q_t} |h_1|^3 \left|\frac{\partial p_1}{\partial t}\right| \leqslant \frac{\varepsilon}{3} \|h_1\|^4_{L^4(Q_t)} + C_\varepsilon \|p_{1,t}\|^4_{L^4(Q_t)},$$
$$\int_{Q_t} |h_1|^3 |\Theta_1| \leqslant \frac{\varepsilon}{3} \|h_1\|^4_{L^4(Q_t)} + C_\varepsilon \|\Theta_1\|^4_{L^4(Q_t)}.$$

Choosing the positive number ε sufficiently small we obtain from (52) the estimate

$$\|h_1\|^3_{L^\infty(0,t;L^3(\Omega)^2)} + \|h_1\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(1 + \|p_{1,t}\|^4_{L^4(Q_t)}).$$

Analogously, we prove that

$$\|h_2\|^3_{L^\infty(0,t;L^3(\Omega)^2)} + \|h_2\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(1 + \|p_{2,t}\|^4_{L^4(Q_t)}).$$

Finally, to prove (50) we multiply Eq. (43) by $\Delta p_1|\Delta p_1|^2$ and integrate over Q_t:

$$A_1\|\Delta p_1\|^4_{L^4(Q_t)} = -\int_{Q_t} \zeta_1 \Delta p_1|\Delta p_1|^2 + \int_{Q_t} \frac{\partial p_1}{\partial t}\Delta p_1|\Delta p_1|^2 + \frac{1}{2}\int_{Q_t} u_1^2 \Delta p_1|\Delta p_1|^2 + \int_{Q_t} \Theta_1 \Delta p_1|\Delta p_1|^2 + g\int_{Q_t} h_1 \Delta p_1|\Delta p_1|^2 + g\frac{\rho_2}{\rho_1}\int_{Q_t} h_2 \Delta p_1|\Delta p_1|^2.$$

Using Gagliardo–Nirenberg's inequality

$$\|u\|^2_{L^8} \leqslant C\|u\|_{L^2}\|\operatorname{div} u\|_{L^4}$$

we can estimate

$$\int_{Q_t} u_1^2 \Delta p_1|\Delta p_1|^2 \leqslant C\|u_1\|_{L^\infty(0,t;L^2(\Omega)^2)}\|\operatorname{div} u_1\|^4_{L^4(Q_t)}$$

and using again Young's inequality we have

$$\left(A_1 - \frac{C}{2}\|u_1\|_{L^\infty(0,t;L^2(\Omega)^2)}\right)\|\Delta p_1\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(\|\zeta_1\|^4_{L^4(Q_t)} + \|p_{1,t}\|^4_{L^4(Q_t)} + \|\Theta_1\|^4_{L^4(Q_t)} + \|h_1\|^4_{L^4(Q_t)} + \|h_2\|^4_{L^4(Q_t)}).$$

Using (49) and the assumptions concerning the stability space described in [3], which allow us to ensure that $A_1 - C/2\|u_1\|_{L^\infty(0,t;L^2(\Omega)^2)} > 0$, we obtain

$$\|\Delta p_1\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(1 + \|p_{1,t}\|^4_{L^4(Q_t)} + \|p_{2,t}\|^4_{L^4(Q_t)}).$$

In the same way, we prove that

$$\|\Delta p_2\|^4_{L^4(Q_t)} \leqslant \mathrm{C}(1 + \|p_{2,t}\|^4_{L^4(Q_t)} + \|p_{1,t}\|^4_{L^4(Q_t)}).$$

Observe that, if we prove that $p_{1,t}$ and $p_{2,t}$ belong to $L^4(Q)$, then we will also have that h_i and $\operatorname{div} u_i$ are in $L^4(Q)$, $i = 1, 2$. This is the goal of the next result.

Theorem 2. *Let $h_{i,0} \in L^3(\Omega)$ and let $u_{i,0} \in H^2(\Omega)^2$ for $i = 1, 2$. Then we have*

$$p_i \in W^{1,4}(Q) \tag{53}$$

and so

$$h_i \in L^4(Q), \tag{54}$$

$$\operatorname{div} u_i \in L^4(Q), \tag{55}$$

$i = 1, 2$.

Proof. The proof of this theorem is very technical and uses the techniques developed in [12].

To show that $p_1 \in W^{1,4}(Q)$ it is enough to prove that $p_{1,t} \in L^4(Q)$. To prove this, we first differentiate Eq. (43) with respect to all independent variables, and obtain the system

$$\frac{\partial u_{p,1}}{\partial t} - A_1 \Delta u_{p,1} + \frac{1}{2}\nabla u_1^2 + \nabla\Theta_1 + g\nabla h_1 + g\frac{\rho_2}{\rho_1}\nabla h_2 = 0, \tag{56}$$

$$\frac{\partial p_{1,t}}{\partial t} - A_1 \Delta p_{1,t} + u_1 u_{1,t} + \Theta_{1,t} + g h_{1,t} + g\frac{\rho_2}{\rho_1} h_{2,t} = \zeta_{1,t}. \tag{57}$$

Next, we multiply (56) by $4u_{p,1}|u_{p,1}|^2$, (57) by $2p_{1,t}$, and integrate over Ω:

$$\frac{d}{dt}\int_\Omega |u_{p,1}|^4 + 4A_1\int_\Omega (\operatorname{div} u_{p,1})\operatorname{div}(u_{p,1}|u_{p,1}|^2)$$
$$= 4\int_\Omega \left(\frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right)\operatorname{div}(u_{p,1}|u_{p,1}|^2). \tag{58}$$

$$\frac{d}{dt}\int_\Omega |p_{1,t}|^2 + 2A_1\int_\Omega |\nabla p_{1,t}|^2$$
$$= 2\int_\Omega (\zeta_{1,t} - u_1u_{1,t} - \Theta_{1,t})p_{1,t} - 2g\int_\Omega h_{1,t}p_{1,t} - 2g\frac{\rho_2}{\rho_1}\int_\Omega h_{2,t}p_{1,t}. \tag{59}$$

Now, some of the terms of previous equations are rewritten:
Note that the second term in (58) can be expressed as

$$4A_1\int_\Omega (\operatorname{div} u_{p,1})\operatorname{div}(u_{p,1}|u_{p,1}|^2)$$
$$= 4A_1\int_\Omega (\operatorname{div} u_{p,1})|u_{p,1}|^2\operatorname{div} u_{p,1} + 4A_1\int_\Omega (\operatorname{div} u_{p,1})u_{p,1}\cdot\nabla|u_{p,1}|^2$$
$$= 4A_1\|u_{p,1}\operatorname{div} u_{p,1}\|_{L^2}^2 + 8A_1\int_\Omega (\operatorname{div} u_{p,1})u_{p,1}\cdot(u_{p,1}\cdot\nabla)u_{p,1},$$

while

$$4\int_\Omega \left(\frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right)\operatorname{div}(u_{p,1}|u_{p,1}|^2)$$
$$= 4\int_\Omega \left(\frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right)|u_{p,1}|^2\operatorname{div} u_{p,1}$$
$$+ 8\int_\Omega \left(\frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right)u_{p,1}\cdot(u_{p,1}\cdot\nabla)u_{p,1}.$$

Substituting $g(h_1 + (\rho_2/\rho_1)h_2)$ by its value in (43) and using the relation

$$\|(u_{p,1}\cdot\nabla)u_{p,1}\|_{L^2} \leqslant C\|u_{p,1}\operatorname{div} u_{p,1}\|_{L^2}$$

we have

$$4\int_\Omega \left(\frac{1}{2}u_1^2 + \Theta_1 + gh_1 + g\frac{\rho_2}{\rho_1}h_2\right)\operatorname{div}(u_{p,1}|u_{p,1}|^2)$$
$$\leqslant \varepsilon\|u_{p,1}\operatorname{div} u_{p,1}\|_{L^2} + C_\varepsilon\int_\Omega (|\zeta_1|^2 + |p_{1,t}|^2 + |\operatorname{div} u_{p,1}|^2)|u_{p,1}|^2.$$

Then, Eq. (58) yields

$$\frac{d}{dt}\|u_{p,1}\|_{L^4}^4 + (4A_1 - \varepsilon)\|u_{p,1}\operatorname{div} u_{p,1}\|_{L^2}^2$$
$$\leqslant 8A_1\int_\Omega |(\operatorname{div} u_{p,1})u_{p,1}\cdot(u_{p,1}\cdot\nabla)u_{p,1}|$$
$$+ C_\varepsilon\int_\Omega (|\zeta_1|^2 + |p_{1,t}|^2 + |\operatorname{div} u_{p,1}|^2)|u_{p,1}|^2. \tag{60}$$

The last two terms in (59) can be treated using the continuity equations $(\mathscr{P})_5$ and $(\mathscr{P})_{11}$, respectively, as follows:

$$-2g\int_\Omega h_{1,t}p_{1,t} = 2g\int_\Omega \mathrm{div}(u_1h_1)p_{1,t} = -2g\int_\Omega u_1h_1\nabla p_{1,t}$$
$$= -2g\int_\Omega u_{p,1}h_1\nabla p_{1,t} - 2g\int_\Omega u_{q,1}h_1\nabla p_{1,t}$$

and

$$-2g\frac{\rho_2}{\rho_1}\int_\Omega h_{2,t}p_{1,t} = 2g\frac{\rho_2}{\rho_1}\int_\Omega \mathrm{div}(u_2h_2)p_{1,t} = -2g\frac{\rho_2}{\rho_1}\int_\Omega u_2h_2\nabla p_{1,t}$$
$$= -2g\frac{\rho_2}{\rho_1}\int_\Omega u_{p,2}h_2\nabla p_{1,t} - 2g\frac{\rho_2}{\rho_1}\int_\Omega u_{q,2}h_2\nabla p_{1,t}.$$

Now, note that we can give an expression for h_2 in which h_1 does not appear if we subtract Eq. (44) from Eq. (43). In a similar way, an expression for h_1 (not containing h_2) is obtained.

Using them, we have that

$$-2g\int_\Omega h_{1,t}p_{1,t} \leqslant \frac{\varepsilon}{2}\|\nabla p_{1,t}\|_{L^2}^2 + C_\varepsilon\int_\Omega S_{1,2}(|u_{p,1}|^2+|u_{q,1}|^2)$$

and

$$-2g\frac{\rho_2}{\rho_1}\int_\Omega h_{2,t}p_{1,t} \leqslant \frac{\varepsilon}{2}\|\nabla p_{1,t}\|_{L^2}^2 + C_\varepsilon\int_\Omega S_{1,2}(|u_{p,2}|^2+|u_{q,2}|^2),$$

where

$$S_{1,2} = \sum_{i=1,2}(|\zeta_i|^2+|p_{i,t}|^2+|\mathrm{div}\,u_{p,i}|^2+|u_i|^4+|\Theta_i|^2).$$

Then, (59) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|p_{1,t}\|_{L^2}^2 + (2A_1-\varepsilon)\|\nabla p_{1,t}\|_{L^2}^2 \leqslant 2\int_\Omega(\zeta_{1,t}-u_1u_{1,t}-\Theta_{1,t})p_{1,t}$$
$$+ C_\varepsilon\int_\Omega S_{1,2}(|u_{p,1}|^2+|u_{q,1}|^2+|u_{p,2}|^2+|u_{q,2}|^2). \tag{61}$$

Adding (60) and (61) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|u_{p,1}\|_{L^4}^4 + \|p_{1,t}\|_{L^2}^2]$$
$$+ (2A_1-\varepsilon)[\|u_{p,1}\mathrm{div}\,u_{p,1}\|_{L^2}^2 + \|\nabla p_{1,t}\|_{L^2}^2]$$
$$\leqslant 8A_1\int_\Omega |(\mathrm{div}\,u_{p,1})u_{p,1}\cdot(u_{p,1}\cdot\nabla)u_{p,1}|$$
$$+ C_\varepsilon\int_\Omega S_{1,2}(|u_{p,1}|^2+|u_{q,1}|^2+|u_{p,2}|^2+|u_{q,2}|^2)$$
$$+ 2\int_\Omega(\zeta_{1,t}-u_1u_{1,t}-\Theta_{1,t})p_{1,t}. \tag{62}$$

Now we want to estimate all the terms on the right-hand side of (62).

We start with the terms of $\int_\Omega S_{1,2}|u_{p,1}|^2$:

As $\zeta_i \in L^\infty(0,T)$ and $u_{p,1} \in L^\infty(0,T;L^2(\Omega)^2)$,

$$\int_\Omega |\zeta_i|^2|u_{p,1}|^2 \leqslant \mathrm{C}.$$

We use Gagliardo–Nirenberg's inequality and $u_{p,1} \in L^\infty(0,T;L^2(\Omega)^2)$ again to get

$$\begin{aligned}\int_\Omega |p_{i,t}|^2|u_{p,1}|^2 &\leqslant \|p_{i,t}\|_{L^4}^2\|u_{p,1}\|_{L^4}^2\\ &\leqslant C^2\|p_{i,t}\|_{L^2}\|\nabla p_{i,t}\|_{L^2}\|u_{p,1}\|_{L^2}\|\mathrm{div}\,u_{p,1}\|_{L^2}\\ &\leqslant \varepsilon\|\nabla p_{i,t}\|_{L^2}^2 + C_\varepsilon\|\mathrm{div}\,u_{p,1}\|_{L^2}^2\|p_{i,t}\|_{L^2}^2. \end{aligned}\tag{63}$$

We also have

$$\begin{aligned}\int_\Omega |\mathrm{div}\,u_{p,i}|^2|u_{p,1}|^2 &\leqslant \|\mathrm{div}u_{p,i}\|_{L^2}\|\mathrm{div}\,u_{p,i}\|_{L^4}\||u_{p,1}|^2\|_{L^4}\\ &\leqslant C^{1/2}\||u_{p,1}|^2\|_{L^2}^{1/2}\|\nabla|u_{p,1}|^2\|_{L^2}^{1/2}\|\mathrm{div}\,u_{p,i}\|_{L^2}\|\mathrm{div}\,u_{p,i}\|_{L^4}\\ &\leqslant \varepsilon\|u_{p,1}\,\mathrm{div}\,u_{p,1}\|_{L^2}^2 + \varepsilon'\|\mathrm{div}\,u_{p,i}\|_{L^2}\|\mathrm{div}\,u_{p,i}\|_{L^4}^2\\ &\quad + C_{\varepsilon\varepsilon'}\|\mathrm{div}\,u_{p,i}\|_{L^2}^2\|u_{p,1}\|_{L^4}^4. \end{aligned}\tag{64}$$

To estimate the term $\int_\Omega |u_i|^4|u_{p,1}|^2$ we first use Young's inequality to write

$$|u_i|^4 \leqslant (|u_{p,i}| + |u_{q,i}|)^4 \leqslant \mathrm{C}(|u_{p,i}|^4 + |u_{q,i}|^4).$$

Also using Gagliardo–Nirenberg's inequality

$$\|u\|_{L^8}^4 \leqslant C'\|u\|_{L^4}^3\|\mathrm{div}\,u\|_{L^4},$$

we have

$$\begin{aligned}\int_\Omega |u_{p,i}|^4|u_{p,1}|^2 &\leqslant \|u_{p,i}\|_{L^8}^4\|u_{p,1}\|_{L^4}^2\\ &\leqslant C'C\|u_{p,i}\|_{L^4}^3\|\mathrm{div}\,u_{p,i}\|_{L^4}\|u_{p,1}\|_{L^2}\|\mathrm{div}\,u_{p,1}\|_{L^2}\\ &\leqslant C'C^{3/2}\|u_{p,i}\|_{L^4}^2\|u_{p,i}\|_{L^2}^{1/2}\|\mathrm{div}\,u_{p,i}\|_{L^2}^{1/2}\|\mathrm{div}\,u_{p,i}\|_{L^4}\|\mathrm{div}u_{p,1}\|_{L^2}\\ &\leqslant \varepsilon'\|\mathrm{div}\,u_{p,i}\|_{L^2}\|\mathrm{div}\,u_{p,i}\|_{L^4}^2 + C_{\varepsilon'}\|\mathrm{div}\,u_{p,1}\|_{L^2}^2\|u_{p,i}\|_{L^4}^4. \end{aligned}\tag{65}$$

Using (36) we know that $u_{q,i} \in L^4(0,T;L^\infty(\Omega)^2)$. Then

$$\int_\Omega |u_{q,i}|^4|u_{p,1}|^2 \leqslant \mathrm{C}\|u_{q,i}\|_{L^\infty}^4,\tag{66}$$

with the $\|u_{q,i}\|_{L^\infty}^4$ integrable function in $[0,T]$.

As $\Theta_i \in L^4(Q)$ and $u_{p,1} \in L^4(Q)^2$, we also have

$$\int_\Omega |\Theta_i|^2|u_{p,1}|^2 \leqslant \|\Theta_i\|_{L^4}^2\|u_{p,1}\|_{L^4}^2,$$

with the second term integrable in $[0,T]$.

Next we estimate the terms in $\int_\Omega S_{1,2}|u_{q,1}|^2$:

The first and the last terms are as easily bounded as those of $\int_\Omega S_{1,2}|u_{p,1}|^2$. The other ones are bounded as follows:

By (42), we also know that $u_{q,i} \in L^\infty(0,T;L^4(\Omega)^2)$. Then

$$\begin{aligned}\int_\Omega |p_{i,t}|^2|u_{q,1}|^2 &\leqslant \|p_{i,t}\|_{L^4}^2\|u_{q,1}\|_{L^4}^2\\ &\leqslant C\|p_{i,t}\|_{L^2}\|\nabla p_{i,t}\|_{L^2}\|u_{q,1}\|_{L^4}^2\\ &\leqslant \varepsilon\|\nabla p_{i,t}\|_{L^2}^2 + C_\varepsilon\|p_{i,t}\|_{L^2}^2 \end{aligned} \tag{67}$$

and

$$\begin{aligned}\int_\Omega |\mathrm{div}\, u_{p,i}|^2|u_{q,1}|^2 &\leqslant \|u_{q,1}\|_{L^\infty}\|u_{q,1}\|_{L^4}\|\mathrm{div}\, u_{p,i}\|_{L^2}\|\mathrm{div}\, u_{p,i}\|_{L^4}\\ &\leqslant \varepsilon'\|\mathrm{div}\, u_{p,i}\|_{L^2}\|\mathrm{div}\, u_{p,i}\|_{L^4}^2 + C_{\varepsilon'}\|u_{q,1}\|_{L^\infty}^2\|\mathrm{div}\, u_{p,i}\|_{L^2}, \end{aligned} \tag{68}$$

where the last term is in $L^1([0,T])$. Finally,

$$\begin{aligned}\int_\Omega |u_i|^4|u_{q,1}|^2 &\leqslant \mathrm{C}\int_\Omega |u_{p,i}|^4|u_{q,1}|^2 + \mathrm{C}\int_\Omega |u_{q,i}|^4|u_{q,1}|^2\\ &\leqslant \mathrm{C}\|u_{p,i}\|_{L^4}^4\|u_{q,1}\|_{L^\infty}^2 + \mathrm{C}\|u_{q,i}\|_{L^4}^4\|u_{q,1}\|_{L^\infty}^2, \end{aligned} \tag{69}$$

with the last term again in $L^1([0,T])$.

The terms in $\int_\Omega S_{1,2}|u_{p,2}|^2$ and $\int_\Omega S_{1,2}|u_{q,2}|^2$ are estimated in the same way as the previous ones.

The first term on the right-hand side of (62) is estimated using relation (64) again and

$$\|(u_{p,1}\cdot\nabla)u_{p,1}\|_{L^2} \leqslant \mathrm{C}\|u_{p,1}\mathrm{div}\, u_{p,1}\|_{L^2}.$$

Now we estimate the last term in (62):

Using (48) and (37) we have that

$$\begin{aligned}\int_\Omega \zeta_{1,t}p_{1,t} &\leqslant \frac{1}{\sqrt{\mathrm{meas}(\Omega)}}\|u_1\|_{L^2}\left(\|\nabla p_{1,t}\|_{L^2} + \left\|\frac{\partial u_{q,1}}{\partial t}\right\|_{L^2}\right)\|p_{1,t}\|_{L^2}\\ &\leqslant \varepsilon\|\nabla p_{1,t}\|_{L^2}^2 + \varepsilon\left\|\frac{\partial u_{q,1}}{\partial t}\right\|_{L^2}^2 + C_\varepsilon\|p_{1,t}\|_{L^2}^2, \end{aligned} \tag{70}$$

with $\varepsilon\|\partial u_{q,1}/\partial t\|_{L^2}^2$ integrable in $[0,T]$.

Next, the term $\int_\Omega u_1u_{1,t}p_{1,t}$ is bounded by

$$\int_\Omega u_1u_{1,t}p_{1,t} \leqslant \varepsilon\|\nabla p_{1,t}\|_{L^2}^2 + \varepsilon\left\|\frac{\partial u_{q,1}}{\partial t}\right\|_{L^2}^2 + C_\varepsilon\int_\Omega |u_1|^2|p_{1,t}|^2.$$

Taking into account (63) and (67) we obtain

$$\int_\Omega u_1u_{1,t}p_{1,t} \leqslant \varepsilon\|\nabla p_{1,t}\|_{L^2}^2 + \varepsilon\left\|\frac{\partial u_{q,1}}{\partial t}\right\|_{L^2}^2 + C_\varepsilon(\|\mathrm{div}\, u_{p,1}\|_{L^2}^2 + 1)\|p_{1,t}\|_{L^2}^2. \tag{71}$$

Finally, using (46) we have

$$\int_\Omega \Theta_{1,t} p_{1,t} \leqslant \|\Theta_{1,t}\|_{L^2}\|p_{1,t}\|_{L^2}$$
$$\leqslant \|\Theta_{1,t}\|_{W^{1,1}}\|p_{1,t}\|_{L^2}$$
$$\leqslant \|\nabla\Theta_{1,t}\|_{L^1}\|p_{1,t}\|_{L^2}$$
$$\leqslant \varepsilon\|\nabla p_{1,t}\|_{L^2}^2 + \varepsilon + C_\varepsilon\|p_{1,t}\|_{L^2}^2. \tag{72}$$

Now we define

$$y_1(t) = \sup_{\tau\in(0,t)} (\|u_{p,1}(\tau)\|_{L^4}^4 + \|p_{1,t}(\tau)\|_{L^2}^2),$$
$$z_1(t) = \|u_{p,1}\,\mathrm{div}\,u_{p,1}(t)\|_{L^2}^2 + \|\nabla p_{1,t}(t)\|_{L^2}^2,$$
$$y_2(t) = \sup_{\tau\in(0,t)} (\|u_{p,2}(\tau)\|_{L^4}^4 + \|p_{2,t}(\tau)\|_{L^2}^2),$$
$$z_2(t) = \|u_{p,2}\,\mathrm{div}\,u_{p,2}(t)\|_{L^2}^2 + \|\nabla p_{2,t}(t)\|_{L^2}^2$$

and

$$\Lambda_{1,2}(t) = 1 + \|\mathrm{div}\,u_{p,1}(t)\|_{L^2}^2 + \|\mathrm{div}\,u_{p,2}(t)\|_{L^2}^2 + \|u_{q,1}(t)\|_{L^\infty}^2 + \|u_{q,2}(t)\|_{L^\infty}^2.$$

Integrating (62) in $(0,t)$ we find that, by virtue of previous inequalities,

$$\begin{aligned} y_1(t) &+ (2A_1-\varepsilon)\int_0^t z_1(\tau)\,\mathrm{d}\tau \\ &\leqslant C\left(1+\varepsilon\int_0^t z_2(\tau)\,\mathrm{d}\tau + \varepsilon'\int_0^t \|\mathrm{div}\,u_{p,1}\|_{L^2}\|\mathrm{div}\,u_{p,1}\|_{L^4}^2\right. \\ &\quad + \varepsilon'\int_0^t \|\mathrm{div}\,u_{p,2}\|_{L^2}\|\mathrm{div}\,u_{p,2}\|_{L^4}^2 \\ &\quad \left. + \int_0^t \Lambda_{1,2}(\tau)y_1(\tau)\,\mathrm{d}\tau + \int_0^t \Lambda_{1,2}(\tau)y_2(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{73}$$

We can obtain the analogous result for $i=2$ after differentiating Eq. (44) with respect to all independent variables. Adding the results for $i=1,2$ it follows that

$$\begin{aligned} y_1(t) &+ y_2(t) + (2A_1-\varepsilon)\int_0^t z_1(\tau)\,\mathrm{d}\tau + (2A_2-\varepsilon)\int_0^t z_2(\tau)\,\mathrm{d}\tau \\ &\leqslant C\left(1+\varepsilon'\int_0^t \|\mathrm{div}\,u_{p,1}\|_{L^2}\|\mathrm{div}\,u_{p,1}\|_{L^4}^2 + \varepsilon'\int_0^t \|\mathrm{div}\,u_{p,2}\|_{L^2}\|\mathrm{div}\,u_{p,2}\|_{L^4}^2\right. \\ &\quad \left. + \int_0^t \Lambda_{1,2}(\tau)y_1(\tau)\,\mathrm{d}\tau + \int_0^t \Lambda_{1,2}(\tau)y_2(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{74}$$

Setting $Q_t = \Omega\times(0,t)$ and using Lemma 9 we have

$$\int_0^t \|\mathrm{div}\,u_{p,i}\|_{L^2}\|\mathrm{div} u_{p,i}\|_{L^4}^2 \leqslant \|\mathrm{div}\,u_{p,i}\|_{L^2(Q_t)}\|\mathrm{div} u_{p,i}\|_{L^4(Q_t)}^2$$
$$\leqslant C(1+\|p_{i,t}\|_{L^4(Q_t)}^2 + \|p_{j,t}\|_{L^4(Q_t)}^2),$$

for $i,j\in\{1,2\}$, $j\neq i$.

Using Gagliardo–Nirenberg's inequality again we can write

$$\|p_{i,t}\|^2_{L^4(Q_t)} \leqslant \left(C^2\int_0^t \|p_{i,t}\|^2_{L^2}\|\nabla p_{i,t}\|^2_{L^2}\right)^{1/2}$$
$$\leqslant C y_i(t)^{1/2}\left(\int_0^t z_i(\tau)\,\mathrm{d}\tau\right)^{1/2} \leqslant \varepsilon\int_0^t z_i(\tau)\,\mathrm{d}\tau + C_\varepsilon y_i(t)$$

for $i = 1, 2$.

Hence, choosing ε and ε' small enough, (74) yields

$$y_1(t) + y_2(t) + \int_0^t z_1(\tau)\,\mathrm{d}\tau + \int_0^t z_2(\tau)\,\mathrm{d}\tau$$
$$\leqslant \mathrm{C}\left(1 + \int_0^t \Lambda_{1,2}(\tau)y_1(\tau)\,\mathrm{d}\tau + \int_0^t \Lambda_{1,2}(\tau)y_2(\tau)\,\mathrm{d}\tau\right). \tag{75}$$

As $\Lambda_{1,2}$ is integrable in $[0, T]$, we can apply Gronwall–Bellman's lemma to conclude that

$$\|u_{p,1}\|^4_{L^\infty(0,T;L^4(\Omega)^2)} + \|u_{p,2}\|^4_{L^\infty(0,T;L^4(\Omega)^2)} + \|p_{1,t}\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|p_{2,t}\|^2_{L^\infty(0,T;L^2(\Omega)^2)}$$
$$+ \|u_{p,1}\operatorname{div} u_{p,1}\|^2_{L^2(Q)} + \|u_{p,2}\operatorname{div} u_{p,2}\|^2_{L^2(Q)} + \|\nabla p_{1,t}\|^2_{L^2(Q)} + \|\nabla p_{2,t}\|^2_{L^2(Q)} \leqslant \mathrm{C}. \tag{76}$$

Then $p_{i,t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ and so, $p_{i,t} \in L^4(Q)$.

We conclude using Lemma 9.

3.3. A uniqueness theorem

The previous estimate is important because it proves that—under suitable hypotheses—$u_{p,i} \in L^4(0, T; W^{1,4}(\Omega)^2)$, and by Lemma 4 we knew that $u_{q,i} \in L^4(0, T; W^{1,4}(\Omega)^2)$. The following result shows that $h_i \in L^\infty(Q)$. This smoothness is absolutely necessary to prove the uniqueness of the solution. In order to prove this we use Lemma 6, that also ensures that $\operatorname{curl} u_{q,i} \in L^4(0, T; W^{1,4}(\Omega))$, and then $\Theta_i \in L^2(0, T; L^\infty(\Omega))$.

Lemma 10. *If $h_{i,0} > 0$, $\log h_{i,0} \in L^\infty(\Omega)$ and $u_{i,0} \in H^2(\Omega)^2$, then we have*

$$h_i \quad \text{and} \quad \frac{1}{h_i} \in L^\infty(Q), \tag{77}$$

for $i = 1, 2$.

This lemma and the following one, which will allow us to prove a uniqueness theorem, are proved in the same way as those of [3] for the one-layer case.

Before stating them, we give some notation:

Let $\{(\bar{u}_1, \bar{h}_1), (\bar{u}_2, \bar{h}_2)\}$ and $\{(\tilde{u}_1, \tilde{h}_1), (\tilde{u}_2, \tilde{h}_2)\}$ be two solutions of $(\mathscr{V})$, with

$$\bar{u}_1 = \bar{u}_{p,1} + \bar{u}_{q,1} = \nabla\bar{p}_1 + \operatorname{Curl}\bar{q}_1, \qquad \bar{u}_2 = \bar{u}_{p,2} + \bar{u}_{q,2} = \nabla\bar{p}_2 + \operatorname{Curl}\bar{q}_2$$

and

$$\tilde{u}_1 = \tilde{u}_{p,1} + \tilde{u}_{q,1} = \nabla\tilde{p}_1 + \operatorname{Curl}\tilde{q}_1, \qquad \tilde{u}_2 = \tilde{u}_{p,2} + \tilde{u}_{q,2} = \nabla\tilde{p}_2 + \operatorname{Curl}\tilde{q}_2.$$

Then, the functions $(u_1,h_1)=(\bar{u}_1-\tilde{u}_1,\bar{h}_1-\tilde{h}_1)$ and $(u_2,h_2)=(\bar{u}_2-\tilde{u}_2,\bar{h}_2-\tilde{h}_2)$ verify

$$u_1=u_{p,1}+u_{q,1}, \qquad u_2=u_{p,2}+u_{q,2},$$

with

$$u_{q,1}=\bar{u}_{q,1}-\tilde{u}_{q,1}, \qquad u_{q,2}=\bar{u}_{q,2}-\tilde{u}_{q,2},$$

$$p_1=\bar{p}_1-\tilde{p}_1, \qquad p_2=\bar{p}_2-\tilde{p}_2.$$

Lemma 11. *The following inequalities*:

$$\|u_{q,i}\|_{L^2}^2 \leqslant \varepsilon\|\nabla p_i\|_{L^2}^2, \tag{78}$$

$$\|\operatorname{curl} u_{q,i}\|_{L^2}^2 \leqslant \varepsilon\|\nabla p_i\|_{L^2}^2, \tag{79}$$

where the arbitrary number ε *is chosen sufficiently small*, *hold for* $i=1,2$.

Now we state the uniqueness theorem:

Theorem 3. *If* $u_{i,0}$ *and* $h_{i,0}$ *verify the hypothesis of Theorem* 2 *and Lemma* 10, *then problem* $(\mathscr{V})$ *has a unique solution* $\{(u_1,h_1),(u_2,h_2)\}$ *such that*

$$\{(u_1,h_1),(u_2,h_2)\}\in[L^4(0,T;W^{1,4}(\Omega)^2)\times L^\infty(Q)]^2. \tag{80}$$

Proof. If $\{(\bar{u}_1,\bar{h}_1),(\bar{u}_2,\bar{h}_2)\}$ and $\{(\tilde{u}_1,\tilde{h}_1),(\tilde{u}_2,\tilde{h}_2)\}$ denote two solutions of $(\mathscr{V})$, we have

$$\frac{\partial\bar{p}_1}{\partial t}-A_1\Delta\bar{p}_1+\frac{1}{2}\bar{u}_1^2+\bar{\Theta}_1+g\bar{h}_1+g\frac{\rho_2}{\rho_1}\bar{h}_2=\bar{\zeta}_1,$$

$$\frac{\partial\bar{h}_1}{\partial t}+\operatorname{div}(\bar{u}_1\bar{h}_1)=0,$$

$$\frac{\partial\bar{p}_2}{\partial t}-A_2\Delta\bar{p}_2+\frac{1}{2}\bar{u}_2^2+\bar{\Theta}_2+g\bar{h}_2+g\bar{h}_1=\bar{\zeta}_2,$$

$$\frac{\partial\bar{h}_2}{\partial t}+\operatorname{div}(\bar{u}_2\bar{h}_2)=0,$$

where $\nabla\bar{\Theta}_1=\operatorname{curl}\bar{u}_1\alpha(\bar{u}_1)$ and $\nabla\bar{\Theta}_2=\operatorname{curl}\bar{u}_2\alpha(\bar{u}_2)$, and

$$\frac{\partial\tilde{p}_1}{\partial t}-A_1\Delta\tilde{p}_1+\frac{1}{2}\tilde{u}_1^2+\tilde{\Theta}_1+g\tilde{h}_1+g\frac{\rho_2}{\rho_1}\tilde{h}_2=\tilde{\zeta}_1,$$

$$\frac{\partial\tilde{h}_1}{\partial t}+\operatorname{div}(\tilde{u}_1\tilde{h}_1)=0,$$

$$\frac{\partial\tilde{p}_2}{\partial t}-A_2\Delta\tilde{p}_2+\frac{1}{2}\tilde{u}_2^2+\tilde{\Theta}_2+g\tilde{h}_2+g\tilde{h}_1=\tilde{\zeta}_2,$$

$$\frac{\partial\tilde{h}_2}{\partial t}+\operatorname{div}(\tilde{u}_2\tilde{h}_2)=0,$$

with $\nabla\tilde{\Theta}_1=\operatorname{curl}\tilde{u}_1\alpha(\tilde{u}_1)$ and $\nabla\tilde{\Theta}_2=\operatorname{curl}\tilde{u}_2\alpha(\tilde{u}_2)$.

Then, the functions $(u_1,h_1)=(\bar u_1-\tilde u_1,\bar h_1-\tilde h_1)$ and $(u_2,h_2)=(\bar u_2-\tilde u_2,\bar h_2-\tilde h_2)$ verify:

$$\frac{\partial p_1}{\partial t}-A_1\Delta p_1+\frac{1}{2}u_1(\bar u_1+\tilde u_1)+(\bar\Theta_1-\tilde\Theta_1)+gh_1+g\frac{\rho_2}{\rho_1}h_2=\bar\zeta_1-\tilde\zeta_1, \tag{81}$$

$$\frac{\partial h_1}{\partial t}+\mathrm{div}(u_1\bar h_1+\tilde u_1 h_1)=0 \tag{82}$$

and

$$\frac{\partial p_2}{\partial t}-A_2\Delta p_2+\frac{1}{2}u_2(\bar u_2+\tilde u_2)+(\bar\Theta_2-\tilde\Theta_2)+gh_2+gh_1=\bar\zeta_2-\tilde\zeta_2, \tag{83}$$

$$\frac{\partial h_2}{\partial t}+\mathrm{div}(u_2\bar h_2+\tilde u_2 h_2)=0. \tag{84}$$

We define the auxiliary functions ψ_1 and ψ_2 as solutions of the Neumann problems

$$\begin{cases}\Delta\psi_1=h_1 & \text{in } \Omega,\\ \dfrac{\partial\psi_1}{\partial n}=0 & \text{on } \Gamma\end{cases} \quad\text{and}\quad \begin{cases}\Delta\psi_2=h_2 & \text{in } \Omega,\\ \dfrac{\partial\psi_2}{\partial n}=0 & \text{on } \Gamma.\end{cases}$$

Now, multiplying (81) by p_1, (82) by ψ_1 and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_1\|_{L^2}^2+A_1\|\nabla p_1\|_{L^2}^2+\frac{1}{2}\int_\Omega p_1u_1(\bar u_1+\tilde u_1)$$
$$-\int_\Omega p_1(\bar\Theta_1-\tilde\Theta_1)+g\int_\Omega p_1\Delta\psi_1+g\frac{\rho_2}{\rho_1}\int_\Omega p_1\Delta\psi_2=0,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi_1\|_{L^2}^2+\int_\Omega\psi_1\mathrm{div}(u_1\bar h_1+\tilde u_1h_1)=0.$$

Adding these equations we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|p_1\|_{L^2}^2+\|\nabla\psi_1\|_{L^2}^2)+A_1\|\nabla p_1\|_{L^2}^2$$
$$=-\frac{1}{2}\int_\Omega p_1u_1(\bar u_1+\tilde u_1)+\int_\Omega p_1(\bar\Theta_1-\tilde\Theta_1)$$
$$+g\int_\Omega\nabla p_1\nabla\psi_1+g\frac{\rho_2}{\rho_1}\int_\Omega\nabla p_1\nabla\psi_2+\int_\Omega(\nabla\psi_1)u_1\bar h_1+\int_\Omega(\nabla\psi_1)\tilde u_1h_1. \tag{85}$$

We define the quantities

$$y_1=\|p_1\|_{L^2}^2+\|\nabla\psi_1\|_{L^2}^2 \quad\text{and}\quad y_2=\|p_2\|_{L^2}^2+\|\nabla\psi_2\|_{L^2}^2$$

and then we estimate the terms on the right-hand side of (85).

Using (78) we have

$$-\frac{1}{2}\int_\Omega p_1u_1(\bar u_1+\tilde u_1)\leqslant\varepsilon(\|\nabla p_1\|_{L^2}^2+\|u_{q,1}\|_{L^2}^2)+C_\varepsilon\|\bar u_1+\tilde u_1\|_{L^\infty}^2\|p_1\|_{L^2}^2$$
$$\leqslant\varepsilon\|\nabla p_1\|_{L^2}^2+C_\varepsilon y_1,$$

with C_ε integrable in time.

Recalling the definition of $\bar{\Theta}_1$ and $\tilde{\Theta}_1$, we can write

$$\int_\Omega p_1(\bar{\Theta}_1 - \tilde{\Theta}_1) \leqslant \varepsilon \|\bar{\Theta}_1 - \tilde{\Theta}_1\|_{L^2}^2 + C_\varepsilon \|p_1\|_{L^2}^2$$

$$\leqslant \varepsilon \|\nabla(\bar{\Theta}_1 - \tilde{\Theta}_1)\|_{L^1}^2 + C_\varepsilon y_1.$$

Using that

$$\nabla(\bar{\Theta}_1 - \tilde{\Theta}_1) = \operatorname{curl} u_{q,1} \alpha(\bar{u}_1) + \operatorname{curl} \bar{u}_{q,1} \alpha(u_1) - \operatorname{curl} u_{q,1} \alpha(u_1)$$

we obtain

$$\|\nabla(\bar{\Theta}_1 - \tilde{\Theta}_1)\|_{L^1}^2 \leqslant \|\bar{u}_1\|_{L^2}^2 \|\operatorname{curl} u_{q,1}\|_{L^2}^2 + \|\operatorname{curl} \bar{u}_{q,1}\|_{L^2}^2 (\|\nabla p_1\|_{L^2}^2 + \|u_{q,1}\|_{L^2}^2) + \|u_1\|_{L^2}^2 \|\operatorname{curl} u_{q,1}\|_{L^2}^2.$$

Finally, with (78) and (79),

$$\|\nabla(\bar{\Theta}_1 - \tilde{\Theta}_1)\|_{L^2}^2 \leqslant \mathrm{C} \|\nabla p_1\|_{L^2}^2$$

and

$$\int_\Omega p_1(\bar{\Theta}_1 - \tilde{\Theta}_1) \leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon y_1.$$

Next, we obtain the estimate

$$g \int_\Omega \nabla p_1 \nabla \psi_1 \leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon \|\nabla \psi_1\|_{L^2}^2 \leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon y_1$$

and

$$g \frac{\rho_2}{\rho_1} \int_\Omega \nabla p_1 \nabla \psi_2 \leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon \|\nabla \psi_2\|_{L^2}^2 \leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon y_2.$$

Using Lemmas 10 and 11 again,

$$\int_\Omega u_1 \bar{h}_1 \nabla \psi_1 \leqslant \varepsilon(\|\nabla p_1\|_{L^2}^2 + \|u_{q,1}\|_{L^2}^2) + C_\varepsilon \|\bar{h}_1\|_{L^\infty}^2 \|\nabla \psi_1\|_{L^2}^2$$

$$\leqslant \varepsilon \|\nabla p_1\|_{L^2}^2 + C_\varepsilon y_1.$$

The last term of (85) is integrated by parts:

$$(\Delta \psi_1 \tilde{u}_1, \nabla \psi_1) = \frac{1}{2}(\operatorname{div} \tilde{u}_1, |\nabla \psi_1|^2) - ((\nabla \psi_1 \cdot \nabla)\tilde{u}_1, \nabla \psi_1).$$

By using Lemma 10, h_1 is a bounded function and then $\nabla \psi_1$ is also a bounded function. Thus, if $|\nabla \psi_1| \leqslant M$ a.e., we can estimate both terms on the right-hand side of previous equality by

$$M^{2\delta} \| |\nabla \psi_1|^{2(1-\delta)} \|_{L^{1/(1-\delta)}} \|\operatorname{div} \tilde{u}_1\|_{L^{1/\delta}} = M^{2\delta} \|\nabla \psi_1\|_{L^2}^{2(1-\delta)} \|\operatorname{div} \tilde{u}_1\|_{L^{1/\delta}},$$

with $\delta \in (0,1)$. Multiplying Eq. (43) by $(\Delta p_1)^{(1-\delta)/\delta}$ and integrating over Ω gives the estimate

$$\|\operatorname{div} \tilde{u}_1\|_{L^{1/\delta}} \leqslant \mathrm{C}.$$

Then

$$(\Delta \psi_1 \tilde{u}_1, \nabla \psi_1) \leqslant K y_1^{1-\delta}.$$

Now, if we choose ε sufficiently small, equality (85) gives

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} \leqslant C_1 y_1 + K_1 y_1^{1-\delta} + C_2 y_2. \tag{86}$$

Multiplying (83) by p_2 and (84) by ψ_2 we can obtain the analogous result for $i=2$:

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} \leqslant C_2 y_2 + K_2 y_2^{1-\delta} + C_1 y_1 \tag{87}$$

and adding (86) and (87) we obtain

$$\frac{\mathrm{d}(y_1+y_2)}{\mathrm{d}t} \leqslant C(y_1+y_2) + K(y_1+y_2)^{1-\delta}$$

and consequently,

$$\frac{\mathrm{d}((y_1+y_2)^{\delta})}{\mathrm{d}t} \leqslant \delta C(y_1+y_2)^{\delta} + \delta K.$$

Gronwall Bellmann's lemma gives the estimate

$$(y_1(t)+y_2(t))^{\delta} \leqslant \delta K\,\mathrm{e}^{\delta C t}$$

and thus,

$$y_1(t)+y_2(t) \leqslant (\delta K)^{1/\delta}\mathrm{e}^{Ct}.$$

The term on the right-hand side converges to zero as δ tends to zero. This proves that $y_1(t)$ and $y_2(t)$ are equal to zero and concludes the proof of the theorem.

3.4. A regularity result

In order to obtain the existence of strong solutions for problem ($\mathscr{P}$), we must estimate the higher derivatives of solutions. As in the previous section, a regularity result can be proved by adapting the regularity result already proved for the one-layer system. This allows us to obtain a regularity $\mathscr{C}^\infty$ for u and h. The most difficult point is to bound the first derivative of the fluid elevation. This is the theorem:

Theorem 4. *If we have* $h_{i,0} \in W^{1,4}(\Omega)$ *then*

$$\nabla u_i \in L^\infty(Q)^4,$$

$$h_i \in W^{1,4}(Q),$$

for $i = 1, 2$.

The proof is carried out using the same techniques as in the one-layer case [3].

4. Conclusion

We have proved an existence and uniqueness theorem for a two-layer shallow water model. It seems really hard to obtain the same result for an n-layer system with this technique because of the data which have to be as small as n is large.

We have also obtained numerical results in a simple case, using the method applied to build the approximated solutions. We use Galerkin's method for the velocity and the characteristics method for the water elevation.

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