

THE CONFIGURATION SPACE OF ARACHNOID MECHANISMS

NIR SHVALB, MOSHE SHOHAM, AND DAVID BLANC

ABSTRACT. The configuration spaces of arachnoid mechanisms are analyzed in this paper. These mechanisms consist of k branches each of which has an arbitrary number of links and a fixed initial point, while all branches end at one common end-point. It is shown that generically, the configuration spaces of such mechanisms are manifolds, and the conditions for the exceptional cases are determined. The configuration space of planar arachnoid mechanisms having k branches, each with two links is analyzed for both the non-singular and the singular cases.

1. INTRODUCTION

Mechanisms and robots consist of links and joints, the actuation of which causes them to move. The type of a mechanism is described by an abstract graph which corresponds to its links and joints, and a specific embedding of this graph in the plane or in 3-space is called a *configuration* of the mechanism. The collection of all such embeddings forms a topological space, called the *configuration space* of the mechanism. For example, the configuration space of a planar mechanism with revolute joints consisting of n rods arranged serially is the n -torus.

In recent years, there has been interest among mathematicians in the study of such spaces, which are of importance in motion planning – that is, moving a mechanism from one given position to another, taking into account various constraints (see for example [MT2]). The topological properties of the configuration space provide insight into practical questions in planning such motions (see [F]) and analysis of some mechanical singularities (see [NM], and [ZFB]).

The main focus had been set on the configuration spaces of a type of mechanism called *polygonal linkage*, which is simply a concatenation of links and hinged joints forming a closed chain. A substantial amount of mathematical literature on polygonal linkage's configuration space has accumulated: Kamiyama, Tezuka and Toma studied Euler characteristics in [K], and homology groups in [KT, KTT]; Trinkle and Milgram constructed a handle-body surgery in [MT1]; and in [Ho], Holcomb studied a special parallel graph mechanism called multi-polygonal linkages, which are three free branches identified at their initial and terminal vertices.

In this paper we analyze a type of mechanism called *arachnoid* which, to the best of our knowledge, has never been dealt with in the literature. This kind of mechanism consists of multiple branches each of which has an arbitrary number of links and a fixed initial point, while all branches end at a common end-point (this type of mechanism resembles some parallel robots which are in practical use). It is shown that generically, the configuration spaces of such mechanisms are manifolds, and

1991 *Mathematics Subject Classification*. Primary 70G25; Secondary 57N99, 70B15.

Key words and phrases. configuration space, work space, robotics, planar mechanism.

the conditions for the exceptional cases are then determined. The configuration space of planar arachnoid mechanisms having k branches, each with two links is fully analyzed, while for the non-manifold cases we analyze the singular configurations.

We now introduce some notation and terminology to describe such mechanism types, and in particular the arachnoid mechanisms which are the subject of this note:

1.1. Definition. For a mechanism M in \mathbb{R}^d , a *branch* (L, \mathbf{x}) of multiplicity n is a sequence $L = (\ell_1, \dots, \ell_n)$ of n positive numbers, together with a point $\mathbf{x} \in \mathbb{R}^d$. We think of L as the lengths of n concatenated rods, having revolute (i.e., rotational) joints at the meeting point of every two consecutive rods, and at the fixed initial point \mathbf{x} , called the *base point* of the branch.

A *branch configuration* $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ for a branch (L, \mathbf{x}) then consists of n vectors in \mathbb{R}^d with the given norms $\|\mathbf{v}_i\| = \ell_i$ ($i = 1, \dots, n$).

Since the configuration space of a branch (L, \mathbf{x}) is independent of the order of the set ℓ_1, \dots, ℓ_n (up to homeomorphism), we can (and shall) assume ℓ_1, \dots, ℓ_n to be in descending order.

1.2. Definition. An *arachnoid mechanism* consists of k branches

$$(\mathcal{L}, \mathcal{X}) = ((L^{(1)}, \mathbf{x}^{(1)}), \dots, (L^{(k)}, \mathbf{x}^{(k)}))$$

with multiplicities $n^{(1)}, \dots, n^{(k)}$. We think of this as a linkage of branches connected by a single revolute joint at their common end point (whose location is not fixed).

An arachnoid mechanism configuration for $(\mathcal{L}, \mathcal{X})$ thus consists of a set

$$\mathcal{V} = (V^{(1)}, \dots, V^{(k)})$$

of branch configurations for \mathcal{L} having a common *end point* $\mathbf{y} = \mathbf{x}^{(i)} + \sum_{j=1}^{n^{(i)}} \mathbf{v}_j^{(i)}$ ($i = 1, \dots, k$).

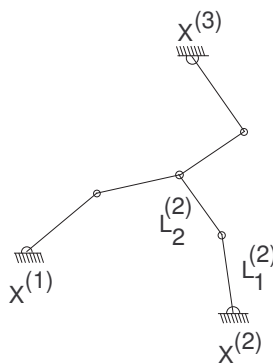


FIGURE 1.1. An arachnoid mechanism with $k = 3$, multiplicities 2.

1.3. Definition. For an arachnoid mechanism $(\mathcal{L}, \mathcal{X})$:

- (1) A branch configuration $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is *aligned* (with direction \mathbf{w}) if each vector $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a scalar multiple of \mathbf{w} .

- (2) A configuration $\mathcal{V} = (V^{(1)}, \dots, V^{(k)})$ of $(\mathcal{L}, \mathcal{X})$ is called a t -node if it has t aligned branch configurations with directions $\mathbf{w}_{i_1}, \dots, \mathbf{w}_{i_t}$ respectively, which are linearly dependent; otherwise \mathcal{V} is called *generic*.

1.4. Definition. The collection $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X})$ of all configurations \mathcal{V} for $(\mathcal{L}, \mathcal{X})$ is called its *configuration space*. It is topologized as a subspace of the appropriate Euclidean space. The space of all such common endpoints \mathbf{y} will be called the *work space* $\mathcal{W} = \mathcal{W}(\mathcal{L}, \mathcal{X})$ for $(\mathcal{L}, \mathcal{X})$. The *work map* $\Phi : \mathcal{C} \rightarrow \mathcal{W}$ assigns to each configuration \mathcal{V} its common endpoint \mathbf{y} .

Organization. In section 2 we show that the configuration space of a generic arachnoid mechanism $(\mathcal{L}, \mathcal{X})$ is a manifold. In section 3 we study planar arachnoid mechanisms for which each branch has 2 joints, and give an explicit formula for the topological type of $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X}_c)$ in the generic case. Finally, in section 4, we analyze the singularities of \mathcal{C} for such planar arachnoid mechanisms in the non-manifold case.

2. GENERIC ARACHNOID MECHANISMS IN \mathbb{R}^d

First, we show that, generically, the configuration space of an arachnoid mechanism is a manifold:

2.1. Theorem. *Let $(\mathcal{L}, \mathcal{X})$ be an arachnoid mechanism in \mathbb{R}^d with k branches of multiplicities $n^{(1)}, \dots, n^{(k)}$, respectively. If all configurations of $(\mathcal{L}, \mathcal{X})$ are generic, then its configuration space \mathcal{C} is a smooth closed orientable manifold of dimension $d(N - k + 1) - N$, where $N = \sum_{i=1}^k n^{(i)}$.*

Proof. We can identify $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X})$ as the pre-image of a certain function $G : \mathbb{R}^{d(N-k+1)} \rightarrow \mathbb{R}^N$, where G is defined as follows:

For each $n \geq 1$ let $g_n : (\mathbb{R}^d)^n \rightarrow \mathbb{R}^{n-1}$ be defined

$$g_n(\mathbf{v}_1, \dots, \mathbf{v}_n) := (|\mathbf{v}_2 - \mathbf{v}_1|^2, \dots, |\mathbf{v}_n - \mathbf{v}_{n-1}|^2),$$

where $|\mathbf{u}| := (\sum_{i=1}^d t_i^2)^{1/2}$ is the length of a vector $\mathbf{u} = (t_1, \dots, t_d) \in \mathbb{R}^d$. Now for each branch $L^{(i)} = (\ell_1^{(i)}, \dots, \ell_{n^{(i)}}^{(i)})$ of \mathcal{L} , let $(\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{n^{(i)}}^{(i)})$ be position vectors of the ends of the $n^{(i)}$ links of a branch configuration, where $\mathbf{v}_0^{(i)} = \mathbf{x}^{(i)}$ (the given base point for this branch). Since in an arachnoid mechanism all branches end at the same point $\mathbf{u} \in \mathbb{R}^d$, we have $\mathbf{v}_{n^{(i)}}^{(i)} = \mathbf{u}$ for all $1 \leq i \leq k$. Thus we have $N - k + 1$ different vectors $\{\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{n^{(i)}}^{(i)}\}_{i=1}^k$, and we define

$$G(\mathbf{v}_1^{(1)}, \dots, \mathbf{v}_{n^{(k)}}^{(k)}) := (g_{n_1}(\mathbf{v}_0^{(1)}, \dots, \mathbf{v}_{n_1}^{(1)}), g_{n_2}(\mathbf{v}_0^{(2)}, \dots, \mathbf{v}_{n_2}^{(2)}), \dots, g_{n_k}(\mathbf{v}_0^{(k)}, \dots, \mathbf{v}_{n_k}^{(k)})),$$

so that $\mathcal{C} = G^{-1}(\vec{\ell})$ for $\vec{\ell} := (\ell_1^{(1)})^2, \dots, (\ell_{n^{(1)}}^{(1)})^2, \dots, (\ell_1^{(k)})^2, \dots, (\ell_{n^{(k)}}^{(k)})^2$. By the Regular Value Theorem (see [Hi, I, Thm. 3.2]), \mathcal{C} will be a smooth manifold if $\vec{\ell}$ is a regular value of G – that is, dG has maximal rank.

Note that (after applying elementary row and column operations), dG has the following $N \times dN$ Jacobian matrix:

$$dG = 2 \begin{pmatrix} A^{(1)} & B^{(1)} & & 0 \\ A^{(2)} & & B^{(2)} & \\ \vdots & & & \ddots \\ A^{(k)} & 0 & & B^{(k)} \end{pmatrix}$$

where each $(n^{(i)} \times d)$ -block $A^{(i)}$ is:

$$A^{(i)} = \begin{pmatrix} \mathbf{v}_{n^{(i)}}^{(i)} - \mathbf{v}_{n^{(i)}-1}^{(i)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and $B^{(i)}$ is:

$$\begin{pmatrix} \mathbf{v}_{n^{(i)}-1}^{(i)} - \mathbf{v}_{n^{(i)}}^{(i)} & 0 & 0 & \dots & 0 & 0 \\ \mathbf{v}_{n^{(i)}-1}^{(i)} - \mathbf{v}_{n^{(i)}-2}^{(i)} & \mathbf{v}_{n^{(i)}-2}^{(i)} - \mathbf{v}_{n^{(i)}-1}^{(i)} & 0 & \dots & 0 & 0 \\ 0 & \mathbf{v}_{n^{(i)}-2}^{(i)} - \mathbf{v}_{n^{(i)}-3}^{(i)} & \mathbf{v}_{n^{(i)}-3}^{(i)} - \mathbf{v}_{n^{(i)}-2}^{(i)} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{v}_2^{(i)} - \mathbf{v}_1^{(i)} & \mathbf{v}_1^{(i)} - \mathbf{v}_2^{(i)} \\ 0 & 0 & 0 & \dots & 0 & \mathbf{v}_1^{(i)} - \mathbf{v}_0^{(i)} \end{pmatrix}$$

an $n^{(i)} \times d(n^{(i)} - 1)$ matrix which can be thought of as the Jacobian matrix for a corresponding closed $n^{(i)}$ -branch. Note that $\text{Rank}(B^{(i)}) \leq n^{(i)}$, and $B^{(i)}$ has less than full rank only when all vectors $\mathbf{v}_{n^{(j)}-1}^{(i)} - \mathbf{v}_{n^{(j)}}^{(i)}$ are collinear for $1 \leq j \leq n^{(i)}$ – so that the i -th branch is aligned. In this case $\text{Rank}(B^{(i)}) = n^{(i)} - 1$, and the non-zero row $\mathbf{w}^{(i)} := \mathbf{v}_{n^{(i)}}^{(i)} - \mathbf{v}_{n^{(i)}-1}^{(i)}$ of $A^{(i)}$ is precisely the direction of the branch.

We can thus divide the matrix dG horizontally into two blocks: (A, B) , where

$$A := \begin{pmatrix} A^{(1)} \\ \vdots \\ A^{(k)} \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} B^{(1)} & 0 \\ & B^{(2)} \\ & & \ddots \\ 0 & & & B^{(k)} \end{pmatrix},$$

and the rank of dG is then given by:

$$(2.1) \quad \text{Rank}(A, B) = \text{Rank}(A) + \text{Rank}(B) - \dim(\text{Col}(A) \cap \text{Col}(B)).$$

Denote by I the set of all indices i for which the i^{th} branch is aligned, so that $\text{Rank}(B) = N - |I|$. Thus if $I = \emptyset$, then dG has maximal rank. If $|I| \neq 0$, let A_I be the sub-matrix of A consisting of the blocks $A^{(i)}$ with $i \in I$. Its rows are therefore spanned by the directions $\{\mathbf{w}^{(i)}\}_{i \in I}$ of the aligned branches. Observe that $\text{Rank}(A) - \text{Rank}(A_I)$ is the dimension of the subspace of $\text{Col}(A)$ consisting

of columns whose entries vanish in the rows indexed by $i \in I$. Since the blocks of B indexed by $i \notin I$ have full rank, we see that

$$\dim(\text{Col}(A) \cap \text{Col}(B)) \leq \text{Rank}(A) - \text{Rank}(A_I)$$

(in fact, equality holds). By (2.1):

$$\text{Rank}(dG) \geq \text{Rank}(A_I) + \text{Rank}(B) = N - |I| + \text{Rank}(A_I),$$

which means that dG has full rank unless $|I| > \text{Rank}(A_I)$. The latter implies that the directions of the aligned branches are linearly dependent – that is, we have a k -node configuration.

Note that $\mathcal{C} = G^{-1}(\vec{\ell})$ is in fact orientable when dG has maximal rank, since in that case it induces an isomorphism between the normal bundle ν to \mathcal{C} in $\mathbb{R}^{d(N-k+1)}$ at any point and the “normal bundle” to $\{\ell\} \hookrightarrow \mathbb{R}^N$. Thus ν (the complement to the tangent bundle $T\mathcal{C}$ in $\mathbb{R}^{d(N-k+1)}$) is orientable, so $T\mathcal{C}$ is, too. Finally, \mathcal{C} is compact since it is a closed subset of the free configuration space, which is a N -torus. \square

2.2. Remark. For an arachnoid mechanism in \mathbb{R}^3 , the matrix dG will be singular for a 2-node configuration (two aligned branches along one line); a 3-node configuration (three aligned branches contained in one plane); or a 4-node configuration (four aligned branches).

3. PLANAR MECHANISMS

From now on we restrict attention to arachnoid mechanisms $(\mathcal{L}, \mathcal{X})$ in the \mathbb{C} -plane (that is, $d = 2$).

3.1. The work space. In this case, each vector v_j in a branch configuration V (of multiplicity n) is determined by its argument θ_j (since $\|v_j\| = \ell_j$), and V can thus be identified with a point $(\theta_1, \dots, \theta_n)$ in the n -torus

$$\mathbf{T}^n = \underbrace{\mathbf{S}^1 \times \dots \times \mathbf{S}^1}_n.$$

Thus if $\nu = n^{(1)} + \dots + n^{(k)}$, then $\mathcal{C}(\mathcal{L}, \mathcal{X}) \subseteq \mathbf{T}^\nu = \prod_{i=1}^k \mathbf{T}^{n^{(i)}}$.

Given such an arachnoid mechanism, we can describe the work space \mathcal{W} as follows: for any branch $L = (\ell_1, \dots, \ell_n)$, let $\beta(L)_{\min}$ denote the minimal value of $|\sum_{j=1}^n \varepsilon_j \ell_j|$, where $\varepsilon_j = \pm 1$ for each $1 \leq j \leq n$; and let $\beta(L)_{\max} := \sum_{j=1}^n \ell_j$ (the maximal value). The work space $W = \mathcal{W}(L, \mathbf{x})$ for the branch L with base point \mathbf{x} (without any constraint on the end point) is then an annulus bounded by circles of radius $\beta(L)_{\min}$ and $\beta(L)_{\max}$, respectively.

If $\mathcal{L} = (L^{(1)}, \dots, L^{(k)})$, with multiplicities $n^{(1)}, \dots, n^{(k)}$, and $\mathcal{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})$, the work space for $(\mathcal{L}, \mathcal{X})$ is $\mathcal{W} = \bigcap_{i=1}^k W^{(i)}$, where $W^{(i)} = \mathcal{W}(L^{(i)}, \mathbf{x}^{(i)})$. Each component of \mathcal{W} is a curvilinear polygon P (not necessarily convex), whose edges $\text{Edge}(P)$ are arcs of the annuli boundary circles $\partial W^{(i)}$, and whose vertices $\text{Vertex}(P)$ are intersection points of such arcs.

3.2. The configuration space. The configuration space for any branch $L = (\ell_1, \dots, \ell_n)$ and base-point \mathbf{x} is an n -torus \mathbf{T}^n , with work map $\phi : \mathbf{T}^n \rightarrow W$.

Note that the fiber $\phi^{-1}(z)$ over any point $z \in \text{Int } W$ is the configuration space for the *closed* chain with links $\ell_0, \ell_1, \dots, \ell_n$, where $\ell_0 := z - \mathbf{x}$. This configuration space has been analyzed in [HR]. On the other hand, if z is on the boundary of the annulus W , then $\phi^{-1}(z)$ is evidently discrete, and in fact consists of a single point (unless it is on the inner circle, and $\beta(L)_{\min}$ can be written as $|\sum_{j=1}^n \varepsilon_j \ell_j|$ in more than one way).

If $\mathcal{L} = (L^{(1)}, \dots, L^{(k)})$, with multiplicities $n^{(1)}, \dots, n^{(k)}$, and $\mathcal{X} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})$, its configuration space is the pullback

$$\mathcal{C}(\mathcal{L}, \mathcal{X}) = \{(\tau_1, \dots, \tau_k) \in \prod_{i=1}^k \mathbf{T}^{n^{(i)}} \mid \phi_1(\tau_1) = \dots = \phi_k(\tau_k) \in \mathcal{W}\}.$$

3.3. Example. Consider an arachnoid mechanism consisting of three branches, each of multiplicity 2, as in Figure 1.1.

The workspace for each free branch is an annulus; let us assume that the three annuli intersect in the shaded lens-shaped component P in Figure 3.1.

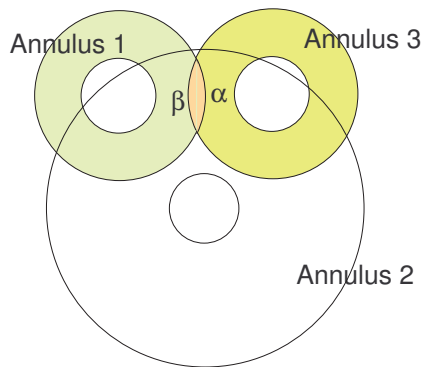


FIGURE 3.1. Polygonal intersection

Now consider an interior point $y \in \text{Int}(P)$: in the corresponding configurations in the fiber $\Phi^{-1}(y)$, each of the three branches can be in one of two positions (branch configurations), usually termed “elbow up” (\mathbf{u}) and “elbow down” (\mathbf{d}), so for each branch we have a copy of $\mathbf{S}^0 = \{\mathbf{u}, \mathbf{d}\}$. Thus the fiber consists of eight points $\mathbf{uuu}, \mathbf{uud}, \dots, \mathbf{ddd}$, thought of as the product $\mathbf{S}^0 \times \mathbf{S}^0 \times \mathbf{S}^0$. Thus $\Phi^{-1}(\text{Int}(P))$ is simply an eight-fold cover of the interior of the lens.

On the other hand, if y is on the edge α of P , which is in the outer boundary of the first annulus, the first branch is completely extended, identifying its \mathbf{u} and \mathbf{d} positions, thus collapsing the first \mathbf{S}^0 to a single point, and generally identifying the copy of α in \mathcal{C} indexed by $\mathbf{u} **$ with the copy indexed by $\mathbf{d} **$, for $** \in \{\mathbf{uu}, \mathbf{ud}, \mathbf{du}, \mathbf{dd}\}$.

Similarly, the copy of β in \mathcal{C} indexed by $** \mathbf{u}$ is identified with that indexed by $** \mathbf{d}$. Therefore, the fiber of $y \in \alpha \cap \beta$ consists of two points. Note that the second \mathbf{S}^0 -factor never collapses, so \mathcal{C} is of the form $\mathbf{S}^0 \times \mathcal{M}$ – i.e., \mathcal{C} has two components, each isomorphic to \mathcal{M} .

To evaluate the Euler characteristic of \mathcal{M} , note that it is obtained from four 2-gons (the lens-shaped intersection in \mathcal{W}) by identifying their 8 edges pairwise (as explained above), identifying the “top” vertex in all the 2-gons to a single point, and similarly for the “bottom” vertex. Thus $\chi(\mathcal{M}) := V - E + F = 4 - 4 + 2 = 2$, so \mathcal{M} is a 2-sphere, and $\mathcal{C} \cong \mathbf{S}^2 \sqcup \mathbf{S}^2$.

3.4. Invariants of annulus arrangements. An annulus (i.e., pair of concentric circles) in the plane is determined by $(\mathbf{x}, \beta_{\min}, \beta_{\max})$, where $\mathbf{x} \in \mathbb{R}^2$ is the center and $0 < \beta_{\min} < \beta_{\max}$ are the radii. Consider a system

$$\langle (\mathbf{x}^{(1)}, \beta(1)_{\min}, \beta(1)_{\max}); \dots ; (\mathbf{x}^{(k)}, \beta(k)_{\min}, \beta(k)_{\max}) \rangle$$

of k such pairs (with distinct centers), and let \mathcal{W} denote the intersection of all the corresponding annuli; this may have several connected components V_1, \dots, V_t . What we have in mind, of course, is the collection of boundary circles for the work space of branches of an arachnoid mechanism.

The boundary ∂V of each component V of \mathcal{W} is a curvilinear planar polygon, not necessarily connected; let $\alpha := \alpha(V)$ denote the number of components of ∂V contained in the interior of its convex hull $\text{conv}(V)$. We set

$$\gamma := \gamma(V) = \begin{cases} 1 & \text{if } \text{conv}(V) \text{ is a disc} \\ 0 & \text{otherwise.} \end{cases}$$

be wholly contained in the interior of some of the annuli; denote the number of such annuli by $\beta := \beta(V)$ ($0 \leq \beta(V) \leq k$), and let the *c-invariant* of V be $c(V) := 2^\beta$. Finally, the *g-invariant* of V is:

$$g(V) := 1 - 2^{k-\beta-3}(|\text{Vertex}(V)| - 2|\text{Edge}(V)| + 4 + 2\gamma - 4\alpha).$$

3.5. Theorem. *Let $(\mathcal{L}, \mathcal{X})$ be a planar arachnoid mechanism with k branches, each of multiplicity 2, and assume that the configuration space $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X})$ has no node configurations; then \mathcal{C} decomposes as the disjoint union of the pre-image under Φ of the components of the workspace \mathcal{W} , and for each such component V , $\Phi^{-1}(V)$ consists of $c(V)$ closed orientable surfaces of genus $g(V)$.*

A special case of this Theorem appears in [E].

Proof. As in example 3.3 above, $\Phi^{-1}(V)$ is obtained from the 2^k curvilinear polygonal “tiles” (i.e., copies of V , corresponding to the “elbow up/elbow down” position of each branch), by identifications of those edges which correspond to the $k - \beta$ “relevant” branches. Since we know from Theorem 2.1 that (each component of) \mathcal{C} is a closed orientable 2-manifold, its type (genus) is determined by the Euler characteristic, which may be computed by calculating how many identifications we have for each vertex or edge of $\Phi^{-1}(V)$.

To orient $\Phi^{-1}(V)$, choose an orientation for some (fixed) tile H_0 . Every other tile H of $\Phi^{-1}(V)$ differs from H_0 in exactly τ of the k possible “elbow up/elbow down” positions, and we reverse its orientation (relative to that of H_0) if and only if τ is odd. \square

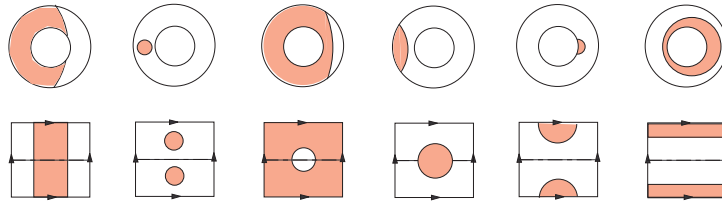


FIGURE 3.2. Work and configuration space intersections of two branches

3.6. *Remark.* As noted in §3.1, the workspace \mathcal{W} was obtained by repeatedly intersecting annuli, which are the workspaces of the individual branches. If we concentrate on the annulus A of the first branch, say, then generically the annuli for each of the remaining branches will intersect with A in one of the six shaded patterns V in the first row of Figure 3.2. In each case we obtain a certain subset $\phi^{-1}(V)$ of the 2-torus \mathbf{T}^2 which is the configuration space for the first branch, where $\phi : \mathbf{T}^2 \rightarrow A$ is the work map for the first branch.

Note that when we further intersect V with a third annulus, the pattern may be more complicated; in particular, three-fold intersections need not be connected, as illustrated by Figure 3.3, which shows the subset of the 2-torus associated with the workspace of Figure 1.1 (without indicating the identifications).

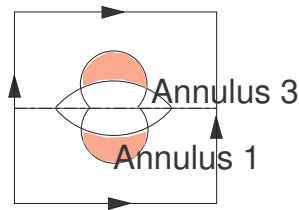


FIGURE 3.3. Subset of \mathbf{T}^2

4. SINGULAR CONFIGURATION SPACES

While the analysis of the configuration space $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X})$ of a mechanism in the non-manifold case is in general difficult, for a planar arachnoid mechanism with branch multiplicity 2 the description of Theorem 3.5 can actually be extended to singular points, corresponding to the node configurations.

4.1. Proposition. *Let $(\mathcal{L}, \mathcal{X})$ be a planar arachnoid mechanism with k branches, each of multiplicity 2, and let \mathcal{V} be a node configuration in $\mathcal{C} = \mathcal{C}(\mathcal{L}, \mathcal{X})$. Then \mathcal{V} has an open neighborhood in \mathcal{C} which is a wedge of $2^{q-\varepsilon}$ 2-dimensional discs with common center \mathcal{V} , where q is the number of aligned branches, $\varepsilon = 1$ if the aligned branches have a common direction, and $\varepsilon = 2$ otherwise.*

Proof. Let $v := \Phi(\mathcal{V})$ be the common end-point of the k -branches in \mathcal{V} , and let U be a small disc containing v in the workspace \mathcal{W} . Since v is necessarily in the boundary of a curvilinear polygonal component of \mathcal{W} , then P , the boundary of U near v , consists of arcs of the boundary circles of the annuli.

- (1) When all of the q aligned branches have a common direction, the centers of the corresponding annuli must be collinear, and P is an arc e of a single boundary circle of such an annulus. Note that $\Phi^{-1}(\text{Int}(U))$ intersects the component of \mathcal{V} in 2^q disjoint discs, which are identified pairwise along $\Phi^{-1}(e)$ so as to yield 2^{q-1} discs whose only common point is $\mathcal{V} \in \Phi^{-1}(v)$.
- (2) Otherwise, P must consist of two arcs e_1, e_2 of distinct boundary circles (whose centers are not collinear with v). Again $\Phi^{-1}(\text{Int}(U))$ intersects the component of \mathcal{V} in 2^q disjoint discs, but now every four of them (corresponding to the four “elbow up/elbow down” positions of the two branches associated to e_1 and e_2 , respectively) are identified in $\Phi^{-1}(U) \subset \mathcal{C}$ along $\Phi^{-1}(e_1)$ and $\Phi^{-1}(e_2)$, forming the four quadrants of a new disc – where again \mathcal{V} is the only point in common. This yields a total of 2^{q-1} discs. \square

REFERENCES

- [E] D. Eldar, “Maps and Machines”, Hebrew Univ. M.Sc. project, in preparation, <http://www.math.toronto.edu/~drobn/People/Eldar>.
- [F] M. Farber, “Instabilities of Robot Motion”, preprint, 2002 cs.RO/0205015.
- [GS] S. Goldberger, M. Shoham and O. Ben-Porat, “Robots in Surgery - Reality or Vision”, *Ob. Gyn. Update* **22** (1997), 4-8.
- [GA] C. Gosselin & J. Angeles, “Singularity analysis of Closed Loop Kinematic Chains”, *IEEE Trans. on Robotics And Automation* **6** (1990), 281-290.
- [HK] J.C. Hausman & A. Knutson, “The Cohomology Ring of Polygon Spaces”, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 281-321.
- [Hi] M.W. Hirsch, *Differential Topology*, Springer-Verlag, Berlin-New York, 1976.
- [Ho] M. Holcomb, “On the Moduli Spaces of Multipolygonal Linkages in the Plane”, preprint, 2003, math.GT/0307001.
- [HR] J.C. Hausman & E. Rodriguez, “The Space of Clouds in an Euclidan Space”, preprint, 2002.
- [K] Y. Kamiyama, “Topology of equilateral polygon linkages in the Euclidean plane modulo isometry group”, *Oski J. Math.* **36** (1999), 731-745.
- [KM] M. Kapovich & J. Millson, “On the moduli space of polygons in the Euclidean plane”, *J. Differential Geom.* **42** (1995), 430-464.
- [KT] Y. Kamiyama & M. Tezuka, “Topology and Geometry of Equilateral Polygon linkages in the Euclidean plane”, *Quart. J. Math. (Oxford)* **50**, 1999, 463-470.
- [KTT] Y. Kamiyama, M. Tezuka & T. Toma, “Homology Of the Configuration spaces of Quasi-equilateral Polygonal Linkages”, *Trans. Am. Math. Soc.* **350** (1998), 4869-4896.
- [MT1] R.J. Milgram & J. Trinkle, “The Geometry of Configuration spaces of Closed Chains in Two and Three Dimensions”, to appear in *Homology, Homotopy & Applications*.
- [MT2] R.J. Milgram & J. Trinkle, “Motion Planning for Planar n -Bar Mechanisms with Revolute Joints”, in *IEEE Int. Conf. on Intelligent Robots and Systems*, IEEE, 2001, 1602-1608.
- [NM] N. Simaan & M. Shoham, “Singularity analysis of a class of Composite serial In-Parallel Robots”, *IEEE Transactions On Robotics and Automation* **17** (2001), 301-311.
- [ZFB] D.S. Zlatanov, R.G. Felton & B. Benhabib, “Classification and Interpretation of Singularities of Redundant Mechanisms”, *Proceedings of the ASME 24th Annual Design Automation Conference, Atlanta, GA, Sept. 1998*, Paper #DETC98/MECH-5896.

E-mail address: shvalbn@tx.technion.ac.il

DEPT. MECH. ENG., THE TECHNION, 32000 HAIFA, ISRAEL

E-mail address: shoham@tx.technion.ac.il

DEPT. MECH. ENG., THE TECHNION, 32000 HAIFA, ISRAEL

E-mail address: blanc@math.haifa.ac.il

DEPT. MATH, U. HAIFA, 31905 HAIFA, ISRAEL