GENERALIZED WEAKLY CONTRACTIONS IN PARTIALLY ORDERED FUZZY METRIC SPACES

S. VAEZZADEH AND S. M. VAEZPOUR

Abstract. In this paper, a concept of generalized weakly contraction mappings in partially ordered fuzzy metric spaces is introduced and coincidence point theorems on partially ordered fuzzy metric spaces are proved. Also, as the corollary of these theorems, some common fixed point theorems on partially ordered fuzzy metric spaces are presented.

1. Introduction and Preliminaries

It is well known that the fuzzy metric space is an important generalization of metric space. Fixed point theory in fuzzy metric spaces can be considered as a part of fuzzy analysis, which is a very dynamic area of mathematical research. Many authors have generalized and extended fixed point, common fixed point, and coincidence point theorems on fuzzy metric spaces [7], [3], [16], and [1]. Mishra et al. [9] obtained several common fixed point theorems for asymptotically commuting maps on fuzzy metric spaces. Singh et al. [13] by introducing the compatibility on fuzzy metric spaces proved some common fixed point theorems. Recently Ciric et al. [4] have presented a concept of monotone generalized contraction in partially ordered probabilistic metric spaces and proved some fixed and common fixed point theorems. The purpose of this work is to present theorems which are the improvement of the results in [4] in partially ordered fuzzy metric spaces.

Definition 1.1. [15] A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous *t–norm* if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $c \leq d$, and $a,b,c,d \in [0,1]$.

Two typical examples of continuous t–norm are $T_P(a,b) = ab$ and $T_M(a,b) = Min(a,b)$.

Now t-norms are recursively defined by $T^1 = T$ and

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in [0,1]$, for all $i \in \{1, 2, \ldots, n+1\}$.

Received: September 2013; Accepted: December 2014

Key words and phrases: Partially ordered fuzzy metric space, Generalized weakly contraction, Fixed point theorem, Common fixed point theorem.

A t-norm T is said to be *of Hadžić type* if the family $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x=1$, that is,

$$\forall\varepsilon\in(0,1)\ \exists\delta\in(0,1): a>1-\delta\Rightarrow T^n(a)>1-\varepsilon\quad (n\geq 1)\,.$$

T_M is a trivial example of a t-norm of Hadžić type, but there exist t-norms of Hadžić type weaker than T_M, [6].

Definition 1.2. [2] A *fuzzy metric space* is a triple (X,F,T), where X is a nonempty set, T is a continuous t–norm, and F is a fuzzy set on $X\times X\times[0,\infty)$ satisfying the following conditions: for all $x,y\in X$ and $s,t>0$,

(I) $F(x,y,0)=0$,

(II) $F(x,y,t)=1$ for all $t>0$, if and only if $x=y$,

(III) $F(x,y,t)=F(y,x,t)$,

(IV) $T(F(x,y,t),F(y,z,s))\leq F(x,z,t+s)$,

(V) $F(x,y,.):[0,\infty)\to[0,1)$ is left-continuous.

Let (X,F,T) be a fuzzy metric space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F(x_n,x,\epsilon)>1-\lambda$ whenever $n\geq N$.

(2) A sequence $\{x_n\}_n$ in X is called *Cauchy sequence* if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F(x_n,x_m,\epsilon)>1-\lambda$ whenever $n,m\geq N$.

(3) A fuzzy metric space (X,F,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Let (X,F,T) be a fuzzy metric space. For each p in X and $\lambda>0$, the strong $\lambda-$*neighborhood* of p is the set

$$N_p(\lambda)\quad=\quad\{q\in X: F(p,q,\lambda)>1-\lambda\},$$

and the strong neighborhood system for X is the union $\bigcup_{p\in V}\mathcal{N}_p$ where $\mathcal{N}_p=\{N_p(\lambda):\lambda>0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 1.3. [2] *If (X,F,T) is a fuzzy metric space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n\to p$ and $q_n\to q$, then $\lim_{n\to\infty}F(p_n,q_n,t)=F(p,q,t)$ for every continuity point t of $F(p,q,.)$.*

Lemma 1.4. [6]*, also see* [8]*. Let (X,F,T) be a fuzzy metric space with T of Hadžić-type and $\{x_n\}$ be a sequence in X such that, for some $k\in(0,1)$,*

$$F(x_n,x_{n+1},kt)\geq F(x_{n-1},x_n,t)\quad(n\geq 1,t>0).$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.5. [4] *If $F,G\in D^+$ and, for some $k\in(0,1)$,*

$$F(kt)\geq\min\{G(t),F(t)\},\forall t>0$$

then

$$F(kt)\geq G(t)\ \forall t>0,$$

where D^+ *is the set of left-continuous and non-decreasing functions on* $\mathbb{R}$ *such that* $F(0) = 0,\ F(\infty) = 1,$ *and* $\lim_{t \to x^-} F(t) = 1.$

Definition 1.6. [11] Let $(X, \leq)$ be a partially ordered set, and $T, S, R : X \to X$ are given mappings, such that $TX \subseteq RX$ and $SX \subseteq RX$. We say that S and T are weakly increasing with respect to R if for all $x \in X$, we have

$$Tx \preceq Sy, \quad \forall y \in R^{-1}(Tx)$$

and

$$Sx \preceq Ty, \quad \forall y \in R^{-1}(Sx).$$

Definition 1.7. [14] Let $(X, \leq)$ be a partially ordered set, and (X, F, T) be a fuzzy metric space. The pair $f, g : X \to X$ is said to be F-compatible if and only if $\lim F(f(gx_n), g(fx_n), t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

2. **Main Results**

In this section we present the following theorem:

Theorem 2.1. *Let* $(X, \leq)$ *be a partially ordered set and* (X, F, T) *be a fuzzy metric space under a t-norm* T *of Hadzic-type. Let* $H, S, R : X \to X$ *be three continuous self-mappings of* X *such that* $H(X) \subseteq R(X),\ S(X) \subseteq R(X)$ *and* H *and* S *are weakly increasing with respect to* R *such that for some* $k \in (0, 1)$,

$$F(S(x), H(y), kt) \geq Min\{F(R(x), R(y), t), F(R(x), S(x), t), F(R(y), H(y), t)\}.$$

for all $t > 0$ *and for all* $x, y \in X$ *for which* $R(x)$ *and* $R(y)$ *are comparable.*

If the pairas $\{H, R\}$ *and* $\{S, R\}$ *are* $F-$*compatible, then H, S, and R have a coincidence point, that is, there exists* $u \in X$ *such that* $Ru = Hu = Su$.

Proof. Let $x_0 \in X$ be an arbitrary point, since $H(X) \subseteq R(X)$, there exists $x_1 \in X$ such that $Rx_1 = Hx_0$. Since $S(X) \subseteq R(X)$, there exists $x_2 \in X$ such that $Rx_2 = Sx_1$.

Continuing this process, we can construct a sequence $\{Rx_n\}$ in X defined by

$$Rx_{2n+1} = Hx_{2n}, Rx_{2n+2} = Sx_{2n+1},\ \forall n \in N. \tag{1}$$

We claim that

$$Rx_n \preceq Rx_{n+1}, \quad \forall n \in N. \tag{2}$$

To this aim, we will use the increasing property with respect to R for the mappings T and S. From (1), we have

$$Rx_1 = Hx_0 \preceq Sy,\ \forall y \in R^{-1}(Hx_0).$$

Since $Rx_1 = Hx_0,\ x_1 \in R^{-1}(Hx_0)$, we get

$$Rx_1 = Hx_0 \preceq Sx_1 = Rx_2.$$

Again,

$$Rx_2 = Sx_1 \preceq Hy,\ \forall y \in R^{-1}(Sx_1).$$

Since $x_2 \in R^{-1}(Sx_1)$, we get

$$Rx_2 = Sx_1 \preceq Hx_2 = Rx_3.$$

Hence, by induction, (2) holds.

Now we show that $\{Rx_n\}$ is a cauchy sequence. By (1) we have,

$$\begin{aligned}
F(Rx_{2n+2}, Rx_{2n+1}, kt) &= \\
F(Sx_{2n+1}, Hx_{2n}, kt) &\geq Min\{F(Rx_{2n+1}, Rx_{2n}, t), F(Rx_{2n+1}, Sx_{2n+1}, t), \\
&\quad F(Rx_{2n}, Hx_{2n}, t)\} \\
&= Min\{F(Rx_{2n+1}, Rx_{2n}, t), F(Rx_{2n+1}, Rx_{2n+2}, t), \\
&\quad F(Rx_{2n}, Rx_{2n+1}, t)\} \\
&= Min\{F(Rx_{2n}, Rx_{2n+1}, t), F(Rx_{2n+1}, Rx_{2n+2}, t)\},
\end{aligned}$$

So by lemma (1.5) we have:

$$F(Rx_{2n+2}, Rx_{2n+1}, kt) \geq F(Rx_{2n+1}, Rx_{2n}, t) \qquad \forall t > 0. \tag{3}$$

also,

$$\begin{aligned}
F(Rx_{2n}, Rx_{2n+1}, kt) &= \\
F(Sx_{2n-1}, Hx_{2n}, kt) &\geq Min\{F(Rx_{2n-1}, Rx_{2n}, t), F(Rx_{2n-1}, Sx_{2n-1}, t), \\
&\quad F(Rx_{2n}, Hx_{2n}, t)\} \\
&= Min\{F(Rx_{2n-1}, Rx_{2n}, t), F(Rx_{2n-1}, Rx_{2n}, t), \\
&\quad F(Rx_{2n}, Rx_{2n+1}, t)\} \\
&= Min\{F(Rx_{2n-1}, Rx_{2n}, t), F(Rx_{2n}, Rx_{2n+1}, t)\},
\end{aligned}$$

So by lemma (1.5) we have:

$$F(Rx_{2n}, Rx_{2n+1}, kt) \geq F(Rx_{2n}, Rx_{2n-1}, t) \qquad \forall t > 0. \tag{4}$$

Now , by (3) and (4) we have:

$$F_{Rx_n, Rx_{n+1}}(kt) \geq F_{Rx_n, Rx_{n-1}}(t) \qquad \forall t > 0.$$

Hence, by lemma (1.4), $\{Rx_n\}$ is a cauchy sequence.
Since $\{Rx_n\}$ is cauchy, there exists $u \in X$ such that

$$\lim_{n\to\infty} Rx_n = u. \tag{5}$$

From the continuity of R, we get

$$\lim_{n\to\infty} R(Rx_n) = Ru. \tag{6}$$

By definition of fuzzy metric space, i.e (IV), we have

$$\begin{aligned}
F(Ru, Hu, t) \geq &T(F(Ru, R(Rx_{2n+1}), \frac{t}{3}), F(R(Hx_{2n}), H(Rx_{2n}), \frac{t}{3}), \\
&F(H(Rx_{2n}), Hu, \frac{t}{3})).
\end{aligned} \tag{7}$$

for all $t > 0$. Since $Rx_{2n} \to u$ and $Hx_{2n} \to u$, by compatibility of R and H, this implies that

$$\lim_{n\to\infty} F(R(Hx_{2n}), H(Rx_{2n}), t) = 1 \quad \forall t > 0. \tag{8}$$

Now from continuity of H and (5) we have,

$$\lim_{n\to\infty} F(H(Rx_{2n}), Hu, t) = 1 \;\; \forall t > 0. \tag{9}$$

Combining (6), (8), and (9), and letting $n \to \infty$ in (7), we obtain

$$F(Ru, Hu, t) \geq 1 \qquad \forall t > 0,$$

that is

$$Ru = Hu. \tag{10}$$

Again, by definition of fuzzy metric space, i.e. (IV), we have

$$\begin{aligned} F(Ru, Su, t) \geq T(F(Ru, R(Rx_{2n+2}), \frac{t}{3}), F(R(Sx_{2n+1}), S(Rx_{2n+1}), \frac{t}{3}), \\ F(S(Rx_{2n+1}), Su, \frac{t}{3})). \end{aligned} \tag{11}$$

for all $t > 0$. On the other hand, we have

$$Rx_{2n+1} \to u, \ Sx_{2n+1} \to u, \ as \ n \to \infty.$$

Since R and S are compatible mappings, this implies that

$$\lim_{n\to\infty} F(R(Sx_{2n+1}), S(Rx_{2n+1}), t) = 1 \quad \forall t > 0 \tag{12}$$

Now from continuity of S and (5) we have,

$$\lim_{n\to\infty} F(S(Rx_{2n+1}), Hu, t) = 1 \ \ \forall t > 0 \tag{13}$$

Combining (6), (12), and (13), and letting $n \to \infty$ in (7), we obtain

$$F(Ru, Su, t) \geq 1 \qquad \forall t > 0,$$

that is

$$Ru = Su. \tag{14}$$

Finally, from (10) and (14), we have

$$Hu = Ru = Su,$$

that is, u is a coincidence point of H, S, and R. This completes the proof. □

Definition 2.2. Let $(X, \leq)$ be a partially ordered set and (X, F, T) be a fuzzy metric space under a t-norm. We say that X is $F-$regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in X with respect to $\preceq$ such that $z_n \to z$ as $n \to \infty$, then $z_n \preceq z$ for all $n \in N$.

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and (X, F, T) be a $F-$regular complete fuzzy metric space under a t-norm T of Hadzic-type. Let $H, S, R : X \to X$ be three self-mappings of X such that $H(X) \subseteq R(X), S(X) \subseteq R(X)$ and H and S are weakly increasing with respect to R such that for some $k \in (0, 1)$,*

$$\begin{aligned} F(S(x), H(y), kt) \geq & Min\{F(R(x), R(y), t), F(R(x), S(x), t), \\ & F(R(y), H(y), t)\}, \end{aligned} \tag{15}$$

for all $t > 0$ and for all $x, y \in X$ for which $R(x)$ and $R(y)$ are comparable. If RX is closed, then H,S, and R have a coincidence point, that is, there exists $u \in X$ such that $Ru = Hu = Su$.

Proof. From the proof of theorem (2.1), we have that $\{Rx_n\}$ is a cauchy sequence. Since RX is closed, there exists $v \in X$ such that

$$\lim_{n\to\infty} Rx_n = Rv. \tag{16}$$

Since $\{Rx_n\}$ is a non-decreasing sequence and X is $F-$regular, it follows from (16) that $Rx_n \preceq Rv$ for all $n \in N$. Now we show that $Rv = Sv$, and also, Rv=Hv. For $x = x_{2n}$ and $y = v$, by inequality (15) we have:

$$F(Sv, Hx_{2n}, kt) \geq Min\{F(Rv, Rx_{2n}, t), F(Rv, Sv, t), F(Rx_{2n}, Hx_{2n}, t)\}, \tag{17}$$

for all $t > 0$. Since $Rx_{2n} \to Rv$ and $Hx_{2n} \to Rv$, and letting $n \to \infty$ in (17) we have

$$F(Sv, Rv, kt) \geq Min\{F(Rv, Rv, t), F(Sv, Rv, t), F(Rv, Rv, t)\},$$

for all $t > 0$. Therefore,

$$F(Sv, Rv, t) \geq F(Rv, Sv, \frac{t}{k}) \quad t > 0.$$

From here we get

$$F(Rv, Sv, t) \geq F(Rv, Sv, \frac{t}{k^n}) \to 1.$$

so, Sv=Rv($*$). Similarly, for $x = x_{2n+1}$ and $y = v$, we obtain

$$\begin{aligned} F(Sx_{2n+1}, Hv, kt) \geq Min\{&F(Rx_{2n+1}, Rv, t), F(Rx_{2n+1}, Sx_{2n+1}, t), \\ &F(Rv, Hv, t)\}, \end{aligned} \tag{18}$$

for all $t > 0$. Since $Rx_{2n+1} \to Rv$ and $Hx_{2n} \to Rv$, and by letting $n \to \infty$ in (18), we have

$$F(Rv, Hv, kt) \geq F(Rv, Hv, t) \quad \forall t > 0.$$

therefore,

$$F(Rv, Hv, t) \geq F(Rv, Hv, \frac{t}{k}) \quad \forall t > 0.$$

so Sv=Rv($**$). Now combining ($*$) and ($**$), we obtain

$$Rv = Sv = Hv$$

Hence, v is a coincidence point of H and S and R. This completes the proof. □

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set, and (X, F, T) be a complete fuzzy metric space under a t-norm T of Hadzic-type. Let H and S be two continuous and weakly increasing self mapping of X such that for some $k \in (0, 1)$*

$$F(S(x), H(y), kt) \geq Min\{F(x, y, t), F(x, S(x), t), F(y, H(y), t)\}$$

for all $x, y \in X$ that are comparable.
Then, H and S have a common fixed point.

Proof. If $R : X \to X$ is the identity mapping in the theorem (2.1), we can easily deduce the result. □

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set, and (X, F, T) be a $F-$regular complete fuzzy metric space under a t-norm T of Hadzic-type. Let H and S be two weakly increasing self mapping of X such that for some $k \in (0, 1)$*

$$F(S(x), H(y), kt) \geq Min\{F(x, y, t), F(x, S(x), t), F(y, H(y), t)\}$$

for all $x, y \in X$ that are comparable.
Then, H and S have a common fixed point.

Proof. If $R : X \to X$ is the identity mapping in theorem (2.3), we can easily deduce the result. □

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S. Vaezzadeh, Department of Mathematics and Computer Science,, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran
E-mail address: sarah_vaezzadeh@yahoo.com

S. M. Vaezpour*, Department of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran
E-mail address: vaez@aut.ac.ir

*Corresponding author