New Solitary Wave Solutions in Higher-Order Wave Equations of the Korteweg – de Vries Type

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In this work we study two partial differential equations that constitute second- and third-order approximations of water wave equations of the Korteweg – de Vries type. In particular, we first study previous results concerning the derivation of solitary wave solutions of the second-order approximation. We then use a simple assumption and find new solitary wave solutions for both equations.

Key words: Solitary Wave Solutions.

1. Introduction

As it is well known, the Korteweg – de Vries (KdV) equation

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} = 0$$

represents a first-order approximation in the study of long wavelength, small amplitude waves of inviscid and incompressible fluids. If one allows the appearance of higher-order terms in α and β, more complicated wave equations can be obtained. Two such equations, including second- and third-order terms, were proposed in [1] and have, respectively, the form

$$\begin{aligned} &u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x \\ &\quad + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) = 0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x \\ &\quad + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) + \alpha^3 \rho_4 u^3 u_x \\ &\quad + \alpha^2\beta(\rho_5 u^2 u_{xxx} + \rho_6 u u_x u_{xx} + \rho_7 {u_x}^3) = 0. \end{aligned} \tag{1.2}$$

Equation (1.1) was examined analytically and numerically in [2 – 4] and it was found that, although it is non-integrable in general, it still possesses solitary wave solutions, which, for small values of the parameters α and β, behave like solitons. New wave solutions of both equations (1.1) and (1.2) were also examined numerically in [5]. Equation (1.1) was further examined in [6 – 8] where new wave and periodic solutions were found. A numerical study of a new wave solution was also presented in [9].

In this paper we first show that all the known solitary wave solutions of equation (1.1) embed in the general category

$$u(x,t) = \frac{b_0 + b_1 e^{\xi} + b_2 e^{2\xi}}{a_0 + a_1 e^{\xi} + a_2 e^{2\xi}}, \tag{1.3}$$

where

$$\xi = b(x - ct),$$

and a_i, b_i, b and c are free or determined parameters. Consequently, we use a simple assumption which reveals new solitary wave solutions for both equations (1.1) and (1.2).

For simplicity, we set $\alpha = \beta = 1$ in (1.1) and (1.2); thus we obtain

$$\begin{aligned} &u_t + u_x + u u_x + u_{xxx} \\ &\quad + \rho_1 u^2 u_x + \rho_2 u u_{xxx} + \rho_3 u_x u_{xx} = 0 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &u_t + u_x + u u_x + u_{xxx} + \rho_1 u^2 u_x + \rho_2 u u_{xxx} \\ &\quad + \rho_3 u_x u_{xx} + \rho_4 u^3 u_x + \rho_5 u^2 u_{xxx} \\ &\quad + \rho_6 u u_x u_{xx} + \rho_7 {u_x}^3 = 0. \end{aligned} \tag{1.5}$$

This is equivalent with applying the transformation

$$u(x,t) = \frac{1}{\alpha} w(X,T) = \frac{1}{\alpha} w\left(\frac{1}{\sqrt{\beta}}x, \frac{1}{\sqrt{\beta}}t\right).$$

2. Previous Results

Let us begin with the solution of (1.1) proposed in [9]. This solution has the form

$$u(x,t) = a_1 + a_2\Lambda(x,t) + a_3\Lambda(x,t)^2, \tag{2.1}$$

where

$$\Lambda(x,t) = \frac{-ab\,\mathrm{sech}^2\xi}{b^2 - ac(1-\tanh\xi)^2}, \quad \xi = \frac{\sqrt{a}}{2}(kx - \omega t), \tag{2.2}$$

and

$$a = \frac{b^2}{4c}, \tag{2.3}$$

while parameters a_i and ω depend on the free parameters ρ_i, b, c and k.

Because of relation (2.3), function $\Lambda(x,t)$ in (2.2) can be written in a more simple form, namely

$$\Lambda(x,t) = \frac{b\,\mathrm{sech}^2\xi}{c(\tanh\xi - 3)(\tanh\xi + 1)} = -\frac{b}{c(2+\mathrm{e}^{2\xi})}.$$

Thus, solution (2.1) can also be written in a more simple form, i. e.

$$u(x,t) = \frac{b_0 + b_1\mathrm{e}^{2\xi} + b_2\mathrm{e}^{4\xi}}{(2+\mathrm{e}^{2\xi})^2}.$$

The above function embeds in the general category (1.3), since it consists only of even powers of e^{ξ}. Moreover, it is easy to verify that all the solitary wave solutions that appear in [3, 5–8] embed in the same category, as they consist of specific combinations of hyperbolic functions. The only exceptions are two unbounded solutions that appear in [7], which have the form

$$u(x,t) = c_1 \pm c_2\frac{\mathrm{sech}\xi}{1\pm\mathrm{sech}\xi} + c_2\frac{\tanh\xi}{1\pm\tanh\xi}$$

and

$$u(x,t) = c_1 \pm c_2\frac{\mathrm{csch}\xi}{1\pm\mathrm{csch}\xi} + c_2\frac{\coth\xi}{1\pm\coth\xi};$$

thus both are of the form

$$u(x,t) = \frac{b_0 + b_1\mathrm{e}^{\xi} + b_2\mathrm{e}^{2\xi} + b_3\mathrm{e}^{3\xi} + b_4\mathrm{e}^{4\xi}}{\mathrm{e}^{2\xi}(a_0 + a_1\mathrm{e}^{\xi} + a_2\mathrm{e}^{2\xi})}.$$

3. New Solutions

In what follows we will be interested only in the case that the solitary wave has zero background, i. e.

$$\lim_{\xi\to\pm\infty} u(x,t) = 0.$$

Motivated by the study presented in [10] we first assume that (1.4) admits a solitary wave solution of the form

$$u(x,t) = \frac{\partial^2}{\partial x^2} f(x,t),$$

where

$$f(x,t) = a_0\log(1 + a_1\mathrm{e}^{\xi} + a_2\mathrm{e}^{2\xi}), \quad \xi = b(x-ct),$$

i. e.

$$u(x,t) = \frac{a_0 b^2\mathrm{e}^{\xi}(a_1 + 4a_2\mathrm{e}^{\xi} + a_1a_2\mathrm{e}^{2\xi})}{(1 + a_1\mathrm{e}^{\xi} + a_2\mathrm{e}^{2\xi})^2}. \tag{3.1}$$

Consequently, we substitute (3.1) in (1.4) and equate to zero the coefficients A_k of $\mathrm{e}^{k\xi}$, $k = 0,\dots,10$.

Relations $A_0 = A_{10} = 0$ imply $c = 1 + b^2$, while relations $A_1 = A_9 = 0$ imply

$$a_2 = \frac{1}{24}{a_1}^2[12 - a_0(1 + b^2(\rho_2 + \rho_3))]. \tag{3.2}$$

Then, relations $A_2 = A_8 = 0$ yield

$$\rho_1 = \Big\{72 - a_0(1 + b^2(\rho_2+\rho_3)) \cdot [18 - a_0(1 + b^2(3\rho_2 + 2\rho_3))]\Big\}\{2{a_0}^2b^2\}^{-1},$$

and relations $A_3 = A_7 = 0$ yield three different solutions for ρ_2, i. e.

$$\text{(i) } \rho_2 = \frac{30 - a_0(5 + 8b^2\rho_3)}{14a_0b^2},$$

$$\text{(ii) } \rho_2 = \frac{6 - a_0(1 + b^2\rho_3)}{a_0b^2},$$

$$\text{(iii) } \rho_2 = \frac{12 - a_0(1 + b^2\rho_3)}{a_0b^2}.$$

We exclude cases (ii) and (iii), since they lead to wave solutions of the form (1.3), with $b_0 = b_2 = 0$. Finally, relations $A_4 = A_5 = A_6 = 0$ imply three different solutions for a_0, namely

$$\text{(i) } a_0 = \frac{6}{1 - 4b^2\rho_3}, \quad \text{(ii) } a_0 = \frac{18}{3 + 2b^2\rho_3},$$

$$\text{(iii) } a_0 = \frac{46}{3 + 2b^2\rho_3}.$$

As before, cases (ii) and (iii) are excluded.

Thus, we conclude with the following results:

$$\rho_1 = 3\rho_3(4b^2\rho_3 - 1), \quad \rho_2 = -2\rho_3,$$

and (1.4) admits the solitary wave solution

$$u(x,t) = \frac{24a_1b^2\mathrm{e}^{\xi}\left[4(1-4b^2\rho_3)+4a_1(1-7b^2\rho_3)\mathrm{e}^{\xi}+a_1{}^2(1-7b^2\rho_3)\mathrm{e}^{2\xi}\right]}{\left[4(1-4b^2\rho_3)+4a_1(1-4b^2\rho_3)\mathrm{e}^{\xi}+a_1{}^2(1-7b^2\rho_3)\mathrm{e}^{2\xi}\right]^2},$$

where

$$\xi = b(x-(1+b^2)t),$$

and ρ_3, a_1, b remain free.

Obviously, the above solution is new, since form (3.1), that was initially assumed, is more general than (1.3). Of course, one could assume more general forms, i. e.

$$u(x,t) = \frac{\mathrm{e}^{\xi}(b_0+b_1\mathrm{e}^{\xi}+b_2\mathrm{e}^{2\xi})}{(1+a_1\mathrm{e}^{\xi}+a_2\mathrm{e}^{2\xi})^2} \tag{3.3}$$

or

$$u(x,t) = \frac{\mathrm{e}^{\xi}(b_0+b_1\mathrm{e}^{\xi}+b_2\mathrm{e}^{2\xi})}{1+a_1\mathrm{e}^{\xi}+a_2\mathrm{e}^{2\xi}+a_3\mathrm{e}^{3\xi}+a_4\mathrm{e}^{4\xi}}. \tag{3.4}$$

However, this would lead to a much more complicate system, due to the nonlinear terms that appear in (1.4).

We now turn to (1.5) and follow the same procedure. Thus, we substitute (3.1) in (1.5) and equate to zero the coefficients A_k of $\mathrm{e}^{k\xi}$, $k = 0,\ldots,14$. This yields the following results:

$$\rho_1 = \frac{126-3a_0(11+b^2(13\rho_2+12\rho_3))+2a_0{}^2(1+b^2(\rho_2+\rho_3))(1+b^2(3\rho_2+2\rho_3))}{3a_0{}^2b^2},$$

$$\rho_4 = -\frac{2[6-a_0(1+b^2(\rho_2+\rho_3))][30-a_0(5+2b^2(7\rho_2+4\rho_3))]}{3a_0{}^3b^4},$$

$$\rho_5 = -\frac{2[6-a_0(1+b^2(\rho_2+\rho_3))][12-a_0(2+b^2(5\rho_2+2\rho_3))]}{9a_0{}^2b^4},$$

$$\rho_6 = \frac{[6-a_0(1+b^2(\rho_2+\rho_3))][24-a_0(4+b^2(7\rho_2-2\rho_3))]}{9a_0{}^2b^4},$$

$$\rho_7 = -\frac{[6-a_0(1+b^2(\rho_2+\rho_3))][6-a_0(1+b^2(\rho_2-2\rho_3))]}{6a_0{}^2b^4},$$

and (1.5) admits the solitary wave solution

$$u(x,t) = \frac{a_0b^2\mathrm{e}^{\xi}(a_1+4a_2\mathrm{e}^{\xi}+a_1a_2\mathrm{e}^{2\xi})}{(1+a_1\mathrm{e}^{\xi}+a_2\mathrm{e}^{2\xi})^2},$$

where

$$\xi = b(x-(1+b^2)t),$$

a_2 is given by (3.2) and ρ_2, ρ_3, a_0, a_1, b remain free.

4. Concluding Remarks

In this paper we have used a simple assumption and have found new solitary wave solutions for equations (1.4) and (1.5), that represent, respectively, second- and third-order approximations of the unidirectional water wave propagation, in the short amplitude α and long wavelength limit β. The procedure followed is quite simple, since it consists only of algebraic manipulations, which can be easily carried out by the use of any computer algebra program.

Since many of the solitary wave solutions of various partial differential equations consist of combinations of hyperbolic functions, we believe that initial assumptions, such as (3.1), (3.3), and (3.4) (or even more general), could be used to reveal new solutions. Of course, this could often lead to a very complicate system. On the other hand, such assumptions can be used even when the equation is non-integrable, thus we cannot use one of the powerful tools that integrability implies, such as Lax pair representation or Bäcklund transformations.

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