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On calculating the lengths of water waves

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ABSTRACT

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A discussion is given of the physical approximations used in obtaining water wave dispersion relations, which relate wave length and height, period, water depth and current. Several known explicit approximations for the wave length are presented, all of which ignore effects of wave height and current. These are compared and are shown to model the usual linear dispersion relation rather more accurately than it describes the physical problem. A simple approximation is obtained:

$$L=(gT^2/2\pi)\{\tanh[2\pi\sqrt{(d/g)}/\mathrm{T}]^{3/2}\}^{2/3}$$

in terms of wave period T, depth d and gravitational acceleration g, which is exact in the limits of short and long waves, and in the intermediate range has an accuracy always better than 1.7%. Explicit approximations which include the effects of current are presented, plus an algorithm based on Newton's method which converges to engineering accuracy in one evaluation, and requires the specification of a value of current, which is a useful reminder that one is obtaining an approximate solution to an approximate problem, and no great effort should go into refining methods or solutions.

INTRODUCTION

The problem of determining the length of a wave, with values of wave height, water depth, period and current known, is essentially one of solving the dispersion relation, which is an equation for the wavelength, but where it appears in such a complicated manner that an explicit solution is not possible. Many attempts have been made to provide explicit approximations for the wavelength, and a great deal of attention has been given to developing and comparing formulae. Sometimes, however, this has been without the recognition that almost always the dispersion relation is only a rough approximation to the physical problem.

The general case of water wave motion is one where disturbances propagate

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in varying directions over water of possible nonuniform density which might be flowing on a current such that the vorticity is nonuniform, over a bed which is not horizontal, where the effect of tides is also to modify the depth, and the disturbances interact nonlinearly in accordance with the surface boundary conditions. It is not possible to solve this general problem analytically. Instead it is convenient to assume that, locally at least, the bed is flat, that the propagation of disturbances is collinear and that they are of infinite length transverse to the direction of propagation such that the flow is two-dimensional, that the fluid is homogeneous and incompressible, and that the boundary layer is small such that irrotational flow theory can be used. Further, it is assumed that the disturbances may be decomposed into a number of different wave trains with a discrete harmonic frequency spectrum. A common assumption is that the waves do not interact nonlinearly or that there is only one component wave train present. With all these assumptions, a dispersion relation can be obtained which is an approximation to the physical problem.

Usually, however, further assumptions are made, in particular that the wave is of such small height that effects of wave height on the wave speed can be ignored, and that the effect of currents on which the wave might be riding can also be neglected. With these further assumptions, the resulting dispersion relation is an even rougher approximation to the physical problem. Nevertheless, there have been many attempts to provide highly-accurate approximations to this dispersion relation. Some clever methods have been devised, which are accurate and useful provided their limitations to certain wavelength values are recognized. However, something of an industry has grown up in producing yet more such formulae. In most cases the amount of computation is of a similar order of magnitude, as is the order of accuracy, but considerable attention has been paid to the comparison of marginal advantages between the various formulae. One really wonders what all the debate has been about. What few of them do, however, is to allow for effects of current or nonlinearity, for which dispersion relations do exist. It should have been recognized that the ability of each formula to approximate the linear dispersion relation is rather better than the latter's ability to approximate the actual problem and that differences between formulae are not very important.

One intention of this paper is to suggest that the problem of approximating the dispersion relation has been well and truly solved. The simple formulae of approximation give acceptable accuracy for many engineering purposes. If higher accuracy is required, it should be recognized that dispersion relations exist which include wave height and current. The problem of solving these numerically is in principle no more difficult than that of solving the linear dispersion relation by numerical means.

It is possible to produce explicit approximations which include the effects of current. Two such formulae are presented here, one of which has been given previously by Hedges (1987). Newton's method can be used to refine the approximation. It is simply programmed and can be used to achieve arbitrary specifiable accuracy. In fact, the inclusion of current yields a minor modification of the method. An algorithm is presented which is recommended for obtaining wavelengths from the linear dispersion relation, whether or not the effects of current are considered. Although in many situations it might be considered unnecessary to incorporate the effects of current in calculations, the act of being forced to specify a value of current, even if it is zero, is a salutatory reminder that one is obtaining an *approximate* solution to an *approximate* problem, and no great effort should go into refining methods or solutions.

THE EFFECTS OF WAVE HEIGHT AND CURRENT

Consider a single train of periodic waves of length L, height H and period T propagating in water of constant depth d, which is incompressible and the fluid motion is irrotational. Let the mean component of current in the direction of wave propagation at any point be $\bar{u}_1$, constant throughout the fluid. Using Stokes' theory to approximate this problem, the dispersion relation to fifth-order in wave height, obtained by Fenton (1985), is:

$$(k/g)^{1/2}\bar{u}_1-\frac{2\pi}{T(gk)^{1/2}}+C_0(kd)+\left(\frac{kH}{2}\right)^2 C_2(kd)+\left(\frac{kH}{2}\right)^4 C_4(kd)=0 \qquad (1)$$

where, instead of the length, it is more convenient to write the equation in terms of the wavenumber $k=2\pi/L$, and where the quantities $C_0(kd)$, $C_2(kd)$, and $C_4(kd)$ are functions only of the wave length/depth ratio (expressed by kd as shown). Formulae for these quantities were given by Fenton, for example, $C_0(kd)=(\tanh kd)^{1/2}$. (In this paper the symbol $\bar{u}_1$ is used for the mean current at a point; c_E was used in the original paper). The equation is, of course, a mathematical approximation even to the idealized physical approximation described above; neglected terms of order $(kH/2)^6$ and higher are not shown.

Provided T,H,d and $\bar{u}_1$ are known, this equation is a transcendental equation for k. It can be solved by any of the usual iterative methods, for example, trial-and-error, bisection, or the secant method. The latter is a sensible alternative to Newton's method, which is difficult here because the functional dependence on k is complicated. The bisection method is the simplest to program.

If the component of the depth-averaged mass transport velocity $\bar{u}_2$ is known instead of the mean current at a point, then a similar formula is obtained, and could be solved using the methods described above (see Fenton, 1985):

$$(k/g)^{1/2}\bar{u}_2 - \frac{2\pi}{T(gk)^{1/2}} + C_0(kd) + \left(\frac{kH}{2}\right)^2 \left(C_2(kd) + \frac{D_2(kd)}{kd}\right) + \left(\frac{kH}{2}\right)^4 \left(C_4(kd) + \frac{D_4(kd)}{kd}\right) = 0 \tag{2}$$

in which the symbol c_S was used for the depth-integrated mean current $\bar{u}_2$. Equations 1 and 2 are valid for waves which are not so high as to be close to breaking and for waves which are not too long. The recommended longest waves are those which are 10 times as long as the water depth. For waves which are longer than this, if there is a single wave train present, the dispersion relation for cnoidal theory should be used. Fenton (1990) has given a simplified fifth-order procedure. To first-order in wave height it can be shown to give the dispersion relation:

$$\frac{\bar{u}_1}{(gd)^{1/2}} + 1 + \left(\frac{H}{md}\right)\left(1 - \frac{3E(m)}{2K(m)} - \frac{m}{2}\right) - \left(\frac{md^2}{3gHT^2}\right)^{1/2} 4K(m) + (\textit{higher-order terms}) = 0 \tag{3}$$

In this relation m is the parameter of elliptic functions, and $K(m)$ and $E(m)$ are elliptic integrals. Having found m, the wavelength follows from cnoidal theory. Here the dispersion relation Eq. 3 (or that for mean mass flux velocity in terms of $\bar{u}_2$ also given by Fenton but identical to this order) is to be solved for the parameter m.

The numerical problem of solving a transcendental equation is essentially the same, whether Eq. 1, 2 or 3 is to be solved. If one actually has the idealized problem as described above, where only one component wave train is present or nonlinear interactions are ignored, and high accuracy is required, then Eqs. 1, 2 or 3 can be solved by standard means. Provided an initial approximation is known, then most methods work. If, however, the problem being solved is one in which various wave trains might be present and the nature of the nonlinear interactions is unknown, then one is forced to assume that the harmonics do not interact and that each phase travels independently of the others. In this case, or if it is recognized that the physical approximations already made are such that seeking high accuracy is not reasonable, then the linearized dispersion relation is used, by neglecting the higher-order terms in Eqs. 1 and 2 (the cnoidal solution is nonlinear, so it is ruled out further here), to give:

$$(k/g)^{1/2}\bar{u} - \frac{2\pi}{T(gk)^{1/2}} + C_0(kd) = 0 \tag{4}$$

In this equation $\bar{u}$ denotes either the component of the mean current at a point $\bar{u}_1$ or of the mass-transport velocity $\bar{u}_2$, as at this lowest-order of approximation the two dispersion relations are the same.

Here, the substitution $C_0(kd) = (\tanh kd)^{1/2}$ is made and the radian frequency $\sigma = 2\pi/T$ used, to give:

$$\bar{u}k - \sigma + \sqrt{gk \tanh kd} = 0 \tag{5}$$

It is possible to obtain explicit approximations in the long and short wave limits. In the latter, short wave (deep water), case, $kd \to \infty$, $\tanh kd \to 1$ and Eq. 5 becomes a quadratic which can be solved for $k^{-1/2}$, which gives the result:

$$k = \frac{4\sigma^2/g}{(1+\sqrt{1+4\bar{u}\sigma/g})^2} \tag{6}$$

as obtained by Hedges (1987). Expressions related to this have been given by Yu (1952) and earlier workers. It is illustrative to consider the effects of current on wavelength using this equation: if the current $\bar{u} = 0$ then:

$$k = \frac{\sigma^2}{g} \tag{7}$$

and the wave speed in this case, denoted by c_0, is given by:

$$c_0 = \frac{L}{T} = \frac{\sigma}{k} = \frac{g}{\sigma} \tag{8}$$

and Eq. 6 can be written as:

$$k = \frac{4\sigma^2/g}{(1+\sqrt{1+4\bar{u}/c_0})^2} \tag{9}$$

Using power series expansions and neglecting second and higher powers of $\bar{u}/c_0$ this can be written as:

$$k = \frac{\sigma^2}{g}\left(1 - 2\frac{\bar{u}}{c_0} + \ldots\right) \tag{10}$$

The factor of two in this equation shows that in the calculation of wave number (and hence wave length) for waves in deep water, the fractional effect of the current is twice the ratio of current to wave speed. This is a possibly unexpected result, and increases the possible need to include current in calculations.

In the other limit, that of long waves, when $kd \to 0$, $\tanh kd \to kd$ and a power series can be written for the last term in Eq. 5. This can be reverted, and elementary but tedious series manipulations give the explicit long wave approximation:

$$kd = \frac{\sigma\sqrt{d/g}}{1+F} + \frac{1}{6}\frac{(\sigma\sqrt{d/g})^3}{(1+F)^4} + \frac{(11-19F)}{360(1+F)^7}(\sigma\sqrt{d/g})^5 + \ldots \tag{11}$$

in which we have introduced the quantity $F=\bar{u}/\sqrt{gd}$, the Froude number, but whose more important significance here is that it is the ratio of the current to the speed of long waves in still water, $\sqrt{gd}$. The approximation that has been made here is that higher powers of σ have been ignored; there has been no approximation in terms of F, the dimensionless current. In this expression it can be seen that the role of current becomes rather more important in the higher-order terms. However to leading order the result is:

$$k=\frac{\sigma/\sqrt{gd}}{1+F}=\frac{\sigma}{\sqrt{gd}+\bar{u}} \qquad (12)$$

From this it is easily shown that, unlike the case for short waves, a change in current yields the same fractional change in wave number and length.

THE LINEAR DISPERSION RELATION FOR ZERO CURRENT

Now, as is widely done, if it is further assumed that there is zero current, or this is assumed in the absence of any information about the current. substituting $\bar{u}=0$ in Eq. 4 gives the lowest level of approximation to the full dispersion relation:

$$\frac{2\pi}{T(gk)^{1/2}}-C_0(kd)=0 \qquad (13)$$

or in the same form as Eq. 5:

$$\sigma-\sqrt{gk\tanh kd}=0 \qquad (14)$$

It is usually one of these two forms which is referred to as "the" dispersion relation, and widely used in practice.

For the long wave case, when $kd\to 0$, $\tanh kd\to kd$ and Eq. 14 becomes:

$$\sigma=k\sqrt{gd} \qquad (15)$$

the well-known result that long waves have a speed $c=\sigma/k=L/T=\sqrt{gd}$. In the short wave (deep water) case, $kd\to\infty$, $\tanh kd\to 1$ and the dispersion relation Eq. 14 becomes:

$$\sigma^2=gk \qquad (16)$$

If the water depth and wave period (i.e., frequency) are known, then, despite, the simplifying physical assumptions of neglecting effects of wave height and current, Eq. 14 is still a transcendental equation for the wavelength L (given by k). Numerical solution of this is no easier in principle than that of solving the more complete dispersion relations, Eqs. 1, 2, 3 or 5.

If the physical problem is well-defined such that high accuracy is sought and effects of nonlinearity and current are included, then either Eq. 1 or 2 should be used, or for long waves the cnoidal theory as described in Fenton

(1990), approximated to first order by Eq. 3. In most cases, however, the approximate relation Eq. 14 can be used. In the past it has not always been recognized that it is an approximation. Here we estimate the effects of nonlinearity and current which have been neglected in going from Eqs. 1 and 2 to Eq. 14. Figure 1 shows "the" dispersion relation, Eq. 14 and various approximations to it (to be described below). The heavy solid line is the linear dispersion relation with no current (Eq. 14). The region shown by vertical lines shows the effect of nonlinearity on the dispersion relation. The upper boundary to this region is Eq. 14, while the lower boundary corresponds to steady periodic waves of greatest height (but with no current) obtained from the results of Cokelet (1977). Hedges (1978, 1987) has plotted this region on a similar diagram. It can be seen that nonlinearities strongly affect the wave length for a given wave period. This is quantified to fifth order by Eq. 1, which is accurate over most of the region and which shows that departure from the linear relation is quadratic in wave height, so that the results for a wave which

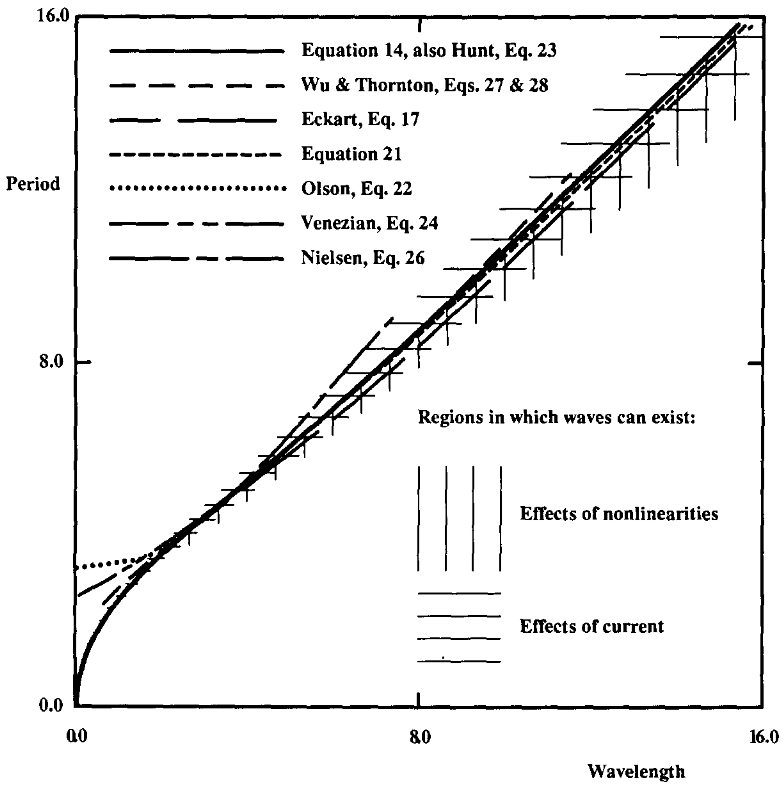

Fig. 1. Plot of the regions where a wave can occur, plus several mathematical approximations to the linearized zero-current relation (Eq. 14). The abscissa is dimensionless wavelength L/d, the ordinate is dimensionless period $T\sqrt{g/d}$.

is half as high as the highest for that wavelength would fall about 1/4 of the width of the shaded region below the solid curve.

The horizontal lines show the effects of current on the wavelength, for a given period. The boundaries of this region are simply those corresponding to a current which is $\pm 1/10$ of the wave speed. This value was arbitrarily chosen; of course in many situations the current may be less than this.

For waves of large amplitude on a favourable current, the effects of nonlinearity and current are cumulative, so that the point corresponding to an actual wavelength and period might be some distance below and to the right of both the hatched regions.

From the above it can be seen that wave height and current have significant effects on the wavelength corresponding to a particular period. In fact, Eq. 14 is at best a rough approximation. It has, however, been the subject of many papers attempting to use it to obtain a mathematical approximation to kd as an explicit function of $\sigma\sqrt{d/g}$. Below some of those approximations are presented.

APPROXIMATIONS TO THE STANDARD DISPERSION RELATION

Figure 1 shows most of the approximations which are described below. In most cases the plotting of the curve has ceased where its deviation from Eq. 14 has become such that the curve is identifiable. Where the curves agree closely with Eq. 14 they are of course not visible.

Eckart's approximation

In 1952, Eckart produced a remarkable irrational approximation which is not well known. His approximate solution (Eckart, 1952) is:

$$k=\frac{\sigma^2}{g}\,[\coth(\sigma^2 d/g)]^{1/2} \qquad (17)$$

Strictly speaking the equality should be replaced by an approximate equality, however, we retain the equality sign here and in other approximate formulae. To examine the behaviour of this approximation in the long wave limit we take the limit $\sigma^2 d/g \to 0$ and use the limiting behaviour of the hyperbolic function to give:

$$k=\sigma/\sqrt{gd} \qquad (18)$$

which is the exact solution of Eq. 15. For deep water or short waves, Eckart's approximation Eq. 17 in the limit $\sigma^2 d/g \to \infty$, becomes:

$$k=\sigma^2/g \qquad (19)$$

an exact solution of Eq. 16. Hence in the two limits, of long waves and short waves (relative to the water depth), Eckart's relation is an exact solution of Eq. 14.

Figure 1 shows Eckart's approximation. The worst fractional error in calculated wavelength is about 5%, for waves which are about 7 times the water depth. However, given that a wave might occur anywhere in the shaded regions of the figure, it would seem that Eckart's result, Eq. 17, is at least as good as Eq. 14, and probably better, in that it errs on the side of over-estimating wave length, thereby mimicking the effects of nonlinearity.

In many practical cases this accuracy is certainly acceptable, given the approximations on which Eq. 14 are based, that the water is of constant depth, that the waves are of small amplitude, that they are periodic, that the flow is irrotational, and that there is no current flowing.

A new approximation

Although, opposed to the idea of generating yet more approximations to Eq. 14, we mention here the fact that all approximations of the following form suggested by Eq. 17, namely:

$$k=\frac{\sigma^2}{g}\left[\coth(\sigma\sqrt{d/g})^{\nu}\right]^{1/\nu}$$

satisfy both long and short wave limits Eqs. 18 and 19. We have experimented with different values of ν (Eckart's approximation, Eq. 17 is for $\nu=2$). We found that the minimum value of the maximum error over all wave lengths in approximating Eq. 14 was for $\nu=1.49$. Rounding to $\nu=3/2$, the approximation:

$$k=\frac{\sigma^2}{g}\left[\coth(\sigma\sqrt{d/g})^{3/2}\right]^{2/3} \tag{20}$$

is obtained. Substituting $k=2\pi/L$ and $\sigma=2\pi/T$ and re-arranging it gives:

$$L=\frac{gT^2}{2\pi}\left[\tanh(2\pi\sqrt{(d/g)}/T)^{3/2}\right]^{2/3}=L_0\left[\tanh(2\pi d/L_0)^{3/4}\right]^{2/3} \tag{21}$$

in which L_0 is the deep-water wavelength, $L_0=gT^2/2\pi$. In either form, this approximation has a maximum error of only 1.7%; it is shown in Fig. 1. It is always closer to Eq. 14 than is Eq. 17, whose maximum error was 5%. It is one of the aims of this paper to emphasize that such accuracy is not necessarily an advantage, however, it is possible that Eq. 20 or 21 might be useful in practice, as a simple expression valid for all wavelengths.

Olson's approximation for longer waves

Olson (1973) obtained the series expansion of Eq. 14, and reverted the series, to give a series approximation, valid for longer waves:

$$\left(\frac{2\pi L}{gT^2}\right)^2=\frac{\sigma^2 d}{g}-\frac{(\sigma^2 d/g)^2}{3}+\frac{(\sigma^2 d/g)^3}{45}+\frac{(\sigma^2 d/g)^4}{189}+$$
$$0.000776014(\sigma^2 d/g)^5-0.000044892(\sigma^2 d/g)^6-$$
$$0.000071391(\sigma^2 d/g)^7-0.000022654(\sigma^2 d/g)^8 \qquad (22)$$

It can be seen on Fig. 1 that this is an excellent approximation to Eq. 14 for waves which are longer than twice the depth.

Hunt's approximation

Hunt (1979) obtained an approximation in Padé approximant form:

$$(kd)^2=(\sigma^2 d/g)^2+\frac{\sigma^2 d/g}{1+\sum_{n=1}^{\infty} d_n(\sigma^2 d/g)^n} \qquad (23)$$

where the d_n are given by $d_1=0.6666666667$, $d_2=0.3555555556$, $d_3=0.1608465608$, $d_4=0.0632098765$, $d_5=0.0217540484$, and $d_6=0.0065407983$. This expression is a remarkably accurate approximation to Eq. 14 over all wave lengths. Indeed, it was derived so as to be exact in both long and short wave limits. It is plotted on Fig. 1, but is everywhere obscured by the solid line which it approximates.

Venezian's approximation for longer waves

Venezian (1980) presented two Padé approximants for the reverted series expansion of Eq. 14. One is particularly simple:

$$kd=\frac{\sigma\sqrt{d/g}}{1-\frac{1}{6}\sigma^2 d/g} \qquad (24)$$

This is accurate for long waves, as can be seen in Fig. 1. It will be seen below that it closely mimics the reverted series approximation to second order. It is not intended to be used for shorter waves, where it loses accuracy, unlike Hunt's rather more complicated expression, Eq. 23 which was intended to be valid in that limit also.

Interpolation in a table

Young and Sobey (1980) adopted the computationally-efficient procedure of using linear interpolation in a table of values of kd and $\sigma\sqrt{d/g}$, originally computed from Eq. 14. While apparently not as appealing to other workers as an explicit formula, it is an efficient and elegant solution to the problem. The setting up of the table is of course a simple exercise.

However, if current is included, then it is necessary to set up a table of corresponding wave lengths and periods for a finite number of current values, and then to use two-dimensional interpolation in this table. Such tables have been given by Jonsson et al. (1971) and Jonsson (1978).

Nielsen's approximations

Nielsen (1982) obtained approximations for expressions occurring in linear wave theory in the limit of long waves and an approximation for Eq. 14 in the short wave limit. For an approximation to Eq. 14 he presented:

$$kd=\sigma\sqrt{d/g}\left[1+\frac{1}{6}\frac{\sigma^2 d}{g}+\frac{11}{360}\left(\frac{\sigma^2 d}{g}\right)^2+\dots\right] \qquad (25)$$

Equation 11 in the case of no current, $F=\bar{u}/gd=0$, is easily shown to reduce to this. It can be simply shown by manipulation of series that this is equivalent to the first three terms of Olson's expression, Eq. 22. It can be verified that Venezian's expression, Eq. 24 has a corresponding power series almost the same as this, but that the number 11 in the last term in Eqn. 25 is replaced by 10, demonstrating the fortuitous closeness of the simple expression, Eq. 24.

For shorter waves Nielsen (1984) presented the approximation:

$$kd=\frac{\sigma^2 d}{g}(1+2e^{-2\sigma^2 d/g}+\dots) \qquad (26)$$

obtained from standard asymptotic approximations to the tanh function for large arguments. This expression is plotted on Fig. 1, and gives good agreement for short waves.

Wu and Thornton's approximations

Wu and Thornton (1986) obtained the approximation:

$$kd=\sigma\sqrt{d/g}\left[1+\frac{\sigma^2 d/g}{6}\left(1+\frac{\sigma^2 d/g}{5}\right)\right] \qquad (27)$$

which is exact in the long wave limit and agrees with Eq. 14 at the point $\sigma\sqrt{d/g}=1$, $kd=1.200$.

For short waves they suggest the approximation:

$$kd=\sigma^2 d/g[1+2t(1+t)] \tag{28}$$

where:

$$t=e^{-2y'},\ y'=\sigma^2 d/g\left(1+1.26e^{-1.84\sigma^2 d/g}\right)$$

Both these relations, Eqs. 27 and 28 are plotted using the same line type on Fig. 1. In the approximate range each is highly accurate, they are only visible on the figure in the other wavelength limit.

Summary of the above approximations

Considering all the above approximations, Fig. 1 shows that as approximations to Eq. 14 they are almost all excellent, provided their limitations are recognized, usually that they be used for waves which are longer or shorter than a certain length. Also, from the above formulae, the amount of computation is a similar order of magnitude in most cases. One really wonders what all the debate and the comparison of marginal advantages between the various formulae has been about.

Hunt's Eq. 23 is very accurate over all wave lengths. Considering the long wave formulae, Olson's Eq. 22, Venezian's Eq. 24, Nielsen's Eq. 25 and Wu and Thornton's Eq. 27 are all highly accurate for waves longer than $L/d\approx 3$ ($T\sqrt{g/d}\approx 4$). Indeed, they are all based on the same reversion of the power series approximation, but with an extra interpolation point in the case of Eq. 27. It is interesting that the inclusion of higher-order terms as in Eq. 22 extends the range of validity only marginally, relative to the simplest expression of them all, Eq. 24.

Turning to the short wave formulae, both Nielsen's Eq. 26 and Wu and Thornton's Eq. 28 are excellent for $L/d<4$ ($T\sqrt{g/d}<5$). For longer waves the former deviates from Eq. 14 much more rapidly than the latter. However in this range the previously-presented long wave formulae are more valid anyway.

It is the main intention here to point out that all the formulae mentioned above (except for Eqs. 17, 20 and 21) have the capacity to represent Eq. 14 very accurately in their appropriate ranges. What none of them do is to allow for any effects of current or nonlinearity. Eckart's irrational approximation Eq. 17 and Eqs. 20 and 21 both mimic the effects of nonlinearity by overestimating wave length, given a value of period.

We suggest that with any of the above formulae the problem of approximating Eq. 14 has been well and truly solved. In fact we strongly urge a moratorium on the production of further formulae. The above simple formulae give acceptable accuracy for many engineering purposes. It should be recognized that the ability of each formula to approximate Eq. 14 is rather better than that of Eq. 14 to approximate the actual problem.

AN ALGORITHM WHICH INCLUDES CURRENT

If high accuracy is required, it should be recognized that the problem of solving a dispersion relation is really one of solving a transcendental equation, as has been done in several of the papers referred to above for the purpose of comparing the proposed approximate formulae. Newton's method can be used to refine the approximation. It is simply programmed and can be used to achieve arbitrary specifiable accuracy. In fact, the inclusion of current yields a minor modification of the method, unlike the complication of including current in explicit approximations, comparing for example, Eq. 11 with Eq. 25.

Equation 5 can be written:

$$f(k)=\bar{u}k-\sigma+\sqrt{gk\tanh kd}=0 \tag{29}$$

If the current, depth and wave period are known, then in this form, Eq. 29 is a transcendental equation for the wavenumber k. Newton's well-known method can be used for the solution of such equations. It has the highly-desirable property of quadratic convergence such that the number of figures of accuracy doubles with each iteration. It is written:

$$k_{n+1}=k_n-\frac{f(k_n)}{f'(k_n)}$$

giving the refined approximation k_{n+1} in terms of approximation k_n. In the case of Eq. 29 this gives:

$$k_{n+1}d=\frac{2\sigma\sqrt{d/g}\sqrt{k_nd\tanh k_nd}-k_nd\tanh k_nd+k_n^2d^2\operatorname{sech}^2k_nd}{2\bar{u}/\sqrt{gd}\sqrt{k_nd\tanh k_nd}+\tanh k_nd+k_nd\operatorname{sech}^2k_nd} \tag{30}$$

Although this looks complicated it can be evaluated with relatively few operations, as $\operatorname{sech}^2k_nd=1-\tanh^2k_nd$, and the tanh function need only be evaluated once per iteration. In the usual case where $\bar{u}$ is small relative to the speed of the waves, then any of the approximations described above might be used to provide a good approximation to give the initial estimate k_0. However, there is some advantage in using formulae which explicitly include the current. We have used these, and recommend the following procedure, *although there are bound to be other approaches of a similar order of computational* *complexity and accuracy*. Searching for better methods would seem to be hardly worthwhile, given the results of the comparisons for zero current.

The following algorithm uses Eqs. 6 or 11 to give an initial approximation, followed by use of Eq. 30 as an explicit approximation to refine the solution. It would be simpler to program to use Eq. 20 or 21 for the initial approximation, but would usually require one more subsequent iteration for finite values of current. The algorithm is:

if $T\sqrt{g/d}<4$ Calculate k_0 from Eq. 6

else, calculate k^0 from Eq. 11

$F=\bar{u}/\sqrt{gd}$

$\omega=\sigma\sqrt{d/g}$

for $n=0$ (and $n=1,2,...$ only if really necessary)

$\alpha=\tanh k_n d$

$\beta=\alpha k_n d$

$\gamma=\sqrt{\beta}$

$$k_{n+1}d=\frac{2\gamma\omega-\beta+(k_n d)^2-\beta^2}{2\gamma F+\alpha+k_n d(1-\alpha^2)}$$

For no current, after a single iteration the maximum error over all wavelengths using this method is 0.001%. With a current of +10% of the long wave speed a single iteration gives a maximum error of 0.02% in the wavelength, whereas with an adverse current of this magnitude a single pass gives a maximum error of 0.4%, and a subsequent pass reduces this to less than 0.001%. Thus, the method converges to engineering accuracy in one iteration. Except for purposes of research or comparison, the second iteration is unlikely to be necessary. This algorithm is similar to that recommended by McKee (1988) for finding the sequence of imaginary values of k which satisfy Eq. 14. He found a simple quadratic approximation to provide an initial estimate and wrote an explicit iteration formula, the equivalent of Eq. 30 as it is evaluated in the above algorithm, which also required only one evaluation of a transcendental function per iteration.

Although in many situations it might be considered unnecessary to incorporate the effects of current in calculations, the act of being forced to specify a value of current, even if it is zero, is a salutatory reminder that one is obtaining an *approximate* solution to an *approximate* problem, and no great effort should go into refining methods or solutions.

CONCLUSIONS

The physical basis for the dispersion relation has been discussed. Several explicit approximations for wave length as a function of other physical variables have been presented and compared, and it has been shown that they model the standard dispersion relation quite accurately. Effects of wave height and current ignored by these methods are rather larger than are differences between the methods. It is asserted that the problem of approximating the standard linear dispersion relation can be regarded as solved to engineering accuracy, and that further effort in that direction is not justified.

Explicit approximations which include the effects of current have been presented, plus an algorithm based on Newton's method which is recommended whether or not the effects of current are considered.

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